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A. W. WICKSTEAD

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REPRESENTATION AND DUALITY OF MULTIPLICATION OPERATORS ON ARCHIMEDEAN RIESZ SPACES

A. W. Wickstead

1. Introduction and definitions

We shall be concerned with three classes of linear operators on Archimedean Riesz spaces. A positive linear operator T on the Riesz space E is a *positive orthomorphism* if whenever $x, y \in E$ and $x \wedge y = 0$ then $x \wedge Ty = 0$. An *orthomorphism* is the difference of two positive orthomorphisms. $P(E)$ will denote the vector space of all orthomorphisms on E, and $P(E)_+$ the cone of positive orthomorphisms.

The *stabiliser* of E is the vector space of all linear operators on E which leave every ideal invariant. We denote this space by $S(E)$, and its positive cone by $S(E)_+$. A linear operator T on E lies in $S(E)$ if and only if for each $x \in E_+$ there is a non-negative real number λ_x such that $-\lambda_x x \leq Tx \leq \lambda_x x$. $Z(E)$ is the subspace of $S(E)$ consisting of those T for which there is a non-negative real λ with $-\lambda x \leq Tx \leq \lambda x$ for all $x \in E_+$. $Z(E)$ is the *ideal centre* of E.

$Z(E)$ appears to have been introduced, for Archimedean ordered vector spaces, by Buck [4] and has received quite a lot of attention recently, especially for ordered topological vector spaces. $P(E)$ was studied, for Archimedean lattice groups, in [3] and [5], where it was shown that if E is represented by Bernau's representation [1] then the elements of $P(E)$ may be described by pointwise multiplication by an extended real valued continuous function. The proofs given there do not lend themselves to application to other representations. In section 3 we shall have need of representing elements of $P(E)$ in this way for other representations of E, where now E is an Archimedean Riesz space. Most of section 2 is denoted to proving that this can be done. Zaanen [9] has already dealt with a number of special cases.

$S(E) \cap P(E)$ and $Z(E) \cap P(E)$ were studied briefly in [3]. We shall see below that in fact $S(E) \subset P(E)$. This is not completely obvious as we did not specify that elements of $S(E)$ were differences of positive elements of $S(E)$. It appears to be an open question whether every linear operator T on E such that $|x| \wedge |Ty| = 0$ whenever $x, y \in E$ with $|x| \wedge |y| = 0$ must lie in $P(E)$.

If E and F are Riesz spaces we denote by $L^{\sim}(E, F)$ the vector space of all differences of positive linear operators from E into F. In particular we write $L^{\sim}(E)$ for $L^{\sim}(E, E)$ and $E^{\sim}$ for $L^{\sim}(E, \mathbb{R})$. $E^{\times}$ will denote the space of *normal integrals* on E, i.e. those $f \in E^{\sim}$ such that $f(x_\gamma) \to 0$ whenever (x_γ) is a net in E directed downward to 0.

If $T \in L^{\sim}(E)$ the formula

$$(T^{\sim} f)(x) = f(Tx) \quad (x \in E, f \in E^{\sim})$$

defines $T^{\sim} \in L^{\sim}(E^{\sim})$. Section 3 is devoted to a study of the duality theory for elements of $P(E)$, $S(E)$ and $Z(E)$. In order to obtain a satisfactory theory it is necessary to assume that $E^{\sim}$ separates the points of E, and hence that E is Archimedean. The situation for $P(E)$ and $Z(E)$ is quite straightforward, but that for $S(E)$ is rather more complicated. The final section contains some examples.

The author is grateful to the referee for the suggestion that he include a proof of Theorem 2.3 and also for supplying the present proof of Theorem 2.5. This replaced a proof that leaned heavily on work published elsewhere by several authors.

2. Representations

If S is a topological space $C^{\infty}(S)$ will denote the set of all continuous extended real valued functions on S which are finite on a dense subset of S. If $f, g \in C^{\infty}(S)$ and $\lambda \in \mathbb{R} \backslash \{0\}$ then $\lambda f : s \mapsto \lambda . f(s)$ and $f \vee g : s \mapsto f(s) \vee g(s)$ lie in $C^{\infty}(S)$. There may or may not be $h \in C^{\infty}(S)$ with $h(s) = f(s) + g(s)$ whenever the sum is defined (we shall say that sums of the form $\infty + (-\infty)$ and products of the form $0.(\pm\infty)$ are not defined). If such an h does exist we denote it by $f + g$.

If E is an Archimedean Riesz space and S a topological space the map $x \mapsto x^{\wedge} : E \to C^{\infty}(S)$ is a *representation* of E if

(1) $E^{\wedge} = \{x^{\wedge} : x \in E\}$ is a vector space and a sublattice of $C^{\infty}(S)$.

(2) $x \mapsto x^{\wedge}$ is a Riesz space isomorphism of E onto $E^{\wedge}$.

The representation is *admissible* if

(3) For each $s \in S$ there is $x \in E$ with $0 < x^{\wedge}(s) < \infty$.

There are always many representations of an Archimedean Riesz space. One of the most useful is that of Bernau [1] which is admissible. We shall make use of a representation for the pair $(E, E^{\sim})$ in the case that $E^{\sim}$ separates the points of E. By an *admissible functional representation* for such a pair we mean a pair of admissible representations of E and $E^{\sim}$ in the same $C^{\infty}(S)$ (where S is extremally disconnected, locally compact and Hausdorff), such that $E^{\sim\wedge}$ is an ideal in $C^{\infty}(S)$ containing the characteristic functions of compact open sets, and which are related as follows. There is a Radon measure μ on S, for which nowhere dense sets are locally μ-negligible, such

$$f(x) = \int_S f^{\wedge} x^{\wedge} \, d\mu \ (x \in E, f \in E^{\sim}).$$

The existence of such a representation is vital to the proofs in section 3. This will be deduced from the results of Fremlin in [8].

Recall that a topological space is *extremally disconnected* if the closure of every open set is open. A compact Hausdorff space which is extremally disconnected is called *Stonian.* A measure μ on the Stonian space S is *normal* if the linear functional $f \mapsto \int f \, d\mu$ lies in $C(S)^{\times}$. Equivalently, if μ is positive, μ vanishes on all the nowhere dense Borel subsets of S. The Stonian space S is *Hyperstonian* if $C(S)^{\times}$ separates the points of $C(S)$.

LEMMA 2.1: *If (X, μ) is a positive finite measure algebra then there is a Hyperstonian space S and a strictly positive normal Radon measure ν on S such that $L^1(\mu)$ and $L^1(\nu)$ are linearly order isomorphic.*

$L^1(\mu)$ is a Banach lattice for its usual norm and order and its Banach dual is isometrically order isomorphic to $L^{\infty}(\mu)$. By [6] there is a Hyperstonian space S such that $L^{\infty}(\mu)$ is isometrically order isomorphic to $C(S)$ and $L^1(\mu)$ to $C(S)^{\times}$. The linear functional $f \mapsto \int f \, d\mu$ on $L^{\infty}(\mu)$ is strictly positive and lies in $L^{\infty}(\mu)^{\times}$. If ν is the measure defining the corresponding element of $C(S)^{\times}$ then ν is strictly positive. An application of the Radon–Nikodym theorem now allows us to identify $C(S)^{\times}$ with $L^1(\nu)$, completing the proof.

The following lemma is essentially proved in [6].

LEMMA 2.2: *Let S be a Hyperstonian space and ν a strictly positive normal Radon measure on S. Every real valued measurable function* *on S coincides, except on a set of ν-measure* 0, *with a unique member of* $C^\infty(S)$.

If S is a locally compact Hausdorff space and μ a Radon measure on S then $\mathcal{M}(S, \mu)$ will denote the space of equivalence classes, under the relation of local μ-almost everywhere equality, of locally μ-integrable real valued functions on S.

THEOREM 2.3: *If E is a Riesz space and $E^\sim$ separates the points of E then there is an admissible functional representation of the pair* $(E, E^\sim)$.

If $x \in E$ then the functional $f \mapsto f(x)$ on $E^\sim$ lies in $E^{\sim\times}$. Thus $E^{\sim\times}$ separates the points of $E^\sim$ and $E^\sim$ is, in the terminology of [8], a preK space. By using Lemma 2.1 in place of Lemma 5 of [8] we can modify the proof of Theorem 6 of [8] to obtain a Riesz space isomorphism $f \mapsto f^\wedge$ of $E^\sim$ onto an ideal $E^{\sim\wedge}$ in $\mathcal{M}(S, \mu)$ which contains the characteristic function of every compact set, where S is a disjoint union of Hyperstonian spaces X_i and μ a strictly positive Radon measure on S the restriction of which to each X_i is normal. It is clear that S is locally compact and extremally disconnected and that nowhere dense subsets of S are locally μ-negligible.

By Theorem 7 of [8] there is a Riesz space isomorphism $X \mapsto X^\wedge$ of $(E^{\sim\wedge})^\times$ into $\mathcal{M}(S, \mu)$ such that

$$X(f^\wedge) = \int_S f^\wedge X^\wedge \, d\mu \quad (f^\wedge \in E^{\sim\wedge}).$$

If we combine the natural embedding of E into $E^{\sim\times}$, the identification of $E^{\sim\times}$ with $(E^{\sim\wedge})^\times$ and this map (all of which are lattice isomorphisms) we obtain a map $x \mapsto x^\wedge$ taking E into a sublattice $E^\wedge$ of $\mathcal{M}(S, \mu)$ such that

$$f(x) = \int_S f^\wedge x^\wedge \, d\mu \quad (x \in E, f \in E^\sim).$$

Lemma 2.2 enables us to choose the unique continuous representative from each equivalence class in $\mathcal{M}(S, \mu)$ so that our elements $x^\wedge$ and $f^\wedge$ may be assumed to lie in $C^\infty(S)$.

Finally, to ensure that $E^\wedge$ is admissible replace S by its open subset $S_0 = \{s \in S : \exists x \in E \text{ with } 0 < x^\wedge(s) < \infty\}$, which is extremally disconnected, and replace each $x^\wedge$ and $f^\wedge$ by their restrictions to S_0. The representations are still isomorphisms for given $x \in E$ and $f \in E^{\sim}$, $S \backslash S_0 \in x^{\wedge-1}(0) \cup x^{\wedge-1}(\infty) \cup x^{-1}(-\infty)$. The final two sets in this union are nowhere dense and thus locally μ-negligible so that $\int_{S \backslash S_0} f^{\wedge} x^{\wedge} \, d\mu = 0$.

The main result of this section is Theorem 2.5. For some representations this result is known (see [3], [5] and [9]). We shall need to know the following lemma.

LEMMA 2.4: *Let E be an Archimedean Riesz space, S a topological space and $x \to x^{\wedge}$ be an admissible representation of E in $C^{\infty}(S)$. Let $T \in P(E)_+$ and set $S_0 = \{s \in S : \exists y \in E_+$ with $0 < y^{\wedge}(s) < \infty$ and $(Ty)^{\wedge}(s) < \infty\}$. If $x \in E$, $s \in S_0$ and $x^{\wedge}(s) = 0$ then $(Tx)^{\wedge}(s) = 0$.*

We may, by considering x^+ and x^- separately, assume $x \in E_+$. Let $y \in E_+$ with $0 < y^{\wedge}(s) < \infty$ and $(Ty)^{\wedge}(s) < \infty$. Let $\epsilon > 0$ and set

$$x_\epsilon = x - (x \wedge \epsilon y), \; y_\epsilon = \epsilon y - (x \wedge \epsilon y)$$

so that $x_\epsilon, y_\epsilon \in E$, $(x_\epsilon)^{\wedge}(s) = x^{\wedge}(s) = 0$ and $0 < (y_\epsilon)^{\wedge}(s) = \epsilon y^{\wedge}(s) < \infty$. As $x_\epsilon \wedge y_\epsilon = 0$, $Tx_\epsilon \wedge y_\epsilon = 0$ so that $(Tx_\epsilon)^{\wedge}(s) = 0$. Now

$$\begin{aligned} 0 \le (Tx)^{\wedge}(s) &= (Tx_\epsilon)^{\wedge}(s) + (T(x \wedge \epsilon y))^{\wedge}(s) \\ &= (T(x \wedge \epsilon y))^{\wedge}(s) \\ &\le \epsilon (Ty)^{\wedge}(s). \end{aligned}$$

This holds for all $\epsilon > 0$ so $(Tx)^{\wedge}(s) = 0$.

THEOREM 2.5: *Let E be an Archimedean Riesz space, S a topological space and $x \to x^{\wedge}$ be an admissible representation of E in $C^{\infty}(S)$. If $T \in P(E)$ there is $q \in C^{\infty}(S)$ such that for each $x \in E$*

$$(Tx)^{\wedge}(s) = q(s) x^{\wedge}(s)$$

for all $s \in S$ for which the product is defined.

We first suppose $T \in P(E)_+$ and let S_0 be the corresponding subset of S defined as in Lemma 2.4. If $s \in S_0$ choose $y \in E_+$ with $0 < y^{\wedge}(s) < \infty$ and $(Ty)^{\wedge}(s) < \infty$. Let $q(s) = (Ty)^{\wedge}(s)/y^{\wedge}(s)$. If $x \in E$ with $|x^{\wedge}(s)| < \infty$ then $x_1 = x - (x^{\wedge}(s)/y^{\wedge}(s))y$ satisfies $x_1^{\wedge}(s) = 0$. By the preceding lemma $(Tx_1)^{\wedge}(s) = 0$, so that

$$(Tx)^{\wedge}(s) = (x^{\wedge}(s)/y^{\wedge}(s))(Ty)^{\wedge}(s) = q(s) x^{\wedge}(s).$$

In particular if $0 < x^\wedge(s) < \infty$ then

$$\frac{(Tx)^\wedge(s)}{x^\wedge(s)} = q(s) = \frac{(Ty)^\wedge(s)}{y^\wedge(s)}$$

so that the definition of $q(s)$ does not depend on the choice of y. Clearly the function q on S_0 is finite valued and continuous.

S_0 is an open subset of S which is dense in S. The latter is because if $s \in S$ and U is any neighbourhood of s in S we can choose $y \in E_+$ with $0 < y^\wedge(s) < \infty$. If $(Ty)^\wedge(s) = \infty$ we can find $s_1 \in U$ with $0 < y^\wedge(s_1) < \infty$ (using the continuity of $y^\wedge$) and $(Ty)^\wedge(s_1) < \infty$ (as $(Ty)^\wedge$ is finite on a dense subset of S).

If we define q for $s_0 \in S \backslash S_0$ by $q(s_0) = \infty$ then q is continuous at s_0. For if we choose $y \in E_+$ with $0 < y^\wedge(s) < \infty$ we must have $(Ty)^\wedge(s) = \infty$. If A is a positive real we can find a neighbourhood V of s_0 on which $y^\wedge(s) < y^\wedge(s_0) + 1$ and $(Ty)^\wedge(s) > A/(y^\wedge(s_0) + 1)$. Hence for $s \in S_0 \cap V$, $q(s) > A$. As S_0 is open it follows that q is continuous on S, and that $q \in C^\infty(S)$ follows from the density of S_0.

It follows by continuity that, for each $x \in E$, $(Tx)^\wedge(s) = q(s)x^\wedge(s)$ holds for all $s \in S$ for which the product is defined.

Finally let $T = T_1 - T_2$ with $T_1, T_2 \in P(E)_+$ and let $q_1, q_2 \in C^\infty(S)$ be the corresponding functions. If $s \in S$, $x \in E$, if at least one of $q_1(s)$ and $q_2(s)$ is finite and if the product is defined then $(Tx)^\wedge(s) = (q_1(s) - q_2(s))x^\wedge(s)$. If $s_0 \in S$ and $q_1(s_0) = q_2(s_0) = \infty$ choose $y \in E_+$ with $0 < y^\wedge(s_0) < \infty$. In any neighbourhood of s_0 there is an s with $q_1(s) - q_2(s)$ defined and equal to $(Ty)^\wedge(s)/y^\wedge(s)$. As $(Ty)^\wedge/y^\wedge$ is continuous at s_0 we can define $q_1(s_0) - q_2(s_0)$ in such a way as to make the function $q = q_1 - q_2$ continuous at s_0. Clearly $q \in C^\infty(S)$ and $(Tx)^\wedge(s) = q(s)x^\wedge(s)$ wherever the product is defined.

REMARK 1: The assumption that the representation be admissible, whilst it possibly may be weakened, cannot in general be omitted. For example, if $E = \{f \in C([0, 1]) : f(0) = 0\}$ then each bounded continuous function on $(0, 1]$ defines an orthomorphism of E, but need not have a continuous extension to $[0, 1]$.

REMARK 2: If we are given any representation $x \to x^\wedge : E \to C^\infty(S)$ then $x \to x^\wedge|_T : E \to C^\infty(T)$, where $T = \{s \in S : \exists x \in E_+ \text{ with } 0 < x^\wedge(s) < \infty\}$, is an admissible representation of E.

REMARK 3: If S is an extremally disconnected compact Hausdorff space the conclusion of the theorem holds for any representation

$x \to x^\wedge : E \to C^\infty(S)$. To see this, use the theorem to find q, defined on the open subset T of S (defined as in Remark 2), with the desired properties. Any extension $\bar{q}$ of q to an element of $C^\infty(S)$ will have the desired property.

REMARK 4: Similar results may be proved for representations as equivalence classes of functions measurable with respect to a Radon measure μ, by identifying these classes with $C^\infty(S)$ for a suitable extremally disconnected compact Hausdorff space S in a well known manner.

REMARK 5: Suppose that $x \mapsto x^\wedge : E \to C^\infty(S)$ is an admissible representation of E and that $q \in C^\infty(S)$ has the property that for all $x \in E$ there is $y_x \in E$ with $y_x^\wedge(s) = q(s)x^\wedge(s)$ wherever the product is defined. The map $x \mapsto y_x$ is well-defined and linear, and clearly if $x \wedge x' = 0$ then $y_x \wedge x' = 0$. $x \mapsto y_x$ lies in $P(E)$ as these remarks also apply to q^+ and q^- as, e.g.

$$q^+(s)x^\wedge(s) = [q(s)x^{+\wedge}(s)]^+ - [q(s)x^{-\wedge}(s)]^-$$

wherever all the products are defined.

REMARK 6: If $x \mapsto x^\wedge : E \to C^\infty(S)$ is an admissible representation and $T \in P(E)$, we shall denote the q of Theorem 2.5 by $T^\wedge$. It is clear that $T \in S(E)$ if and only if $T^\wedge$ is bounded on the support of $x^\wedge$ for each $x \in E$, and that $T \in Z(E)$ if and only if $T^\wedge$ is bounded.

The final result we need in this section is:

PROPOSITION 2.6: *For every Archimedean Riesz space E, $S(E) \subset P(E)$.*

Let $x \mapsto x^\wedge$ be an admissible representation of E in some $C^\infty(S)$, which always exists. If $s \in S$ the set $\{x \in E : x^\wedge(s) = 0\}$ is an ideal in E. Thus if $T \in S(E)$ and $x^\wedge(s) = 0$ we have $(Tx)^\wedge(s) = 0$. A simplified version of the proof of Theorem 2.5 now yields $q \in C^\infty(S)$ with

$$(Tx)^\wedge(s) = q(s)x^\wedge(s) \quad (x \in E)$$

for all $s \in S$ for which the product is defined. The result now follows from Remark 5.

In general the spaces $P(E)$, $S(E)$ and $Z(E)$ are all distinct (see section 4).

3. Duality

For the whole of this section E will denote a Riesz space whose order dual, $E^{\sim}$, separates its points, so that in particular E is Archimedean. We consider an element of $L^{\sim}(E)$ and ask when it, or one of its adjoints, lies in one of the spaces we have been considering. Our first result is very easily proved.

PROPOSITION 3.1: *If $T \in L^{\sim}(E)$ then $T \in Z(E)$ if and only if $T^{\sim} \in Z(E^{\sim})$.*

$$\begin{aligned} T \in Z(E) &\Leftrightarrow \exists \lambda \geq 0 \text{ with } -\lambda x \leq Tx \leq \lambda x \quad (x \in E_+) \\ &\Leftrightarrow \exists \lambda \geq 0 \text{ with } -\lambda f(x) \leq f(Tx) = (T^{\sim} f)(x) \leq \lambda f(x) \\ &\qquad\qquad \text{for all } x \in E_+ \text{ and } f \in E^{\sim}_+ \\ &\Leftrightarrow \exists \lambda \geq 0 \text{ with } -\lambda f \leq T^{\sim} f \leq \lambda f \quad (f \in E^{\sim}_+) \\ &\Leftrightarrow T^{\sim} \in Z(E^{\sim}). \end{aligned}$$

Before proving the next result we need a lemma.

LEMMA 3.2: *Let $(E, E^{\sim})$ have an admissible functional representation in $C^{\infty}(S)$ and let μ be the corresponding measure. If $\phi \in C^{\infty}(S)_+$ and $|\int_S x^{\wedge} \phi \, \mathrm{d}\mu| < \infty$ for all $x \in E$ then there is $f \in E^{\sim}_+$ with $\phi = f^{\wedge}$.*

Define $h \in E^{\sim}_+$ by $h(x) = \int_S x^{\wedge} \phi \, \mathrm{d}\mu$ $(x \in E)$. If $g \in E^{\sim}_+$ and $g^{\wedge} \leq \phi$ then $g(x) = \int_S x^{\wedge} g^{\wedge} \, \mathrm{d}\mu \leq \int_S x^{\wedge} \phi \, \mathrm{d}\mu = h(x)$ for all $x \in E_+$. Thus $h \geq g$ and hence $h^{\wedge} \geq g^{\wedge}$. We know $\phi(s) = \sup \{\psi(s);\ \psi \in C^{\infty}(S),\ \mathrm{supp}\,(\psi)$ is compact, ψ is bounded$\} = \sup \{g^{\wedge}(s) : g \in E^{\sim}_+,\ g^{\wedge} \leq \phi\}$ for each $s \in S$, so that $h^{\wedge} \geq \phi$. As $E^{\sim\wedge}$ is an ideal in $C^{\infty}(S)$ we must have $\phi = f^{\wedge}$ for some $f \in E^{\sim}$ (in fact $f = h$).

THEOREM 3.3: *If $T \in L^{\sim}(E)$ then $T \in P(E)$ if and only if $T^{\sim} \in P(E^{\sim})$.*

Suppose $T \in P(E)$. Represent $(E, E^{\sim})$ admissibly in $C^{\infty}(S)$, with μ the corresponding measure. By Theorem 2.5 there is $T^{\wedge} \in C^{\infty}(S)$ with $(Tx)^{\wedge}(s) = T^{\wedge}(s) x^{\wedge}(s)$ for all $s \in S$ for which the product is defined. In particular this fails to hold on a nowhere dense set only, which is locally μ-negligible. As S is extremally disconnected and locally compact there is, for each $f \in E^{\sim}_+$, $\phi \in C^{\infty}(S)$ with $\phi(s) = T^{\wedge}(s) f^{\wedge}(s)$ except, again, on a locally μ-negligible set. Clearly we have, if $x \in E$,

$$\begin{aligned}\int_S \phi(s)x^\wedge(s)\,\mathrm{d}\mu(s) &= \int_S T^\wedge(s)f^\wedge(s)x^\wedge(s)\,\mathrm{d}\mu(s)\\ &= \int_S (Tx)^\wedge(s)f^\wedge(s)\,\mathrm{d}\mu(s)\\ &= f(Tx)\\ &= (T^\sim f)(x)\end{aligned}$$

which is finite. It follows from Lemma 3.2 that there is $g_f \in E_+^\sim$ with $(g_f)^\wedge = \phi$. We have now a linear map $f \mapsto g_f = g_{f^+} - g_{f^-}$ on $E^\sim$, and $(g_f)^\wedge(s) = T^\wedge(s)f^\wedge(s)$ whenever the product is defined. It follows from Remark 5 that $f \mapsto g_f$ lies in $P(E^\sim)$. But $g_f(x) = \int_S \phi(s)x^\wedge(s)\,\mathrm{d}\mu(s) = (T^\sim f)(x)$ for all $x \in E$, so $g_f = T^\sim f$ and $T^\sim \in P(E^\sim)$.

If $T^\sim \in P(E^\sim)$ then, as $E^{\sim\sim}$ separates the points of $E^\sim$, $T^{\sim\sim} \in P(E^{\sim\sim})$. Let $\pi : E \to E^{\sim\sim}$ be the natural injection, so that $\pi(Tx) = T^{\sim\sim}(\pi x)$ $(x \in E)$. We know π is a lattice homomorphism, so if $x, y \in E$ with $x \wedge y = 0$ we have $\pi x \wedge \pi y = 0$. Hence $0 = \pi x \wedge T^{\sim\sim}\pi y = \pi x \wedge \pi(Ty) = \pi(x \wedge Ty)$, so that $x \wedge Ty = 0$. Thus $T \in P(E)$.

Bigard [2], has shown that if $T \in P(E)$ then $T^\times \in P(E^\times)$, where $T^\times$ denotes the restriction of $T^\sim$ to $E^\times$.

It is not true that $T \in S(E)$ if and only if $T^\sim \in S(E^\sim)$, and an example to show this will be given in section 4. The positive result we do have requires us to go the second dual.

THEOREM 3.4: *If $T \in L^\sim(E)$ then $T \in S(E)$ if and only if $T^{\sim\sim} \in S(E^{\sim\sim})$.*

Again we let $\pi : E \to E^{\sim\sim}$ be the natural injection. This time we choose an admissible functional representation of the pair $(E^\sim, E^{\sim\sim})$ in $C^\infty(S)$, with μ the associated measure. We certainly have that $T^{\sim\sim} \in S(E^{\sim\sim})$ implies $T \in S(E)$, so we shall assume $T \in S(E)_+$ and prove that $T^{\sim\sim} \in S(E^{\sim\sim})_+$. As $S(E)$ is positively generated this will prove the result.

We know $T \in P(E)$ so $T^\sim \in P(E^\sim)$ and $T^{\sim\sim} \in P(E^{\sim\sim})$ by Theorem 3.3. There is thus a function $T^\wedge \in C^\infty(S)$ such that $T^\sim$ and $T^{\sim\sim}$ are represented in $C^\infty(S)$ by multiplication by $T^\wedge$ at all points of S for which the product is defined. We know that if $x \in E_+$ there is $\lambda_x \geq 0$ with $0 \leq \pi(Tx) = T^{\sim\sim}(\pi x) \leq \lambda_x(\pi x)$. Thus $T^\wedge$ is bounded on the support of $(\pi x)^\wedge$ for each $x \in E$. We must prove that $T^\wedge$ is bounded on the support of $X^\wedge$ for each $X \in E_+^{\sim\sim}$.

Let $B_n = \{s \in S : n-1 < T^\wedge(s) < n+1\}^-$, an open and closed subset of S since $T^\wedge$ is continuous and S is extremally disconnected. Set $A_n = B_n \cap \operatorname{supp}(X^\wedge)$ which is again open and closed. Let $P = \{n \in \mathbb{N} : A_n \neq \emptyset\}$. We claim P is a finite set.

If $n \in P$ choose $s_n \in A_n$ with $X^\wedge(s_n) > 0$ and $n-1 < T^\wedge(s_n) < n+1$. Choose $f_n \in E_+$ with $f_n^\wedge(s_n) > 0$, possible by the admissibility of the representation. $P(E^\sim)$ is a lattice so we may form

$$g_n^\wedge = [-|T^\wedge(s_n)I_E - T^\sim| + \alpha_n I_E]^+ \cdot f_n \geq 0,$$

where α_n is chosen so that $n+1-T^\wedge(s_n)$, $T^\wedge(s_n)-(n-1) > \alpha_n > 0$. We have

$$g_n^\wedge(s) = [-|T^\wedge(s_n) - T^\wedge(s)| + \alpha_n]^+ f_n^\wedge(s)$$

whenever the product is defined. If $g_n^\wedge(s) > 0$ then $|T^\wedge(s_n) - T^\wedge(s)| < \alpha_n$ so that $n+1-T^\wedge(s) = [n+1-T^\wedge(s_n)] + [T^\wedge(s_n) - T^\wedge(s)] > 0$, and similarly $T^\wedge(s)-(n-1) > 0$. Thus $\operatorname{supp}(g_n^\wedge) \subset B_n$. Also $g_n^\wedge(s_n) = \alpha_n f_n^\wedge(s_n) > 0$.

Note that $X(g_n) > 0$. For let K be a non-empty compact open subset of S, containing s_n, with $X^\wedge \geq \beta\chi_K$ and $g_n \geq \gamma\chi_K$ for some $\beta, \gamma > 0$. $\chi_K = Y^\wedge$ for some $0 \neq Y \in E_+^{\sim\sim}$, and $\mu(K) > 0$ for else $Y(h) = \int_S h^\wedge Y^\wedge \, d\mu = \int_K h^\wedge \, d\mu = 0$ for all $h \in E^\sim$. Then $X(g_n) = \int_S X^\wedge g_n^\wedge \, d\mu \geq \beta \cdot \gamma \cdot \mu(K) > 0$.

If $x \in E$ then $\operatorname{supp}(\pi x)^\wedge \cap B_n = \emptyset$ for all but finitely many n, as $T^\wedge$ is bounded on $\operatorname{supp}(\pi x)^\wedge$. Thus $g_n(x) = (\pi x)(g_n) = \int_S (\pi x)^\wedge g_n^\wedge \, d\mu = 0$ for all but finitely many n, since $\operatorname{supp}(g_n^\wedge) \subset B_n$. Thus the series $\Sigma_{n\in P} X(g_n)^{-1} g_n(x)$ converges for all $x \in E$. We may thus define $h \in E_+^\sim$ by $h(x) = \Sigma_{n\in P} X(g_n)^{-1} g_n(x)$. If F is a finite subset of P, then for $x \in E_+$,

$$h(x) \geq \sum_{n\in F} X(g_n)^{-1} g_n(x),$$

so that

$$h \geq \sum_{n\in F} X(g_n)^{-1} g_n.$$

Thus

$$X(h) \geq \sum_{n\in F} X(g_n)^{-1} X(g_n) = |F|.$$

It follows that P must be a finite set.

Thus there is $m \in \mathbb{N}$ with

$$T^{\wedge}(\operatorname{supp}(X^{\wedge})) \subset [0, m] \cup \{\infty\}.$$

As $T^{\wedge} \in C^{\infty}(S)$ continuity shows that $T^{\wedge}(\operatorname{supp}(X^{\wedge})) \subset [0, m]$, completing the proof.

Finally we answer the question "when is it true that $T \in S(E)$ and $T^{\sim} \in S(E^{\sim})$?"

THEOREM 3.5: *If $T \in L^{\sim}(E)$ the following are equivalent*:

(1) $T \in Z(E)$

(2) $T \in S(E)$ *and* $T^{\sim} \in S(E^{\sim})$.

The proof is similar to that of Theorem 3.4, the proof of (1) $\Rightarrow$ (2) being obvious. Represent $(E, E^{\sim})$ admissibly in $C^{\infty}(S)$ with μ the corresponding measure. By Theorem 2.5 there is $T^{\wedge} \in C^{\infty}(S)$ with $(Tx)^{\wedge}(s) = T^{\wedge}(s)x^{\wedge}(s)$ and (by the proof of Theorem 3.3) $(T^{\sim}f)^{\wedge}(s) = T^{\wedge}(s)f^{\wedge}(s)$ $(x \in E,\ f \in E^{\sim})$ whenever the products are defined. We also know $T^{\wedge}$ is bounded on the sets $\operatorname{supp}(x^{\wedge})$ $(x \in E)$ and $\operatorname{supp}(f^{\wedge})$ $(f \in E^{\sim})$.

Let $S_n = \{s \in S : n - 1 < T^{\wedge}(s) < n + 1\}^{-}$, an open and closed set, $P = \{n \in \mathbb{N} : S_n \neq \emptyset\}$. We need only show that P is a finite set. Let K_n be a non-empty compact open subset of S_n for each $n \in P$. For each such n there is $f_n \in E^{\sim}_{+}$ with $f^{\wedge}_n = \chi_{K_n}$. Define $F \in E^{\sim}_{+}$ by

$$F(x) = \sum f_n(x) = \sum_{n \in P} \int_K x^{\wedge}\, \mathrm{d}\mu \quad (x \in E),$$

the sum converging as $\operatorname{supp}(x^{\wedge})$ meets only finitely many S_n, and hence only finitely many K_n. If $x \in E_+$, $n \in P$ then $F(x) \geq f_n(x)$, so $F \geq f_n$ and $F^{\wedge} \geq \chi_{K_n}$. Thus $\operatorname{supp}(F^{\wedge}) \cap S_n \neq \emptyset$ for all $n \in P$. But $T^{\wedge}$ is bounded on $\operatorname{supp}(F^{\wedge})$, so we must have P finite and the proof is complete.

4. Examples

Let Ω denote the space of all real sequences, $\Phi = \{(x_n) \in \Omega : x_n = 0$ for all but finitely many $n\}$, $m = \{(x_n) \in \Omega : \exists K \geq 0$ with $-K \leq x_n \leq K \forall n \in \mathbb{N}\}$. It is easy to verify that we have the following identifications:

$$Z(\Omega) = S(\Omega) = m, P(\Omega) = \Omega,$$
$$Z(\Phi) = m, S(\Phi) = P(\Phi) = \Omega.$$

This shows that we need have neither $Z(E) = S(E)$ nor $S(E) = P(E)$ in general. Also since $\Omega^{\sim}$ may be identified with Φ and $\Phi^{\sim}$ with Ω this shows that neither $S(E)$ nor $\{T \in L^{\sim}(E): T^{\sim} \in S(E^{\sim})\}$ need include the other.

An exception occurs when E is a Banach lattice. This will follow from the next proposition, special cases of which have been proved by Bigard ([2], Théorème 8) and Flösser ([7], Satz 1.15).

PROPOSITION 4.1: *Let E be a normed lattice and $T \in P(E)$. T is norm bounded with $\|T\| \le \lambda$ if and only if $-\lambda I_E \le T \le \lambda T_E$.*

Suppose $-\lambda I_E \le T \le \lambda I_E$ and $x \in E$. By using a representation of T as a multiplication operator it is clear that $|Tx| = |T(|x|)|$. Hence

$$-\lambda |x| \le T(|x|) \le \lambda |x|$$

implies

$$|Tx| = |T(|x|)| \le \lambda |x|,$$

so that

$$\|Tx\| \le \lambda \|x\|$$

and hence

$$\|T\| \le \lambda.$$

Suppose conversely that $\|T\| \le \lambda$, then $T^{\sim}$ leaves E^*, the normed dual of E, invariant and $\|T^{\sim}|_{E^*}\| \le \lambda$. Represent $(E, E^{\sim})$ admissibly in $C^{\infty}(S)$ and use Theorems 2.5 and 3.3 to find $T^{\sim\wedge} \in C^{\infty}(S)$ such that, for all $f \in E^{\sim}$, $(T^{\sim}f)^{\wedge}(s) = T^{\sim\wedge}(s)f^{\wedge}(s)$ whenever the product is defined. If we prove that $-\lambda I_{E^*} \le T^{\sim}|_{E^*} \le \lambda I_{E^*}$ the proof of Proposition 3.1 will show that $-\lambda I_E \le T \le \lambda I_E$.

We prove $T^{\sim}|_{E^*} \le \lambda I_{E^*}$, the proof that $-\lambda I_{E^*} \le T^{\sim}|_{E^*}$ being similar. If this fails we can find $g \in E^*_+$ with $T^{\sim}g \not\le \lambda g$. It follows that

$$A = \{s \in S: T^{\sim\wedge}(s) > \lambda + \epsilon, g^{\wedge}(s) > 0\}$$

is non-empty for some $\epsilon > 0$. Let B be a non-empty open and closed subset of A. As E^* is a lattice ideal in $E^\sim$, there is $h \in E^*$ with $h^\wedge = \chi_B \cdot g^\wedge$. It follows from the representation that

$$T^\sim h \geq (\lambda + \epsilon)h > 0,$$

and hence that $\|T^\sim h\| \geq (\lambda + \epsilon)\|h\|$, contradicting $\|T^\sim\| \leq \lambda$.

COROLLARY 4.2: *If E is a Banach lattice then $Z(E) = S(E) = P(E)$.*

This follows from the proposition as all positive linear operators on a Banach lattice, and hence their differences, are norm bounded.

Theorem 3.5 may be rephrased as $\{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} \cap S(E) = Z(E)$. It is natural to ask what $\{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} + S(E)$ is, especially as in many of the examples that first come to mind it is precisely $P(E)$. This is not the case in general as the following example shows.

EXAMPLE 4.3: *Let $E = \{(a_n) \in \Omega : (n^p a_n) \in l_1$ for all $p \in \mathbb{N}\}$. E is a Riesz space, $E^\sim$ separates the points of E and $\{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} + S(E) \neq P(E)$.*

That E is a Riesz space is easy to check, and $E^\sim$ separates its points as the functionals $(a_n) \to a_m$ lie in $E^\sim$ for each $m \in \mathbb{N}$. If $(a_n) \in E$ then also $(na_n) \in E$, so $(a_n) \to (na_n)$ is an element of $P(E)$. This does not lie in $Z(E)$. We claim that $S(E) = \{T \in L^\sim(E) : T^\sim \in S(E^\sim)\} = Z(E)$, which will certainly prove the claim.

The sequence $(n^{-n})_{n=1}^\infty$ lies in E and has support the whole of $\mathbb{N}$. Thus if $T \in S(E)$ is represented by multiplication co-ordinatewise by the sequence (t_n) then $(t_n) \in m$ by Remark 6 and hence $T \in Z(E)$. Similarly if $T \in L^\sim(E)$ and $T^\sim \in S(E^\sim) \subset P(E^\sim)$ then $T \in P(E)$. Represent T as multiplication by the sequence (t_n). $F : (a_n) \to \Sigma_{n=1}^\infty a_n$ lies in $E^\sim$, so there is $\lambda \geq 0$ with $-\lambda F \leq T^\sim F \leq \lambda F$. If e_n is the sequence with 1 in its n'th position and zero elsewhere, which certainly lies in E, then

$$-\lambda = -\lambda F(e_n) \leq (T^\sim F)(e_n) = F(Te_n) = t_n \leq \lambda F(e_n) = \lambda.$$

Thus $(t_n) \in m$ and $T \in Z(E)$.

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Department of Pure Mathematics,
The Queen's University of Belfast,
Belfast BT7 1NN.
Northern Ireland