COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 36, nº 1 (1978), p. 101-112

<http://www.numdam.org/item?id=CM_1978__36_1_101_0>

COMPOSITIO MATHEMATICA, Vol. 36, Fasc. 1, 1978, pag. 101–112
Noordhoff International Publishing
Printed in the Netherlands

CONICS IN CHARACTERISTIC 2

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Abstract

The purpose of this work is to show that, over an algebraically closed field of characteristic 2, the number of smooth conics tangent to 5 general conics is only 51, (versus 3264 in every other characteristic) and that each solution occurs with multiplicity one.

1. Introduction

Enumerative problems concerning conics – and in particular that of determining the number, 3264, of smooth conics tangent to 5 general ones – have had a rich history. The reader is referred to the article of H.G. Zeuthen and M. Pieri, "Géometrie énumerative", Encyclopédie des sciences Mathématiques, tome III, vol. 1, fasc. II, pp. 260–331, Leipzig (1915), and to the forthcoming work of S.L. Kleiman, "Chasles' Enumerative Theory of Conics, a Historical Introduction", to appear in the volume of the Mathematical Association of America on Algebraic Geometry, edited by A. Seidenberg.

We review here some of the features of the classical theory, which extend without surprises to every characteristic $\neq 2$. Then we explain the changes needed to treat the case of char. 2.

The set of conics tangent to a fixed smooth conic is a hypersurface D of degree d ($=6$ in char. $\neq 2$, and $=3$ in char. 2) in the projective 5-space P^5 parametrizing all the conics. If one tries a naïve solution to the problem, one will obtain d^5 for the number of intersections of 5 such divisors (as J. Steiner (1848) and J. Bischoff (1859) actually did). This number, however, has no enumerative significance because each

* Supported by the CNPq. (Brazil).

of these hypersurfaces contains the surface parametrizing the double lines. What is in fact required is the number of isolated intersections of these hypersurfaces with the open subset of smooth conics. The correct answer was first found by Chasles (1864), essentially by introducing the variety B of complete conics. B is, in char. $\neq 2$, the closure of the graph of the duality correspondence that sends a smooth conic to its dual conic. It turns out that B is just the blowup of P^5 along the Veronese surface of double lines and that the proper transforms of 5 general translates of the hypersurface D intersect properly. Moreover, this intersection is actually transversal and lies over the open subset of P^5 of smooth conics. The correct solution, namely 3264, can be computed in the numerical equivalence ring of B as the 5-fold self-intersection of the class of the proper transform of D.

In characteristic 2, however, the classical duality no longer holds. Nevertheless, we will see that the very pathology can be used to our advantage. We replace the classical duality correspondence by the mapping of a smooth conic to its strange point (see 2 and 3 below). In section 4, we introduce the variety of complete conics and identify it with the blowup of P^5 along the linear subspace of double lines. We review in section 5 the action of $Gl(3, k)$ on P^5 and in the following section we show this action extends naturally to the complete conics. In section 7, we employ the techniques of [3] to prove a "char. p-transversality" statement needed in the last section to justify the claim of multiplicity one. Section 8 contains an elementary derivation of the degree of D and the verification that D passes doubly through the subspace of double lines. Finally, in section 9, we compute the number 51 of smooth conics tangent to 5 general conics.

2. Strange points (cf. [4], p. 76)

Fix an algebraically closed ground field k of characteristic 2.

The most curious feature of the theory is the presence of a point common to all tangent lines to a fixed conic. Indeed, let

$$F \equiv aX_1^2 + bX_2^2 + cX_3^2 + dX_1X_2 + eX_1X_2 + fX_2X_3 = 0 \tag{1}$$

denote the equation of an arbitrary conic. The tangent line of F at a point (x, y, z) is given by

$$(dy + ez)X_1 + (dx + fz)X_2 + (ex + fy)X_3 = 0.$$

Obviously the point (f, e, d) lies on that line. We are, of course, assuming that (x, y, z) is not a singular point of our conic, and in particular, the coefficients f, e, d are not all zero. This amounts to saying F is not the square of a linear form, i.e., the conic is not a double line.

We call (f, e, d) the *strange point* of F, and denote it by $\mathrm{st}(F)$.

3. The dual map

The projective 5-space P^5 parametrizes the conics. We will often denote a conic and its equation by the same letter and say "the conic F" instead of "conic with equation $F = 0$". We adopt the homogeneous coordinates (a, b, c, d, e, f), in that order, for the conic F as in (1).

Let $L \subset P^5$ denote the linear subspace

$$L\colon f = e = d = 0. \tag{2}$$

The map

$$\begin{aligned} \mathrm{st}\colon (P^5 - L) &\longrightarrow P^2 \\ F &\longrightarrow \mathrm{st}(F) \end{aligned}$$

will be called the *dual map*. In homogeneous coordinates, we have,

$$\mathrm{st}(a, \ldots, f) = (f, e, d).$$

4. Complete conics

Let $\mathscr{A} \subset (P^5 - L) \times P^2$ denote the graph of the dual map. The closure B of $\mathscr{A}$ in $P^5 \times P^2$ will be called the *variety of complete conics*. It plays a central role below.

Since st is a linear projection with centre L, and B is the closure of its graph, B is the blowup of P^5 along L. We have the following diagram:

$$\begin{tikzcd}
(P^5 - L) \times P^2 \arrow[d] & P^5 \times P^2 \arrow[l, "\subset"'] \arrow[d] & \\
\mathscr{A} \arrow[d, "\wr"'] & B \arrow[l, "\subset"'] \arrow[r] \arrow[d, "p"] & E = p^{-1}(L) \arrow[d] \\
P^5 - L & P^5 \arrow[l, "\subset"'] \arrow[r] & L
\end{tikzcd} \tag{3}$$

where p is the projection. It can be shown that B is cut out in $P^5 \times P^2$ by the bi-homogeneous equations,

$$dX_2 + eX_3 = dX_1 + fX_3 = eX_1 + fX_2 = 0. \tag{4}$$

The subvariety $E = p^{-1}(L)$ is the exceptional divisor. Recalling equations (2) which define L, one finds

$$E = L \times P^2 \hookrightarrow P^5 \times P^2.$$

Intuitively, the exceptional divisor solves the indeterminacy of assigning a strange point to a double line. Precisely, the 2nd projection $q: B \to P^2$ extends the dual map, once we have identified $P^5 - L$ with $\mathscr{A} \hookrightarrow B$ as in diagram (3).

We remark for later use that the class of E in the Chow ring $A(B)$ is given by

$$[E] = p^*[H] - q^*[H'], \tag{5}$$

where H (resp. H') is a hyperplane in P^5 (resp. a line in P^2). In fact, (5) is an immediate consequence of the finer equality of divisors,

$$p^{-1}(H) = q^{-1}(H') + E,$$

which simply means that the product of local equations for E and $q^{-1}(H')$ is a local equation for $p^{-1}(H)$. This can easily be verified with the help of (2) and (4).

5. The group action on P^5

The general linear group

$$\underset{\sim}{G} = Gl(3, k)$$

acts on P^2 and hence on the linear systems of plane curves. In particular, we have a natural action of $\underset{\sim}{G}$ on our P^5 of conics. Explicitly, given an element g in $\underset{\sim}{G}$ and a conic F in P^5, the equation of the image of $\{F = 0\}$ under g is

$$F\left(\sum g'_{1j}X_j, \sum g'_{2j}X_j, \sum g'_{3j}X_j\right) = 0,$$

where (g'_{ij}) is the inverse matrix of g.

It is well-known (and very easy to check) that $\underset{\sim}{G}$ has 3 orbits in P^5:

(i) $P^5 - S$, the complement of the divisor S which parametrizes the singular conics;

(ii) $S - L$, the variety of the singular conics consisting of 2 distinct lines; and

(iii) L, the linear subspace (2) parametrizing the double lines.

(5.1) PROPOSITION: *Fix some F in $T = P^5 - S$. Then the scheme-theoretic fibres of the map*

$$m\colon \underset{\sim}{G} \longrightarrow T, \quad g = (g_{ij}) \longrightarrow g^{-1}F = F\Big(\sum g_{1j}X_j, \sum g_{2j}X_j, \sum g_{3j}X_j\Big),$$

are integral.

PROOF: Any two fibres are isomorphic, because $\underset{\sim}{G}$ acts transitively on T. For the same reason, we may assume $F = X_1^2 + X_2X_3$. Denote by $\underset{\sim}{G}_{(F)}$ the fibre over F. Set-theoretically, $\underset{\sim}{G}_{(F)}$ is the subgroup of all elements in G that leave F fixed. Restricting m over the open subset T_a, complement of the hyperplane $\{a = 0\}$, we may write, for each $g = (g_{ij})$ in $m^{-1}(T_a)$,

$$(6) \qquad m(g) = \Big(\Big(\sum g_{1j}X_j\Big)^2 + \Big(\sum g_{2j}X_j\Big)\Big(\sum g_{3j}X_j\Big)\Big)\Big/(g_{11}^2 + g_{21}g_{31}).$$

The equations defining $\underset{\sim}{G}_{(F)}$ in $\underset{\sim}{G}$ are obtained by matching the coefficients of $m(g)$ above with those of F. Let $\phi_\nu(g_{ij}) = 0$ denote these equations (there are 5 of these since the coefficient of X_1^2 has been normalized to 1). We will show $\underset{\sim}{G}_{(F)}$ is (a) smooth (hence reduced) and (b) irreducible. Since $\underset{\sim}{G}_{(F)}$ acts transitively on itself, smoothness follows from the explicit verification that the jacobian matrix $(\partial\phi_\nu/\partial g_{ij})$ has a rank 5 submatrix at $g =$ identity. This can be done conveniently by differentiating (6) implicitly. As to the irreducibility, recall that any smooth conic is isomorphic to P^1, and therefore, the action of $\underset{\sim}{G}_{(F)}$ on the conic F yields a homomorphism of algebraic groups,

$$h\colon \underset{\sim}{G}_{(F)} \longrightarrow PGL(2, k) \quad (= \text{automorphisms group of } P^1),$$

(see [1], prop. on p. 265). The kernel of h is the subgroup of scalar multiples of the identity matrix, because any element of $\underset{\sim}{G}$ that fixes more than 4 points, no 3 of which are collinear, induces the identity. It follows that the fibres of h are irreducible. Counting dimensions, one sees that h is dominating; hence, h is surjective ([1], (1.4), p. 88). Since h is generically flat ([EGAIV$_2$] 6.9.1), it is actually flat (because $\underset{\sim}{G}_{(F)}$ acts transitively on $PGL(2, k)$). By ([EGAIV$_2$] 2.3.5 (iii)), it follows that $\underset{\sim}{G}_{(F)}$ is irreducible.

6. The group action on B

Since any automorphism g in $\underset{\sim}{G}$ preserves both tangency and lines, it follows that for any F in $P^5 - L$, the strange point of gF is $g(\mathrm{st}(F))$. In other words, the dual map is $\underset{\sim}{G}$-equivariant. Thus, its graph $\mathscr{A} \subset P^5 \times P^2$ is invariant under the action of $\underset{\sim}{G}$ on $P^5 \times P^2$. By continuity, it follows that B is also $\underset{\sim}{G}$-invariant.

Let us now describe the orbits of $\underset{\sim}{G}$ in B. Observing that $p: B \to P^5$ is $\underset{\sim}{G}$-equivariant, and that p identifies $B - E$ with $P^5 - L$, we conclude that $B - E$ is $\underset{\sim}{G}$-invariant, and the orbits of $\underset{\sim}{G}$ in $B - E$ are identified with those of $\underset{\sim}{G}$ in $P^5 - L$, namely, $S - L$ and $P^5 - S$ (see 5).

It remains to investigate the orbits of $\underset{\sim}{G}$ in E. Recall that a point of E is a pair (F, x) where F is a double line and x is an arbitrary point in P^2. Let $\underset{\sim}{G}_{(F)}$ denote the stabilizer of F, i.e., the set of g in $\underset{\sim}{G}$ such that $gF = F$. It is clear that $\underset{\sim}{G}_{(F)}$ has precisely 2 orbits in P^2: (i) the line l of which F is the double; and (ii) $P^2 - l$. Therefore, the orbits of G in E are

$$E_c = \{(F, x) \mid x \text{ is in } F\}$$

and

$$E_0 = E - E_c.$$

7. Miscellaneous properties of group actions

Throughout this section, the characteristic of k is arbitrary.

Let $\underset{\sim}{G}$ be an integral algebraic group, and V an integral variety with a $\underset{\sim}{G}$-action $\underset{\sim}{G} \times V \to V$. Assume there are only finitely many orbits.

(7.1) Lemma: *Exactly one of the orbits is open in V.*

Proof: Each orbit is open in its closure ([1], p. 98). Now, V is irreducible and equal to the finite union of the closures of the orbits, so V is the closure of some orbit. There is only one open orbit because orbits are disjoint and V is irreducible.

(7.2) PROPOSITION: *Let $W_1, \ldots, W_n$ be (not necessarily distinct nor irreducible) locally closed subvarieties of V. Suppose each W_i intersects properly each of the orbits, and*

$$r \underset{\mathrm{df}}{=} \left(\dim V - \sum \operatorname{codim}(W_i, V)\right) \leq 0.$$

Then: (i) There exists an open dense subset U of $\underset{\sim}{G}^{\times n}$ such that, for every (g_i) in U, the intersection $\cap(g_i W_i)$ is finite, is empty if $r < 0$, and lies in the open orbit.

(ii) If each W_i is integral, and if, for some point x_0 in the open orbit V_0, the map

$$m_{x_0}: \underset{\sim}{G} \longrightarrow V_0, \quad g \longmapsto g^{-1}x_0,$$

has integral fibres, then there exist integers s, e and an open dense subset U of $\underset{\sim}{G}^{\times n}$ such that, for all (g_i) in U, the intersection $\cap(g_i W_i)$ has precisely s distinct points and the multiplicity at each is 1 in characteristic 0 and p^e in characteristic $p > 0$.

PROOF: Observe that, once (i) is proven, we may replace V by V_0 and W_i by $W_i \cap V_0$ to prove (ii). On the other hand, since $\underset{\sim}{G}^{\times n}$ is irreducible (because $\underset{\sim}{G}$ is and k is algebraically closed), we may, also in the proof of (i), replace V by an orbit and W_i by an irreducible component. Thus, we may as well assume the action is transitive and W_i integral.

Consider the fibre product diagram:

$$\begin{array}{llll} & Z \longrightarrow \underset{\sim}{G}^{\times n} \times W_1 \times \cdots \times W_n, & & ((g_i), (x_i)) \\ & q' \big\downarrow \qquad \big\downarrow q & & \quad \big\downarrow \\ (7) & V \longrightarrow V^{\times n} & & \quad (g_i x_i) \\ & & & \\ & x \longmapsto (x, \ldots, x). & & \end{array}$$

By ([EGA IV$_2$], 6.9.1), q is generically flat. Since $\underset{\sim}{G}^{\times n}$ acts transitively on $V^{\times n}$ (and compatibly with q), q is flat. Since q is also surjective, all of its fibres are equidimensional, with dimension

$$\dim(\underset{\sim}{G}^{\times n} \times W_1 \times \cdots \times W_n) - \dim(V^{\times n}) = \dim(\underset{\sim}{G}^{\times n}) + r - \dim(V).$$

Hence Z is equidimensional, with

$$\begin{aligned}\dim(Z) &= \dim(V) + (\dim(\underset{\sim}{G}^{\times n}) + r - \dim(V))\\ &= \dim(\underset{\sim}{G}^{\times n}) + r.\end{aligned}$$

Denote by $p: Z \to \underset{\sim}{G}^{\times n}$ the projection map. A minute of reflection should convince us all that the fibre $p^{-1}(g_i)$ is precisely

$$V \underset{V^{\times n}}{\times} ((g_i) \times W_1 \times \cdots \times W_n) = \cap (g_i W_i).$$

There exists an open dense subset of $\underset{\sim}{G}^{\times n}$ over which the fibres of p are either empty or of the dimension

$$\dim(Z) - \dim(\underset{\sim}{G}^{\times n}) = r,$$

(e.g., by generic flatness). This completes the proof of (i).

Next we show that the additional hypotheses of (ii) imply Z is in fact integral. By ([EGA IV$_2$], 2.3.5 (iii), and 3.3.5), it suffices to show the fibres of q' (see diagram (7)) are integral. Since q' is the pullback of q, we are reduced to verifying the fibres of $\underset{\sim}{G} \times W_i \to V$ are integral for each W_i. Now, it can easily be seen that the fibre over any x in V is isomorphic to the fibred product $\underset{\sim}{G} \underset{V}{\times} W_i$ defined by the diagram,

$$\begin{array}{ccc} \underset{\sim}{G} \underset{V}{\times} W_i & \longrightarrow & W_i \\ \downarrow & & \downarrow \\ \underset{\sim}{G} & \xrightarrow{m_x} & V, \end{array} \quad m_x(g) = g^{-1}x.$$

Since m_x and m_{x_0} differ by an automorphism, every m_x has integral fibres. Since m_x is flat (same argument as for q above), it follows that $\underset{\sim}{G} \underset{V}{\times} W_i$ is flat and has integral fibres over W_i, whence is integral. Thus Z is integral, as we claimed.

By the lemma below, there exists an open dense subset U of $\underset{\sim}{G}^{\times n}$ such that, for every (g_i) in U, the number of distinct points in $p^{-1}(g_i)$ (resp. the multiplicity at each) equals the separable (resp. inseparable) degree of p (both are zero if $\dim(p(Z)) < \dim(\underset{\sim}{G}^{\times n})$). Since the inseparable degree is 1 in char. 0 and a power of p in char. $p > 0$, (ii) follows.

Remark: The multiplicity referred to in the last paragraph as well as in the lemma below is the naïve one, namely, the length of the artinian local ring of the fibre at each point. One still must verify that it coincides with the intersection-theoretic one, obtained from the alternating sum of Tor's. Now, all the higher Tor's vanish provided each g_iW_i is Cohen-Macaulay at the points of their intersection ([5], Cor. p. V-20). Shrinking U, we may actually assume (in view of (i)) that the intersection lies in the smooth locus of each g_iW_i. Therefore, the naïve multiplicity is right.

We include a proof of the result below as no convenient reference could be found.

LEMMA: *Let $f: X \to Y$ be a finite surjective map of integral schemes of finite type over k. Then there exists an open dense subset U of Y such that the geometric fibre of f over each y in U has s (= separable degree of f) distinct points and the multiplicity at each is the inseparable degree of f.*

PROOF: Denote by S the singular locus of X. Replacing Y by $Y\text{-}f(S)$, we may assume X smooth, and in particular, normal. Now, let X_s denote the normalization of Y in the separable closure of its function field $R(Y)$ in $R(X)$. Since X is the normalization of Y in $R(X)$, we get a factorization $X \to X_s \to Y$ for f. Thus, we may assume f is either separable or purely inseparable. In both cases, by factoring f through the normalization of Y in some intermediate field, the result follows easily by induction on the degree.

8. Conics tangent to a smooth conic

Return to char. k = 2. We will derive the equation of the subvariety D of P^5 parametrizing the conics tangent to a given smooth conic. Since $\underset{=}{G} = G1(3, k)$ acts transitively on the set of smooth conics, (and of course the action preserves tangency), it is enough to work it out for the conic

$$F_0 \equiv X_1^2 + X_2X_3,$$

which will remain fixed for the rest of this article.

A conic F in $P^5 - L$ is tangent to F_0 iff some point of $F \cap F_0$ lies on a line with st(F) and st$(F_0) = (1, 0, 0)$. Hence we have to eliminate X_1, X_2 and X_3 from the equations

$$F(X_1, X_2, X_3) = F_0(X_1, X_2, X_3) = \begin{vmatrix} f & e & d \\ 1 & 0 & 0 \\ X_1 & X_2 & X_3 \end{vmatrix} = 0$$

On the complement of the hyperplane $\{e=0\}$, we have

$$X_3 = dX_2/e, \quad X_1^2 = X_2X_3 = X_2^2 d/e.$$

Substituting in $F \equiv aX_1^2 + \cdots + fX_2X_3$, we get

$$X_2^2\left(\frac{ad}{e} + b + \frac{cd^2}{e^2} + d\sqrt{\frac{d}{e}} + e\frac{d}{e}\sqrt{\frac{d}{e}} + f\frac{d}{e}\right) = 0,$$

which requires

$$\theta \equiv ade + be^2 + cd^2 + def = 0. \tag{8}$$

Set (provisorily) D' = divisor of zeros of θ. We have seen that $D = D'$ off $\{e=0\}$. Since neither D nor D' contains $\{e=0\}$ (e.g., look at $X_3^2 + X_1X_2$), it follows that $D = D'$.

(8.1): Let us now consider the tangency of complete conics. Looking at the homogeneous equation (8) of D, we find

$$\theta \in (f, e, d)^2 - (f, e, d)^3.$$

Since (f, e, d) is the homogeneous ideal of L (see (2)), the multiplicity of L in D is exactly 2. It follows from the general theory of blowing-ups that the pull-back of D to B has the form

$$p^{-1}(D) = 2E + \tilde{D}, \tag{9}$$

where $\tilde{D}$, the proper transform of D in B, is the closure of $p^{-1}(D-L)$ in B. ((9) means that, locally on B, the ideal of $p^{-1}(D)$ is generated by an element of the form $\epsilon^2\tilde{\delta}$, where ϵ (resp. $\tilde{\delta}$) generates the ideal of E (resp. $\tilde{D}$), and ϵ does not divide $\tilde{\delta}$.)

What is $E \cap \tilde{D}$? In other words, when is a double line F plus an assigned strange point x a limit of elements of $p^{-1}(D-L)$? We claim that *a n.a.s.c. for* (F, x) *in* $L \times P^2$ *to lie in* $\tilde{D}$ *is that* x *belong to a line through* $\mathrm{st}(F_0)$ *and a point in* $F \cap F_0$.

Indeed, since for any F in $D-L$, there exists a point x' in $F \cap F_0$ such that x', $\mathrm{st}(F)$ and $\mathrm{st}(F_0)$ are collinear, namely, the point of tangency, it follows by continuity that the condition is necessary.

Conversely, assume the condition. We will prove that (F, x) lies in $\tilde{D}$ by explicitly exhibiting a pencil $\{A_t\} \subset D$ such that: (a) $A_0 = F$ and, (b) for each $t \neq 0$, A_t is in $D-L$ and $\mathrm{st}(A_t) = x$.

To construct the pencil, we consider two cases:

(i) F intersects F_0 in only one point;

(ii) F intersects F_0 in two distinct points.

Recalling that the stabilizer $\underset{\sim}{G}_{(F_0)}$ induces the full group of automorphisms of F_0 (see proof of 5.1) and observing that $\tilde{D}$ is invariant under $\underset{\sim}{G}_{(F_0)}$, we may assume in case (i) (resp. (ii)) that F is twice the tangent line $X_2 = 0$ (resp. the transversal line $X_1 = 0$). Hence $F \cap F_0$ is $\{(0, 0, 1)\}$ (resp. $\{(0, 1, 0), (0, 0, 1)\}$). Thus x is of the form $(f, 0, d)$ (resp. $(f, e, 0)$ or $(f, 0, d)$). Consider the pencils:

$$ {}_1A_t = X_2^2 + t(dX_1X_2 + fX_2X_3); \quad {}_2A_t = X_1^2 + t(eX_1X_3 + fX_2X_3) $$

and

$$ {}_3A_t = X_1^2 + t(dX_1X_2 + fX_2X_3). $$

With the help of equation (8), one sees at once that ${}_iA_t$ lies in D for $i = 1, 2, 3$ and all t, and also that (a) and (b) hold.

9. The 51 smooth conics tangent to 5 general conics

We will prove now that 5 general translates of $\tilde{D}$ (see (9)) intersect properly on B in 51 points. Moreover, the intersection actually lies over $P^5 - S$, and the multiplicity of each point in that intersection is one.

To apply the results on group actions of Section 7, we must verify that $\tilde{D}$ intersects properly each of the orbits of $\underset{\sim}{G} = Gl(3, k)$ in B. But this is clear, because each of the orbits is irreducible and $\tilde{D}$ is a divisor which does not contain any of them (see the explicit descriptions in 6 and 8.1). In view of (5.1), we may apply (7.2, (ii)) with $V = B$ and $W_i = \tilde{D}$: there exist an open dense subset $U \subset G^{\times 5}$ and integers s and e such that, for each (g_i) in U, the intersection $g_1\tilde{D} \cap \cdots \cap g_5\tilde{D}$ lies over $P^5 - S$, has s distinct points and the multiplicity at each is 2^e. Hence, the weighted number of points is $s2^e$. However, the computation below gives $s2^e = 51$, so $e = 0$ as asserted.

Let us finally compute the weighted number of intersections of 5 general translates of $\tilde{D}$. Since $\underset{\sim}{G}$ is isomorphic to an open subset of an affine space, the translates of $\tilde{D}$ lie all in the same class in the rational equivalence ring $A(B)$. Thus, what is required is the degree of the self-intersection $[\tilde{D}]^5$.

By (9), we have $[\tilde{D}] = p^*[D] - 2[E]$. Since D is a hypersurface of degree 3 (by (8)), its class in $A(P^5)$ is $3[H]$. Recalling the expression (5), then observing the relation $[H']^i = 0$ for $i > 2$, we get

$$\begin{aligned}[\tilde{D}]^5 &= (p^*[H] + 2q^*[H'])^5 \\ &= p^*[H]^5 + 10p^*[H]^4q^*[H'] + 40p^*[H]^3q^*[H']^2.\end{aligned}$$

Using the projection formula, and the invariance of degree under p_*, we get:

$$\deg[\tilde{D}]^5 = 1 + 10 + 40 = 51.$$

We have used the formulas

$$\deg([H']q_*(p^*[H]^4)) = \deg([H']^2q_*(p^*[H]^3)) = 1,$$

which hold because the restriction of $q: B \to P^2$ to the pullback of a linear subspace of P^5 disjoint from L, is an isomorphism onto a linear subspace of P^2.

Acknowledgment

We would like to thank Steven L. Kleiman for many helpful suggestions.

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(Oblatum 15-XI-1976)

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