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THE TRANSCENDENCE OF CERTAIN QUASI-PERIODS ASSOCIATED WITH ABELIAN FUNCTIONS IN TWO VARIABLES

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1. Introduction

Let Λ be a lattice in the complex space $\mathbb{C}^2$. An Abelian function with respect to Λ is a meromorphic function $f(\mathbf{z})$ of the variable $\mathbf{z} = (z_1, z_2)$ such that

$$f(\mathbf{z} + \boldsymbol{\omega}) = f(\mathbf{z}) \tag{1}$$

for all $\boldsymbol{\omega}$ in Λ. The totality of such functions form a finitely generated field $\mathfrak{F}$ whose transcendence degree over $\mathbb{C}$ is well-known not to exceed 2 (see for example [9], p. 78). Throughout this paper we shall assume that this transcendence degree is exactly 2. Hence there exists a vector $\mathbf{A} = (A_1, A_2)$ of algebraically independent Abelian functions A_1 and A_2, and a function B algebraic over $\mathbb{C}(\mathbf{A})$, such that

$$\mathfrak{F} = \mathbb{C}(\mathbf{A}, B). \tag{2}$$

Further, from (1) it is clear that the differential operators $\partial/\partial z_1$, $\partial/\partial z_2$ take the field $\mathbb{C}(\mathbf{A}, B)$ into itself.

The object of this paper is to prove the transcendence of certain numbers associated with $\mathfrak{F}$, and consequently we shall suppose $\mathfrak{F}$ defined with reference to the field $\mathbf{A}$ of algebraic numbers in the following sense. First, the function B is algebraic over $\mathbf{A}(\mathbf{A})$, and secondly the operators $\partial/\partial z_1$, $\partial/\partial z_2$ take the field $\mathbf{A}(\mathbf{A}, B)$ into itself. We then say that $\mathfrak{F}$ is algebraically defined with respect to its generators $\mathbf{A}$ and B.

It is easy to construct examples of such Abelian function fields. An important example for this paper arises from taking Λ as the Cartesian product of lattices Λ_1, Λ_2 each in $\mathbb{C}$. If $\wp_1(z)$, $\wp_2(z)$ are the Weierstrass elliptic functions associated with these lattices the corresponding Abelian function field is given by the right-hand side of (2) with

$$A_1(\mathbf{z}) = \wp_1(z_1),\ A_2(\mathbf{z}) = \wp_2(z_2),\ B(\mathbf{z}) = \wp_1'(z_1) + \wp_2'(z_2). \tag{3}$$

Furthermore if the invariants of $\wp_1(z)$, $\wp_2(z)$ are algebraic numbers the above generators provide an algebraic definition of this function field.

Our main concern in this paper is with a slightly different class of functions. We define a quasi-periodic function with respect to Λ as a meromorphic function $f(\mathbf{z})$ such that for each $\boldsymbol{\omega}$ in Λ the difference

$$\eta(f, \boldsymbol{\omega}) = f(\mathbf{z} + \boldsymbol{\omega}) - f(\mathbf{z}) \tag{4}$$

is constant. This plainly implies that $\partial f/\partial z_1$, $\partial f/\partial z_2$ lie in $\mathbb{C}(\mathbf{A}, B)$. In addition we say that f is algebraically defined if these derivatives lie in $\mathbb{A}(\mathbf{A}, B)$. Clearly any linear function $\alpha_1 z_1 + \alpha_2 z_2$ is quasi-periodic, and if α_1, α_2 are algebraic numbers it is also algebraically defined. The existence of less trivial examples, at least in the Cartesian case mentioned above, can be seen as follows. If $\zeta_1(z)$, $\zeta_2(z)$ are the Weierstrass zeta functions associated with $\wp_1(z)$, $\wp_2(z)$ respectively, then any combination such as

$$f(\mathbf{z}) = \alpha_1 z_1 + \alpha_2 z_2 + \beta_1 \zeta_1(z_1) + \beta_2 \zeta_2(z_2)$$

is quasi-periodic; also f is algebraically defined if α_1, α_2, β_1, β_2 are themselves algebraic numbers. Notice that if ω_1, ω_2 are periods of Λ_1, Λ_2 respectively, and η_1, η_2 are the corresponding quasi-periods, the quasi-period attached to f and the period $\boldsymbol{\omega} = (\omega_1, \omega_2)$ of Λ is given by

$$\eta(f, \boldsymbol{\omega}) = \alpha_1\omega_1 + \alpha_2\omega_2 + \beta_1\eta_1 + \beta_2\eta_2. \tag{5}$$

More generally quasi-periodic functions may be constructed from integrals of the second kind (see p. 121 of [8]) or by logarithmically differentiating theta functions (see (7) below).

Let now Λ be a lattice such that the corresponding Abelian function field $\mathfrak{F}$ is algebraically defined with respect to certain generators $\mathbf{A}$ and B. Fix a non-zero period $\boldsymbol{\omega} = (\omega_1, \omega_2)$ of Λ, and let H be a quasi-periodic function with respect to Λ, algebraically defined with respect to the same generators. In this paper we shall determine when the quasi-period $\eta(H, \boldsymbol{\omega})$ is transcendental.

Partial results on this problem were first obtained in 1941 by Schneider [8]. His theorems, which are stated under slightly different hypotheses for Abelian functions on $\mathbb{C}^n$, imply the following when $n = 2$. If H is not itself an Abelian function, and $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are suitably chosen periods of Λ, then at least one of $\eta(H, \boldsymbol{\omega}_1)$, $\eta(H, \boldsymbol{\omega}_2)$ is transcendental.

In this connexion there is an assertion in [6] which ought to be corrected. On taking $H(\mathbf{z}) = z_1$, say, in this result of Schneider we see that the first components of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ cannot both be algebraic. Similarly for the second components; from the generalization in $\mathbb{C}^n$ Schneider deduced his well-known theorem on values of the beta function. The result quoted in [6] and mistakenly attributed to Schneider, that the components of a single period cannot all be algebraic, appears to have been first proved by Lang in [3].

Now $\eta(H, \boldsymbol{\omega})$ can quite easily be an algebraic number, for if H is itself an Abelian function this quasi-period is zero. Also if $\alpha_1\omega_1 + \alpha_2\omega_2 = 0$ for algebraic α_1, α_2, we have $\eta(H, \boldsymbol{\omega}) = 0$ when

$$H(\mathbf{z}) = \alpha_1 z_1 + \alpha_2 z_2. \tag{6}$$

Accordingly we call $\boldsymbol{\omega}$ special if its components are linearly dependent over $\mathbf{A}$. Our theorem implies that the above are the only examples of algebraic quasi-periods.

THEOREM: *If H is not an Abelian function and $\boldsymbol{\omega}$ is not special then $\eta(H, \boldsymbol{\omega})$ is transcendental.*

If $\boldsymbol{\omega}$ is special we can use results of some previous papers to investigate the transcendence of the quasi-periods. For then Theorem II of [6] shows that Λ is essentially a Cartesian product, and it is not difficult to deduce that any quasi-period is given by the expression on the right of (5). It follows from Baker's theorem on two Weierstrass zeta functions [1] that $\eta(H, \boldsymbol{\omega})$ is either zero or transcendental. Finally the generalization of [1] proved in [7] enables us to determine exactly which quasi-periods can vanish.

We recover Theorem I of [6] simply by making the choice (6) in our present theorem. In fact we shall establish our theorem by extending the methods used in [6]. Under the assumption that $\eta(H, \boldsymbol{\omega})$ is an algebraic number, we construct for each large integer k a non-zero polynomial in H and $\boldsymbol{A}$ with many zeros. If H is not in $\mathfrak{F}$, this auxiliary function is not identically zero. The main difficulty in carrying out the extrapolation concerns certain division values associated with H. By using a well-known but not entirely trivial lower bound for polynomials in algebraic numbers we obtain estimates for these division values which, although they are probably far from best possible, just suffice for our limited purposes.

We then borrow an elimination procedure from [7] to remove the quasi-periodic part of the auxiliary function. For each k this yields a non-zero homogeneous polynomial in $\mathbf{A}[\mathbf{x}]$ with small absolute value at $\boldsymbol{x} = \boldsymbol{\omega}$. We conclude from the familiar transcendence criterion of Gelfond that if $\omega_1 \neq 0$ then ω_2/ω_1 must be algebraic; i.e., the period $\boldsymbol{\omega}$ is special.

Following Schneider, the result of this paper, as well as those of [6], may be stated in terms of Jacobians. Let $F(x, y)$ be a polynomial with algebraic coefficients such that the equation $F(x, y) = 0$ represents a curve $\mathscr{C}$ of genus 2. We call $\mathscr{C}$ special if there exists a rational map from $\mathscr{C}$ onto an elliptic curve. It is not hard to see that $\mathscr{C}$ is special if and only if its Jacobian variety splits into a product, and so the study of such curves may be reduced to the consideration of two elliptic curves as in [6]. Hence we shall suppose that $\mathscr{C}$ is not special.

Now on $\mathscr{C}$ there exist non-zero differentials of the first kind having the form $\varphi = R(x, y)dx$ where $R(x, y)$ is a rational function with algebraic coefficients. Let $\mathscr{L}$ be any closed path on the associated Riemann surface of $\mathscr{C}$. The theorems of [6] imply that the integral of φ around $\mathscr{L}$ is transcendental if $\mathscr{L}$ is not trivial in the sense of homology.

Similarly there exist differentials σ of the second kind on $\mathscr{C}$ also defined by rational functions with algebraic coefficients. The theorem of the present paper shows that if σ is not the differential of a rational function, the integral of σ around a non-trivial closed path $\mathscr{L}$ is transcendental.

As an example consider the curve $\mathscr{C}$ corresponding to $F(x, y) = y^2 + x^6 - x$. Two linearly independent differentials of the first kind are given by

$$\varphi_1 = dx/y, \quad \varphi_2 = xdx/y.$$

Their periods can be rapidly calculated by applying the endomorphisms $x \to \epsilon x$, $y \to \epsilon^3 y$ ($\epsilon^5 = 1$) of $\mathscr{C}$, and it follows by inspection that $\mathscr{C}$ is not special. The differentials

$$\sigma_1 = x^3 dx/y, \quad \sigma_2 = x^4 dx/y$$

are of the second kind; furthermore if α_1, α_2, β_1, β_2 are not all zero then

$$\sigma = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \beta_1\sigma_1 + \beta_2\sigma_2$$

is not the differential of a rational function, since not all of its periods vanish. From a familiar argument one of these periods η is given by the formula

$$\tfrac{1}{2}\eta = \int_0^1 (\alpha_1 + \alpha_2 x + \beta_1 x^3 + \beta_2 x^4) y^{-1} dx.$$

in which y assumes only non-negative values. Hence if the coefficients α_1, α_2, β_1, β_2 are algebraic numbers, the period η is a transcendental number. This leads to a result on the beta function $B(x, y)$ as follows.

If the positive integers m, n vary subject only to $m \not\equiv -n$ modulo 5, the values $B(\frac{1}{5}m, \frac{1}{5}n)$ span a vector space over $\mathbf{A}$ of dimension at most 5. Our theorem implies, after a simple calculation, that this dimension is exactly 5; equivalently the numbers $B(\frac{1}{2}, \frac{1}{5}m)$ $(1 \leq m \leq 5)$ are linearly independent over $\mathbf{A}$. It would be pleasant to include the number π by removing the restriction on m and n, but all I can prove is the weaker result that $\eta + \pi$ is either zero or transcendental.

Finally I am much indebted to Michel Waldschmidt for correcting a misconception of mine about Lemma 4.1.

2. Normalization and preliminary lemmas

Suppose $\mathfrak{F}$ is algebraically defined with respect to the generators A and B, and let H be an algebraically defined quasi-periodic function. By multiplying B by a function in the ring $\mathbf{A}[A]$ if necessary, we may assume that B is integral over this ring. Also the partial derivatives of H, A, B can be expressed as rational functions of A, B with a common denominator C in $\mathbf{A}[A]$. It follows easily that the operators $\partial/\partial z_1$, $\partial/\partial z_2$ take H into the ring $\mathbf{A}[A, B, C^{-1}]$ and also take this ring into itself.

Our first task is to find a convenient representation of all these functions as quotients of entire functions. By a theta function with respect to the lattice Λ we mean an entire function $\theta(\mathbf{z})$ such that for each $\boldsymbol{\omega}$ in Λ there exist constants a_1, a_2, b such that

$$\theta(\mathbf{z} + \boldsymbol{\omega}) = \theta(\mathbf{z}) \exp(a_1 z_1 + a_2 z_2 + b). \tag{7}$$

This implies (cf. Lemma 1 of [6]) that for some c independent of $\mathbf{z}$ we have the estimate

$$|\theta(\mathbf{z})| < c^{R^2}, \tag{8}$$

where

$$R = \max(1, |z_1|, |z_2|). \tag{9}$$

More generally, if θ is an entire function satisfying (8) and (9) we shall say that θ has growth order at most 2.

LEMMA 2.1: *There is a non-zero theta function θ such that θH, θA_1, θA_2, θB, θC^{-1} are entire functions of growth order at most* 2.

PROOF: Let us temporarily denote the operator $\partial/\partial z_i$ by ∂_i $(i = 1, 2)$.

Since $\partial_1 H$, $\partial_2 H$ lie in $\mathfrak{F}$, we can find (e.g., [9], p. 91) a non-zero theta function θ such that the functions

$$\theta\partial_1 H,\ \theta\partial_2 H,\ \theta A_1,\ \theta A_2,\ \theta B,\ \theta C^{-1}$$

are entire. We proceed to show, using the unique factorization property ([9], p. 11) of the ring $\mathfrak{T}$ of power series convergent near the origin, that θH is also entire.

Let $\boldsymbol{a}$ be an arbitrary point of $\mathbf{C}^2$. There exist mutually non-associate non-unit primes $p_1, \ldots, p_m$, $q_1, \ldots, q_n$ of $\mathfrak{T}$ and a unit u of $\mathfrak{T}$ together with positive integers $\mu_1, \ldots, \mu_m$, $\nu_1, \ldots, \nu_n$ such that

$$q_1^{\nu_1} \ldots q_n^{\nu_n} H(\boldsymbol{z}+\boldsymbol{a}) = u p_1^{\mu_1} \ldots p_m^{\mu_m}$$

for $\boldsymbol{z}$ near $\mathbf{0}$. Thus for $i = 1, 2$ the expressions

$$Q = q_1^{\nu_1+1} \ldots q_n^{\nu_n+1}, \quad P_i = Q\partial_i H(\boldsymbol{z}+\boldsymbol{a})$$

lie in $\mathfrak{T}$, and in fact

$$P_i = u p_1^{\mu_1} \ldots p_m^{\mu_m} q_1 \ldots q_n \left\{ u^{-1}\partial_i u + \sum_{j=1}^{m} \mu_j p_j^{-1}\partial_i p_j - \sum_{k=1}^{n} \nu_k q_k^{-1}\partial_i q_k \right\}. \tag{10}$$

Because $\theta(\boldsymbol{z}+\boldsymbol{a})\partial_i H(\boldsymbol{z}+\boldsymbol{a})$ is holomorphic at $\boldsymbol{z} = \mathbf{0}$, we deduce that Q divides $\theta(\boldsymbol{z}+\boldsymbol{a})P_i$. If now Q does not divide $\theta(\boldsymbol{z}+\boldsymbol{a})$, neither does $q_k^{\nu_k+1}$ for some k, and hence q_k divides each P_i. It follows from (10) that q_k divides each $\partial_i q_k$. But this is easily seen to be impossible since q_k is not a unit. Therefore Q does indeed divide $\theta(\boldsymbol{z}+\boldsymbol{a})$, and so $\theta(\boldsymbol{z}+\boldsymbol{a})H(\boldsymbol{z}+\boldsymbol{a})$ is holomorphic at $\boldsymbol{z} = \mathbf{0}$. On recalling that $\boldsymbol{a}$ is arbitrary, we conclude that $\varphi = \theta H$ is entire.

To calculate the growth order of φ, let $\boldsymbol{z}$ be a point of $\mathbf{C}^2$, and suppose for the moment that H is holomorphic at $\boldsymbol{z}$. If $\boldsymbol{\omega}$ is a period of Λ nearest to $\boldsymbol{z}$, the point $\boldsymbol{u} = \boldsymbol{z} - \boldsymbol{\omega}$ lies in some compact set independent of $\boldsymbol{z}$. Hence if the c's denote positive constants also independent of $\boldsymbol{z}$, we have from Lemma 1 of [6]

$$|\theta(\boldsymbol{z})| \leq c_1^{R^2}|\theta(\boldsymbol{u})| < c_2^{R^2}$$

with R given by (9). Thus

$$|\varphi(\boldsymbol{z})| \leq c_1^{R^2}|\theta(\boldsymbol{u})H(\boldsymbol{z})| \leq c_1^{R^2}|\theta(\boldsymbol{u})H(\boldsymbol{u})| + c_2^{R^2}|\eta(H, \boldsymbol{\omega})|.$$

Now $\theta(\boldsymbol{u})H(\boldsymbol{u}) = \varphi(\boldsymbol{u})$ is bounded and plainly

$$|\eta(H, \boldsymbol{\omega})| < c_3 R,$$

whence

$$|\varphi(\boldsymbol{z})| \leq c_4^{R^2}.$$

This inequality continues to hold when the restriction on $\boldsymbol{z}$ is removed, and so φ has growth order at most 2. Since the other functions θA_1, θA_2, θB, θC^{-1} are themselves theta functions, this completes the proof of the lemma.

Let J denote the Jacobian determinant of $\boldsymbol{A}$ with entries $\partial A_j/\partial z_i$ $(i, j = 1, 2)$. Since J is not identically zero ([9], p. 77), there exists a point $\boldsymbol{a}$ at which none of C, θ, J vanishes. Then $\boldsymbol{A}$ is holomorphic at $\boldsymbol{a}$, and by slightly shifting $\boldsymbol{a}$ we can ensure that $\boldsymbol{A}(\boldsymbol{a})$ has algebraic coordinates. The effect of translating the variable $\boldsymbol{z}$ by $\boldsymbol{a}$ is to take $\boldsymbol{a} = \boldsymbol{0}$ in the preceding remarks; supposing this done it follows that A_1, A_2, B, C^{-1} are holomorphic at the origin and take algebraic values there, while $\theta(\boldsymbol{0})$, $J(\boldsymbol{0})$ are both non-zero. Furthermore H is holomorphic at the origin and its quasi-periods are unchanged. They remain so even if we subtract an arbitrary constant from H; thus we may assume that $H(\boldsymbol{0}) = 0$. Plainly none of these operations affects the algebraic definition of $\mathfrak{F}$ or H.

The non-vanishing of $J(\boldsymbol{0})$ has the following useful consequence.

LEMMA 2.2: *If $P(\boldsymbol{x})$ is a non-zero polynomial of total degree at most L, the function $P(\boldsymbol{A}(\boldsymbol{z}))$ cannot have a zero of order greater than L at the origin.*

PROOF: See Lemma 2 of [6].

Since we intend to construct auxiliary polynomials in H, A_1, A_2 we need to know when these functions are algebraically independent. The next lemma shows the obvious necessary condition also to be sufficient.

LEMMA 2.3: *If H is not in $\mathfrak{F}$ then the functions H, A_1, A_2 are algebraically independent.*

PROOF: If H, A_1, A_2 are algebraically dependent let m denote the degree of H over $\mathfrak{F}$. Thus there are functions $f_1, \ldots, f_m$ in $\mathfrak{F}$ such that

$$H^m + f_1 H^{m-1} + \cdots + f_m = 0. \tag{11}$$

For $i = 1, 2$ we apply the operator $\partial/\partial z_i$ to (11), obtaining

$$g_0 H^{m-1} + g_1 H^{m-2} + \cdots + g_{m-1} = 0$$

where $g_0, \ldots, g_{m-1}$ also lie in $\mathfrak{F}$. Hence this must be a trivial equation for H. In particular its first coefficient

$$g_0 = \partial(mH + f_1)/\partial z_i$$

vanishes for $i = 1, 2$ and so H differs from $-m^{-1}f_1$ by only a constant. Thus in fact $m = 1$ and H lies in $\mathfrak{F}$.

The second stage of normalization involves the introduction of an algebraic number field. Let K be an algebraic number field containing $A_1(\mathbf{0})$, $A_2(\mathbf{0})$, $B(\mathbf{0})$ such that B is integral over $K[\mathbf{A}]$, C lies in $K[\mathbf{A}]$, and the operators $\partial/\partial z_1$, $\partial/\partial z_2$ take H into $K[\mathbf{A}, B, C^{-1}]$ and this ring into itself. If I is the ring of integers of K, we may even suppose that B is integral over $I[\mathbf{A}]$, C lies in $I[\mathbf{A}]$, and these operators take H into $\mathfrak{R} = I[\mathbf{A}, B, C^{-1}]$ as well as $\mathfrak{R}$ into itself; for if necessary we replace B and C by integral multiples of themselves. The action of differential operators on the ring $\mathfrak{R}[H]$ is now particularly easy to examine in detail. For the rest of this section $c, c_1, \ldots$ will denote positive constants depending only on the differential equations implied by the above remarks. Also d denotes the degree of B over $I[\mathbf{A}]$.

LEMMA 2.4: *Let P be a polynomial in $I[x_0, \mathbf{x}, y, z]$ with degree at most L_0 in x_0 and total degree at most L. If ∂ is a differential operator of order not exceeding k we have*

$$\partial P(H, \mathbf{A}, B, C^{-1}) = Q(H, \mathbf{A}, B, C^{-1}),$$

where Q in $I[x_0, \mathbf{x}, y, z]$ has degree at most L_0 in x_0 and total degree at most $L + ck$. Furthermore if the coefficients of P have sizes not exceeding S, those of Q have sizes not exceeding $(cL)^k k!S$.

PROOF: A straightforward induction on k; compare Lemma 3 of [6] and Lemma 5 of [7]. Note that the degree of P in the variable x_0 corresponding to the function H does not increase on differentiation. This remark is the basis of the elimination procedure of [7], which is modified for our present situation in the following lemma.

LEMMA 2.5: *Let P be a polynomial in $I[x_0, \mathbf{x}]$ of degree at most L_0 in x_0 and total degree at most L. Suppose the function*

$$\Phi(\mathbf{z}) = P(H(\mathbf{z}), \mathbf{A}(\mathbf{z}))$$

is not identically zero. Then there exists a non-zero polynomial Q in $I[\mathbf{x}]$ of total degree at most cL_0L with the following property. The function

$$\Psi(\mathbf{z}) = Q(\mathbf{A}(\mathbf{z}))$$

has a zero at each point where H and $\mathbf{A}$ are holomorphic and Φ has a zero of order at least $2L_0 + 1$. Furthermore if the coefficients of P have sizes not exceeding S, those of Q have sizes not exceeding $(cL)^{cL_0L}S^{cL_0}$.

PROOF: Using Lemma 6 of [7] we deduce the existence of a polynomial R in $I[\boldsymbol{x}, y, z]$ of total degree $M \leq c_1 L_0 L$ such that the function

$$f(z) = R(\boldsymbol{A}, B, C^{-1})$$

is not identically zero yet has the required zeros. Also the coefficients of R have sizes at most $(c_1 L)^{c_1 L_0 L} S^{c_1 L_0}$. To obtain Ψ we would like to clear of denominators and take a suitable norm; this can be formally carried out as follows. The function $g = C^M f$ lies in $I[\boldsymbol{A}, B]$, and we can write

$$gB^r = h_{r,0} + h_{r,1}B + \cdots + h_{r,d-1}B^{d-1} \quad (0 \leq r \leq d-1) \tag{12}$$

for $h_{r,s}$ in $I[\boldsymbol{A}]$. From these equations we construct the characteristic polynomial of g over $I[\boldsymbol{A}]$. Since this is a power of the corresponding minimal polynomial, its constant term Ψ is not identically zero. On the other hand Ψ vanishes at all zeros of g where $\boldsymbol{A}$ is holomorphic. The lemma follows on noting that up to sign Ψ is the determinant of the functions $h_{r,s}$ $(0 \leq r, s \leq d-1)$, and its coefficients are readily estimated from (12).

3. Heights of division values

In this section we fix a non-zero period $\boldsymbol{\omega}$ of Λ and study the values of Abelian and quasi-periodic functions at the points $r\boldsymbol{\omega}/q$ for coprime integers $q \geq 1, r$. For simplicity we shall sometimes restrict q to the set $\mathcal{Q}$ consisting of all odd integers $q \geq 3$ such that $\theta(r\boldsymbol{\omega}/q) \neq 0$ for all r prime to q. This condition implies that the functions H, $\boldsymbol{A}$, B, C^{-1} are holomorphic at the corresponding points $r\boldsymbol{\omega}/q$. Since $\theta(\boldsymbol{0}) \neq 0$ the function $\theta(z\boldsymbol{\omega})$ is not identically zero, and so it is clear from (7) that $\mathcal{Q}$ contains all sufficiently large odd integers.

We retain the normalizations introduced in the previous section, and for brevity we denote the field $K(\boldsymbol{A}, B)$ by $\mathfrak{F}(K)$. The following simple result is very useful.

LEMMA 3.1: *The coefficients in the power series expansions of H about the origin lie in K. Furthermore a function in $\mathfrak{F}$ holomorphic at the origin has this property if and only if it belongs to $\mathfrak{F}(K)$.*

PROOF: If P lies in $I[x_0, \boldsymbol{x}, y, z]$ then by putting $\boldsymbol{z} = \boldsymbol{0}$ in Lemma 2.4 we see that $P(H, \boldsymbol{A}, B, C^{-1})$ lies in $\mathfrak{T}(K)$ the ring of power series convergent near $\boldsymbol{0}$ with coefficients in K. Hence so does H and any function in $\mathfrak{F}(K)$ holomorphic at $\mathbf{0}$. Conversely any function in $\mathfrak{F}$ and $\mathfrak{T}(K)$ can be expressed as

$$(f_0 + f_1 B + \cdots + f_{d-1} B^{d-1})/f_d, \; f_i = P_i(\mathbf{A}) \quad (0 \le i \le d),$$

for coprime polynomials $P_0, \ldots, P_d$. Normalizing P_d to have some coefficient unity, we conclude as in the proof of Lemma 6.2 of [4] that $P_0, \ldots, P_d$ lie in $K[\mathbf{x}]$.

We shall frequently make use of the device analogous to L'Hôpital's rule employed on p. 505 of [5] and p. 102 of [6] to ensure that certain denominators do not vanish. The positive constants $c, c_1, \ldots$ will depend only on the period $\boldsymbol{\omega}$ and the differential equations underlying the normalizations of section 2. The first three lemmas of this section deal with division values of functions in $\mathfrak{F}(K)$.

LEMMA 3.2: *Suppose f in $\mathfrak{F}(K)$ can be written as the quotient g_1/g_2 of non-zero functions g_1, g_2 in $I[\mathbf{A}, B]$ of total degrees at most L in $\mathbf{A}$, B, and let S be an upper bound for the sizes of the coefficients in g_1, g_2. Let $q \ge 3$, r be coprime integers such that f is holomorphic at $r\boldsymbol{\omega}/q$. Then the number $f(r\boldsymbol{\omega}/q)$ is an algebraic number of degree at most cq^4 and height at most $(cq^L S)^{cq^4}$.*

PROOF: This estimate is implicit in the proof of Lemma 7 of [6], and so we give no more than a sketch of the argument. We know that $f(\mathbf{z})$ is a rational function of $\mathbf{A}(\mathbf{z})$, $B(\mathbf{z})$ of total degree at most L. From the multiplication formula for $\mathfrak{F}$ given as Lemma 4 of [6] the functions $A_1(q\mathbf{z})$, $A_2(q\mathbf{z})$ can be expressed as rational functions of $\mathbf{A}(\mathbf{z})$, $B(\mathbf{z})$ of total degrees at most $c_1 q^2$. Let $P(\mathbf{x}, y)$ be a polynomial with undetermined coefficients whose total degree in $\mathbf{x}$ is formally CLq^2 and whose degree in y is formally Cq^4. If $C < c_2$ is a sufficiently large integer, we can use a linear forms lemma to choose the coefficients of P as integers of I, not all zero, such that

$$P(\mathbf{A}(q\mathbf{z}), f(\mathbf{z})) = 0. \tag{13}$$

Further, by keeping track of the magnitudes of all the algebraic quantities occurring, we find that the sizes of the coefficients of P can be supposed not to exceed $(c_3 q^L S)^{c_3 q^4}$. Finally the substitution $\mathbf{z} = r\boldsymbol{\omega}/q$ in (13) after using the L'Hôpital device yields an algebraic equation for $\xi = f(r\boldsymbol{\omega}/q)$ over I. We derive an equation over the rational field $\mathbb{Q}$ almost immediately, and with it the required estimates for the degree and height of ξ.

One consequence of this lemma will be that for each q in $\mathfrak{Q}$ there exists an algebraic number field K_q of degree at most cq^4 which contains all the numbers $f(r\omega/q)$ as r and f vary subject to the conditions of the lemma. To see this, we fix r prime to q and temporarily write $K_{q,r}$ for the field generated over K by the numbers $f(r\omega/q)$ as f varies over all functions in $\mathfrak{F}(K)$ holomorphic at $r\omega/q$.

LEMMA 3.3: *For q in $\mathfrak{Q}$ and r prime to q the field $K_{q,r}$ is generated over K by $A_1(r\omega/q)$, $A_2(r\omega/q)$, $B(r\omega/q)$, and it is independent of r.*

PROOF: Using the L'Hôpital device if necessary to evaluate quotients, we immediately verify that $K_{q,r}$ has the generators indicated. Next let r, r' be integers prime to q, and denote by s any integer such that $r \equiv sr'$ modulo q. Then if $f(z)$ is any of A_1, A_2, B the function $f'(z) = f(sz)$ lies in $\mathfrak{F}(K)$ by Lemma 3.1, and so

$$f(r\omega/q) = f'(r'\omega/q)$$

lies in $K_{q,r'}$. Consequently $K_{q,r}$ is contained in $K_{q,r'}$, and this by symmetry implies the last assertion of the lemma.

For q in $\mathfrak{Q}$ we denote the fields $K_{q,r}$ by K_q, and we write d_q for the degree of K_q. The next lemma summarises all we require on division values of Abelian functions. It gives a rather curious decomposition of the division value α into constituents which need in no sense be 'simpler' than α. Such a reduction seems to be a necessary precaution before using the lower bound of Lemma 4.2.

LEMMA 3.4: *For q in $\mathfrak{Q}$ we have $d_q < cq^4$. Let r be prime to q, and suppose f in $\mathfrak{F}(K)$ is holomorphic at $r\omega/q$. Then for some constant C which may depend on f we can write*

$$f(r\omega/q) = \alpha_1 + \alpha_2 \tag{14}$$

where α_1, α_2 are algebraic numbers of K_q with degrees exactly d_q and heights at most q^{Cq^4}.

PROOF: Let κ be an algebraic number such that $K = \mathbb{Q}(\kappa)$. The preceding lemma shows that, as in the proof of Lemma 1.6 of [4], we can find integers m_1, m_2, n with absolute values at most q^{c_1} such that the number

$$\gamma = m_1A_1(r\omega/q) + m_2A_2(r\omega/q) + nB(r\omega/q) + \kappa \tag{15}$$

generates $K_q = K_{q,r}$ over $\mathbb{Q}$. Since γ is the value at $r\omega/q$ of the function

$$g(z) = m_1A_1(z) + m_2A_2(z) + nB(z) + \kappa,$$

it follows from Lemma 3.2 that its degree does not exceed cq^4. Since $f(r\omega/q)$ lies in K_q, there is similarly a non-zero integer k of absolute value at most q^{c_2} such that the number

$$\delta = f(r\omega/q) - k\gamma \tag{16}$$

also generates K_q over $\mathbb{Q}$. Thus (14) is satisfied with

$$\alpha_1 = k\gamma, \quad \alpha_2 = \delta,$$

and these numbers have degrees d_q. The formulae (15), (16) present them as the values at $r\omega/q$ of the functions $kg(z)$, $f(z) - kg(z)$ respectively; hence Lemma 3.2 gives the desired estimate for their heights.

Finally we turn to the division values associated with the quasi-periodic function H normalized as in section 2. In the Cartesian situation described in section 1, these are related to certain values of the corresponding Weierstrass zeta functions. Such numbers were first studied in detail by Baker in his paper [1], and his methods suggest the approach we take here. The following decomposition is similar to that of the preceding lemma but slightly more elaborate.

LEMMA 3.5: *For q in $\mathcal{Q}$ and r prime to q we can write*

$$m\{H(r\omega/q) - r\eta(H, \omega)/q\} = m_1\alpha_1 + \cdots + m_{2p}\alpha_{2p} \tag{17}$$

for some $p \leq q$, where $\alpha_1, \ldots, \alpha_{2p}$ are algebraic numbers of K_q with degrees exactly d_q and heights at most q^{cq^4}, and $m \neq 0$, $m_1, \ldots, m_{2p}$ are rational integers with absolute values at most 2^q.

PROOF: We consider the function

$$D(z) = H(2z) - 2H(z).$$

Plainly this is an Abelian function, and from Lemma 3.1 it lies in $\mathfrak{F}(K)$. We have for any positive integer p

$$H(2^p z) - 2^p H(z) = \sum_{i=0}^{p-1} 2^{p-1-i} D(2^i z), \tag{18}$$

in which every term is holomorphic at $r\omega/q$ because q is odd; since $q \geq 3$ this last condition also implies that we can choose $p \leq q$ such that $2^p = 1 + kq$ for some integer $k \neq 0$. Hence, writing $\eta = \eta(H, \omega)$ we obtain

$$H(2^p r\omega/q) = H(r\omega/q) + kr\eta,$$

and (18) evaluated at $r\omega/q$ yields

$$kq\{H(r\omega/q) - r\eta/q\} = -\sum_{i=0}^{p-1} 2^{p-1-i} D(2^i r\omega/q).$$

The lemma now follows on applying the previous lemma with $f = D$ to split up the division values $D(2^i r\omega/q)$.

The decomposition (17) is somewhat delicate. We cannot, for example, absorb the integers m_i into the algebraic numbers α_i, as this might give rise to numbers of height c^{q^5}. In fact it is implicit in Lemma 3.5 that $H(r\omega/q) - r\eta/q$ is an algebraic number of K_q whose height does not exceed q^{cq^5}. Upper bounds of the form c^{q^κ} with some absolute constant $\kappa \geq 6$ are much easier to obtain, and these suffice to prove the result on $\eta + \pi$ mentioned in section 1. For the present proof, however, we need $\kappa < 16/3$. In any case we shall not use such estimates explicitly, since we prefer to apply (17) directly to the auxiliary function.

4. Additional lemmas on algebraic numbers

We require two results well-known in the theory of transcendental numbers. Unfortunately they are usually stated with reference to the ring $\mathbb{Z}$ of rational integers. For convenience we shall use instead their generalizations to the ring I of integers of an algebraic number field K. In either case there is no difficulty in proving the desired extension by the obvious procedure of 'taking the norm'.

Throughout this section the positive constants $c, c_1, \ldots$ will depend only on K. Also $\|P\|$ will denote the maximum of the absolute values of the coefficients of a polynomial P. The first result is a lower bound for a particular type of algebraic number, and is due to Feldman. It gives a substantial, and for our purposes indispensable, improvement on the usual norm estimate. We suppose $\alpha_1, \ldots, \alpha_m$ are algebraic numbers of exact degrees $d_1, \ldots, d_m$ and heights at most $H_1, \ldots, H_m$ respectively, and we denote by D the degree of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$.

LEMMA 4.1: *Let P in $\mathbb{Z}[x_1, \ldots, x_m]$ have degree at most L_i in x_i $(1 \leq i \leq m)$, and put $\pi = P(\alpha_1, \ldots, \alpha_m)$. Then if $\pi \neq 0$ we have*

$$|\pi| \geq \|P\|^{1-D} \prod_{i=1}^{m} (2^{d_i} H_i)^{-DL_i/d_i}.$$

PROOF: This is virtually Lemma 2 of [2].

The extension to I may be stated as follows.

LEMMA 4.2: *Let P in $I[x_1, \ldots, x_m]$ have degree at most L_i in x_i $(1 \le i \le m)$ and coefficients of sizes at most S, and put $\pi = P(\alpha_1, \ldots, \alpha_m)$. Then if $\pi \neq 0$ we have*

$$\log|\pi| > -cD\left\{\log S + \sum_{i=1}^{m} L_i(1 + d_i^{-1}\log H_i)\right\}.$$

PROOF: As suggested above, we consider the norm of P from $K(x_1, \ldots, x_m)$ to $\mathbb{Q}(x_1, \ldots, x_m)$.

The other result of this section is due to Gelfond, apart from the precise constants.

LEMMA 4.3: *Let ξ be a complex number, and for some integer n let $t_n, t_{n+1}, \ldots$ be real numbers tending to infinity whose successive ratios t_{k+1}/t_k tend to 1. If for each $k \ge n$ there exists a non-zero polynomial P_k in $\mathbb{Z}[x]$ of degree at most t_k with $\|P_k\| \le e^{t_k}$ such that*

$$|P_k(\xi)| \le e^{-7t_k^2},$$

then ξ is an algebraic number.

PROOF: Take $c = d = \frac{7}{6}$ in Theorem 5.1.1 of [10].

For later convenience we record the version over I in terms of homogeneous polynomials.

LEMMA 4.4: *Let ξ_1, ξ_2 be complex numbers, and for some integer n let $t_n, t_{n+1}, \ldots$ be real numbers tending to infinity whose successive ratios t_{k+1}/t_k tend to 1. If for each $k \ge n$ there exists a non-zero homogeneous polynomial P_k in $I[x_1, x_2]$ of degree at most t_k, with coefficients of sizes at most e^{t_k}, such that*

$$|P_k(\xi_1, \xi_2)| < e^{-ct_k^2},$$

then ξ_1, ξ_2 are linearly dependent over $\mathbb{A}$.

PROOF: Straightforward.

5. The auxiliary function

Let $\mathfrak{F}$ be an algebraically defined Abelian function field normalized as in section 2, and let H be a quasi-periodic function similarly normalized. Suppose that H does not lie in $\mathfrak{F}$, and let $\boldsymbol{\omega} = (\omega_1, \omega_2)$ be a non-zero period of the lattice Λ. We shall prove our theorem by showing that if $\eta = \eta(H, \boldsymbol{\omega})$ is algebraic, the period $\boldsymbol{\omega}$ is special.

Thus we assume η is an algebraic number; we take the field K introduced in section 2 so large as to contain η. Let ϵ be a small absolute positive constant not exceeding $\frac{1}{100}$. For an integer $k \geq 1$ we let

$$L_0 = [k^{6\epsilon}], \quad L = [k^{1-2\epsilon}]$$

and

$$h = [k^{\epsilon}].$$

In the estimates that follow, the positive constants $c, c_1, \ldots$ will be independent of k; in particular we take c so large that the accompanying calculations are valid whenever $k > c$.

LEMMA 5.1: *There is a non-zero polynomial P in $\mathbb{Z}[x_0, \mathbf{x}]$, of degree at most L_0 in x_0 and total degree at most L in $\mathbf{x}$, such that $\|P\| < k^{c_1 k}$ and the function*

$$\Phi(\mathbf{z}) = P(H(\mathbf{z}), \mathbf{A}(\mathbf{z}))$$

has a zero of order at least k at each point $s\boldsymbol{\omega}$ with s a non-negative integer not exceeding h.

PROOF: This is an exercise in the use of well-known lemmas on solutions of homogeneous linear equations. We express the derivatives of $\Phi(\mathbf{z})$ in terms of $H(\mathbf{z})$, $\mathbf{A}(\mathbf{z})$, $\mathbf{B}(\mathbf{z})$, $(C(\mathbf{z}))^{-1}$ by means of Lemma 2.4. The essential point is that $H(s\boldsymbol{\omega}) = s\eta$ for any integer s, and so the resulting equations for the coefficients of P have coefficients in K. The calculations are similar to those appearing in the proof of Lemma 5 of [6].

LEMMA 5.2: *The function*

$$\varphi(\mathbf{z}) = (\theta(\mathbf{z}))^{2L}\Phi(\mathbf{z})$$

is entire, and if ∂ is a differential operator of order at most k we have for all complex numbers z

$$|\partial\varphi(z\boldsymbol{\omega})| < k^{c_2 k} c_2^{L|z|^2}.$$

PROOF: We use Cauchy's integral formula and the growth estimates of Lemma 2.1 (see also Lemma 6 of [6]).

To describe the extrapolation procedure we recall the set $\mathcal{Q}$ defined in section 3. Also let $\lambda \leq 1$ be a real number such that whenever $|z| \leq \lambda$ we have

$$|\lambda| > c_3^{-1}, \quad |\theta(z\boldsymbol{\omega})| > c_3^{-1}. \tag{19}$$

For real numbers $Q \geq 1$, $S \geq 1$ we denote by $\mathcal{T}(Q, S)$ the collection of rationals $t = s$ and $t = s + r/q$ for integers q in $\mathcal{Q}$, r, s with

$$q \leq Q, \quad 1 \leq r \leq \lambda q, \quad 0 \leq s \leq S. \tag{20}$$

Elementary number theory shows that $\mathcal{T}(Q, S)$ contains at least $c_4^{-1}Q^2S$ distinct numbers t (cf. p. 103 of [6]). At the corresponding points $t\boldsymbol{\omega}$ the function θ does not vanish and H, $\mathbf{A}$, B, C^{-1} are holomorphic.

The following lemma gives the lower bound which was the sole objective of the calculations of section 3.

LEMMA 5.3: *If t lies in $\mathcal{T}(Q, S)$ and ∂ is a differential operator of order at most k such that $\xi = \partial\Phi(t\boldsymbol{\omega}) \neq 0$, then*

$$|\xi| > (kQS)^{-c_5kQ^4}Q^{-c_5L_0Q^5}.$$

PROOF: If t is an integer the inequality is immediate since no division values are involved and a straightforward norm argument suffices. We can therefore suppose that $t = s + r/q$ for integers q in $\mathcal{Q}$, r, s satisfying (20) with q, r coprime. Now by Lemma 2.4 the function $\partial\Phi(\mathbf{z})$ can be expressed as a polynomial in H, $\mathbf{A}$, B, C^{-1}; this polynomial has degree at most L_0 in H and total degree at most c_6k, while from Lemma 5.1 its coefficients are in I with sizes at most k^{c_6k}. Thus by putting $\mathbf{z} = t\boldsymbol{\omega}$ and rewriting $H(t\boldsymbol{\omega})$ in terms of

$$\alpha_0 = H(t\boldsymbol{\omega}) - t\eta,$$

we obtain a polynomial F such that

$$(nq)^{L_0}\xi = F(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \tag{21}$$

where n is a denominator for η and

$$\alpha_1 = A_1(t\boldsymbol{\omega}), \quad \alpha_2 = A_2(t\boldsymbol{\omega}), \quad \alpha_3 = B(t\boldsymbol{\omega}), \quad \alpha_4 = (C(t\boldsymbol{\omega}))^{-1}.$$

The polynomial F has degree at most L_0 in the first variable, total degree at most c_7k, and since $s \leq S$ its coefficients are in I with sizes not exceeding $(c_7QS)^{L_0}k^{c_7k}$. We now use the decompositions of Lemmas 3.4 and 3.5 to write

$$\alpha_i = \alpha_{i,1} + \alpha_{i,2} \quad (1 \leq i \leq 4)$$

and

$$m\alpha_0 = m_1\alpha_{0,1} + \cdots + m_{2p}\alpha_{0,2p}$$

for some $p \leq q$ and integers $m \neq 0, m_1, \ldots, m_{2p}$ of absolute values at most 2^Q. The $\alpha_{i,j}$ are algebraic numbers of K_q with degrees exactly d_q and heights at most $Q^{c_8 Q^4}$. Thus (21) yields a polynomial G such that

$$\pi = (mnq)^{L_0}\xi = G(\alpha_{0,1}, \ldots, \alpha_{0,2p}, \alpha_{1,1}, \ldots, \alpha_{4,2}). \tag{22}$$

This polynomial has degree at most L_0 in each of the first $2p$ variables, and degree at most $c_9 k$ in each of the last eight. Furthermore its coefficients are in I with sizes not exceeding $W = (c_9^Q S)^{L_0} k^{c_9 k}$. The lower bound of Lemma 4.2 was needed for just this kind of situation. All the $\alpha_{i,j}$ lie in K_q whose degree is d_q by definition. It follows that since $\pi \neq 0$ and $q \leq Q$ we have

$$\log|\pi| > -c_{10} d_q\{\log W + (k + L_0 Q)(1 + d_q^{-1} Q^4 \log Q)\}.$$

Because $d_q < c_{11} Q^4$ we deduce from this

$$\log|\pi| > -c_{12} Q^4\{\log W + (k + L_0 Q)\log Q\},$$

and the desired inequality is now immediate from (22). We observe that if neither Q nor S exceeds a bounded power of k, this lower bound becomes $k^{-c_{13}Q^4(k+L_0Q)}$; and when further $Q \leq k/L_0$ this is simply $k^{-c_{14}kQ^4}$, which is as precise as the estimate of Lemma 7 of [6]. We also see that the weak results on the division values of H are offset by the smallness of L_0.

After these preliminaries we can put into operation the extrapolation procedure as follows.

LEMMA 5.4: *For any non-negative integer $n \leq 8\epsilon^{-1} - 100$ the function $\Phi(z)$ has a zero of order at least $k/2^n$ at each point $t\omega$ with t in $\mathcal{T}(h^{n/8}, h^{1+n/4})$.*

PROOF: The result is true for $n = 0$ by Lemma 5.1. As in the proof of Lemma 8 of [6], it will suffice to obtain a contradiction from the validity of the present lemma for some non-negative integer $m < 8\epsilon^{-1} - 100$ and the existence of a counterexample

$$\xi = \partial\Phi(t\omega) \neq 0$$

to the lemma for $n = m + 1$. Thus if we write

$$Q_n = h^{n/8}, \quad S_n = h^{1+n/4}, \quad T = [k/2^{m+1}]$$

then t lies in $\mathcal{T}(Q_{m+1}, S_{m+1})$ and ∂ is a differential operator of order at most T. From the lemma for $n = m$ the function

$$\chi(z) = \partial\varphi(z\boldsymbol{\omega})$$

has a zero of order at least T at each point of $\mathcal{T}(Q_m, S_m)$, and we have seen that there are $N \geq c_{14}^{-1}Q_m^2 S_m$ of these. We deduce from the maximum modulus principle and Lemma 5.2 that

$$|\chi(t)| < 4^{-NT}k^{c_{15}k}c_{15}^{LS_{m+1}^2},$$

and since $k \geq Lh^2$ a simple calculation shows the right hand side of this to be at most 3^{-NT}. If ξ is a counterexample with the order of ∂ minimal a similar calculation, as in [6], gives the upper bound $|\xi| < 2^{-NT}$. But this contradicts the lower bound

$$|\xi| > k^{-c_{16}kQ_{m+1}^4}$$

arising from Lemma 5.3 and the observation $Q_{m+1} < k/L_0$.

The final lemma of this section summarises the information we need to extract from the preceding lemma. For brevity we put $\mathcal{T} = \mathcal{T}(k^{1-15\epsilon}, k^{1-40\epsilon})$.

LEMMA 5.5: *The function $\Phi(z)$ has a zero of order at least $2L_0 + 1$ at each point $t\boldsymbol{\omega}$ with t in $\mathcal{T}$.*

PROOF: Immediate; note there is a significant reduction in the range of the integer s.

6. Conclusion of the proof

We commence by eliminating the function H from Φ.

LEMMA 6.1: *There is a non-zero polynomial Q in $I[\mathbf{x}]$ of total degree $M \leq c_{17}L_0L$ with coefficients of sizes at most $k^{c_{17}kL_0}$, such that the function $\Psi(z) = Q(\mathbf{A}(z))$ has a zero at each point $t\boldsymbol{\omega}$ with t in $\mathcal{T}$.*

PROOF: From Lemma 2.3 the function Φ is not identically zero, and so Lemma 2.5 supplies the required polynomial Q.

Next we put $\psi(z) = \Psi(z\boldsymbol{\omega})$, and for a non-negative integer m we denote the m-th derivative of $\psi(z)$ by $\psi^{(m)}(z)$.

LEMMA 6.2: *For each non-negative integer $m \leq M$ we have*

$$|\psi^{(m)}(0)| < e^{-k^{3-75\epsilon}}.$$

PROOF: Let N denote the number of distinct points of $\mathscr{T}$, so that $N > c_{18}^{-1}k^{3-70\epsilon}$. If $t_1, \ldots, t_N$ are these points, the function

$$\chi(z) = (\theta(z\boldsymbol{\omega}))^M \psi(z) \prod_{i=1}^{N} (z - t_i)^{-1}$$

is entire. Recalling the number λ of (19), and comparing $\chi(z)$ on the circles centred at the origin with radii λ and $R = k^{1-39\epsilon}$, we deduce that on the smaller circle

$$|\psi(z)| < 3^{-N} k^{c_{21}kL_0} c_{21}^{MR^2} < 2^{-N}.$$

The inequalities of the lemma are now immediate from Cauchy's integrals taken around this smaller circle.

To complete the proof of the theorem we observe that since $\boldsymbol{\omega} = (\omega_1, \omega_2)$ we have for each $m \leq M$

$$\psi^{(m)}(0) = \sum n(\partial) \omega_1^{m_1} \omega_2^{m_2} \partial \Psi(\mathbf{0}),$$

where

$$\partial = (\partial/\partial z_1)^{m_1} (\partial/\partial z_2)^{m_2}, \quad n(\partial) = m!/(m_1! m_2!),$$

and the sum is over all non-negative integers m_1, m_2 with $m_1 + m_2 = m$. Lemma 2.4 shows that as m varies, the coefficients $\partial\Psi(\mathbf{0})$ are algebraic numbers of K with sizes at most $k^{c_{22}kL_0}$ and a common denominator not exceeding c_{22}^M; and from Lemma 2.2 not all of them vanish. Hence there is an integer $m \leq M$ such that $\psi^{(m)}(0)$ can be written as a non-zero homogeneous polynomial in ω_1 and ω_2. By the preceding lemma this immediately provides us with a non-zero homogeneous polynomial F in $I[x_1, x_2]$ of degree at most $M < k^{1+7\epsilon}$, with coefficients of sizes at most $k^{c_{23}kL_0} < e^{k^{1+7\epsilon}}$, such that

$$|F(\omega_1, \omega_2)| < e^{-k^{3-80\epsilon}}.$$

But recall that the construction of the auxiliary function depends on the parameter $k > c$. Therefore we can apply Lemma 4.4 with $t_k = k^{1+7\epsilon}$ to conclude that ω_1, ω_2 are linearly dependent over $\mathbf{A}$. Thus $\boldsymbol{\omega}$ is special and the theorem is established.

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