

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 35, n° 2 (1977), p. 129-137

http://www.numdam.org/item?id=CM_1977__35_2_129_0

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CURVATURE ON HOLOMORPHIC PLANE CURVES II

Linda Ness

Introduction

Let $C: f(y, z) = 0$ be a holomorphic curve with an ordinary double point at P and let $C_t: f + t = 0$. Let B be an open ball centered at P which is so small that $B \cap C = C_1 \cup C_2$ where C_1 and C_2 are non-singular connected holomorphic curves and $C_1 \cap C_2 = \{P\}$. For sufficiently small t , $C_t \cap B$ is nonsingular.

As in Part I [3] we will assume that all holomorphic curves are endowed with the metric induced by the Fubini-Study metric on \mathbb{C}^2 . The Fubini-Study metric on \mathbb{C}^2 is given by

$$\frac{2}{1 + |y|^2 + |z|^2} ((1 + |z|^2) dy d\bar{y} - \bar{y} z dy d\bar{z} - y \bar{z} dz d\bar{y} + (1 + |y|^2) dz d\bar{z})$$

where y and z denote the usual coordinates on \mathbb{C}^2 .

Here C_1 and C_2 are Riemannian surfaces as well as Riemann surfaces. In Part I we obtained a formula for the Gaussian curvature and studied the Gaussian curvature on $C_t \cap B$ in the more general case that C had an ordinary singularity at P . In Part II we will study the closed geodesics on $C_t \cap B$ and the lines of constant Gaussian curvature on $C_t \cap B$, when C has an ordinary double point at P .

From Part I [3] we recall

THEOREM 1: *Let $C \subset \mathbb{C}^2$ be a holomorphic curve defined by $f(y, z) = 0$. The Gaussian curvature at a nonsingular point of C is given by*

$$(1) \quad K(y, z) = 2 - \frac{(1 + |y|^2 + |z|^2)^3 |H(f)|^2}{(|f_y|^2 + |f_z|^2 + |yf_y + zf_z|^2)^3}$$

where

$$(2) \quad H(f) = 2f_{yz}f_yf_z - f_{yy}f_z^2 - f_{zz}f_y^2$$

($H(f)$ is the affine hessian of f .)

We now state the first main result in Part II. For convenience in referral, we number the theorems consecutively throughout Parts I and II.

THEOREM 3: *Let D be a small solid cone with vertex at P and with the line tangent to C_1 at P as its axis. Let D_2 be the image of D_1 under the unitary transformation of \mathbb{C}^2 that fixes P and interchanges the tangents to C_1 and C_2 . Let $\tilde{C}_t = C_t \cap B - D_1 - D_2$. Given $M > 0$ there exists $\epsilon > 0$ such that if $-\epsilon < t < \epsilon$*

- (i) K on $\tilde{C}_t < -M < 0$.
- (ii) \tilde{C}_t is homeomorphic to a cylinder.
- (iii) \tilde{C}_t is geodesically convex.
- (iv) In \tilde{C}_t there is a smooth closed geodesic Γ and every other smooth closed geodesic in C_t consists in going around Γ more than once.

The second main result in Part II is a picture of the curves of constant curvature on $B \cap C_t$ in the case that neither branch of C_0 has a flex at P .

1. Proof of Theorem 3

After a unitary transformation we may assume that $P = (0, 0)$ and $z = 0$ is tangent to C at P . We give the proof for the case that the other tangent at P is $y = 0$. The proof of the more general case is a straightforward generalization of the following argument. Hence we are assuming that

$$(3) \quad f(y, z) = yz + ay^3 + by^2z + cyz^2 + dz^3 + \text{higher order terms}$$

and that in B , f may be factored as $f = f_1 \cdot f_2$ where

$$(4) \quad f_1(y, z) = z + ay^2 + R_1(y, z)$$

$$(5) \quad f_2(y, z) = y + dz^2 + R_2(y, z).$$

Here R_1 and R_2 have terms of order 2 or more; R_1 contains no y^2 term and R_2 contains no z^2 term. Let $C_t: f_t = 0 \cap B$. In this case the cones

are defined by

$$D_1 = \{(y, z): |z/y| \leq \alpha\} \quad D_2 = \{(y, z): |y/z| \leq \alpha\}$$

for some small positive number α .

Part (i) follows immediately from Corollary 2 of Theorem 2.

PROOF OF ii: In the equation defining C_t substitute $z = \beta y$. This is an analytic change of coordinates on \tilde{C}_t . Then

$$\tilde{C}_t: 0 = t + \beta y^2 + y^3(a + b\beta + c\beta^2 + d\beta^3)$$

+ terms of order at least 4 in y .

$$\text{In } \tilde{C}_t, \beta = \frac{t}{y^2} \cdot h(y, \beta)$$

where h is analytic and bounded away from 0. Thus \tilde{C}_t is topologically equivalent to the annulus

$$\sqrt{|\alpha|} \leq |\sqrt{\beta}| < \frac{1}{\sqrt{|\beta|}} \text{ via } (y, z) \rightarrow \sqrt{\beta} = \frac{z}{y}$$

PROOF OF iii: The following is well-known [4]:

LEMMA 1: *Suppose M is a compact Riemannian surface and $D \subset M$ is open and connected. D is geodesically convex if a geodesic exists through every boundary point of D such that all the points of D in a neighborhood of the boundary point lie on one side of the geodesic.*

Hence it suffices to prove that the geodesic curvature on the boundaries of \tilde{C}_t has the correct constant sign.

LEMMA 2: *Let E_t denote the boundary curve of \tilde{C}_t defined by*

$$(6) \quad E_t: f(y, \alpha e^{i\theta} y) + t = 0 \quad 0 \leq \theta \leq 2\pi.$$

For t sufficiently small the sign of the geodesic curvature k_g on E_t is constant and is given by the sign of $(\alpha^2/1 + \alpha^2) - \frac{1}{2}$.

PROOF: We may take y to be a local coordinate in a neighborhood

of E_t . Then we may view E_t as the image of a real analytic function

$$\theta \rightarrow y(\theta) \quad 0 \leq \theta \leq 2\pi.$$

Let $h(y)dyd\bar{y}$ give the induced metric. Then with respect to the arc length parameter s

$$K_g = \text{Im} \frac{h_y dy}{h ds} + h \frac{d^2y d\bar{y}}{ds^2 ds}$$

This follows by rewriting the classical formula [5] for k_g in terms of complex coordinates. With respect to the parameter

$$k_g = \frac{1}{\sqrt{h} \cdot |y'|} \text{Im} \left(\frac{h_y}{h} y' + \frac{y''}{y'} \right) \text{ where } y' = \frac{dy}{d\theta}$$

We will show the sign of $\sqrt{h} \cdot |y'|k_g$ is given by the sign of $\alpha^2/1 + \alpha^2 - \frac{1}{2}$ for t sufficiently small by showing

$$(7) \quad \frac{y''}{y'} \rightarrow \frac{-i}{2} \text{ as } t \rightarrow 0$$

$$(8) \quad \frac{h_y y'}{h} \rightarrow \frac{+i\alpha^2}{1 + \alpha^2}$$

We may assume that in a neighborhood of the boundary curve, C_t is the graph of a holomorphic function $z_t(y)$. Then one can calculate using $z = \alpha e^{i\theta}y$ on E_t and writing $z_t(y) = z(y)$

$$y' = \frac{i\alpha e^{i\theta}y}{z_y - \alpha e^{i\theta}}$$

$$\frac{y''}{y'} = i + 2 \frac{y'}{y} - \frac{(y')^2 z_{yy}}{i\alpha e^{i\theta}y}$$

Now as $t \rightarrow 0$, $z_y \rightarrow -\alpha e^{i\theta}$ on E_t so $\frac{y'}{y} \rightarrow i/2$.

Also as $t \rightarrow 0$, $yz_{yy} \rightarrow 2\alpha e^{i\theta}$ hence $\frac{y''}{y'} \rightarrow i/2$.

To evaluate (8) recall that

$$h = \frac{1}{(1 + |y|^2 + |z|^2)^2} (1 + |z_y|^2 + |z - yz_y|^2).$$

Hence $h(y, \alpha e^{i\theta}y) \rightarrow 1 + |\alpha|^2$ as $t \rightarrow 0$. By calculating h_y , one can show $h_y y' \rightarrow -i\alpha^2$ as $t \rightarrow 0$. Q.E.D.

Finally we must check the meaning of the sign of k_g on E_t . Project E_t onto the complex line $z = 0$; the tangent and geodesic curvature vectors of the projected curve $\pi(E_t)$ are just the projections of the tangent and curvature vectors of E_t . For t sufficiently small we may assume

$$\pi(E_t): 0 = t + \alpha e^{i\theta} y^2 \text{ and } y' = \frac{-iy}{2}$$

Hence $\pi(E_t)$ is swept out in the clockwise direction. By the lemma k_g is negative on E_t . Thus the curvature vector points toward the origin.

The other bounding curve is the image of E_t under a unitary transformation which preserves geodesic curvature. Thus the signs are compatible.

PROOF OF iv: We will prove the following more general

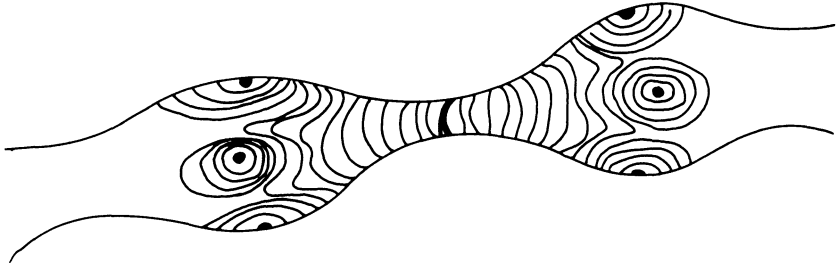
PROPOSITION: *Consider a geodesically convex subdomain M of a Riemannian surface such that $\pi_1(M) \cong \mathbb{Z}$ and the Gaussian curvature is bounded away from zero by a negative constant. Then there exists a unique smooth closed simple geodesic Γ and every other smooth closed geodesic in M consists in going around Γ more than once.*

PROOF: It is known that if N is a compact Riemannian manifold, then every free homotopy class of loops has a minimal length member which is a smooth closed geodesic. The proof of this result as given in [1] is valid when the hypothesis of compactness is replaced by geodesic convexity. To prove uniqueness, suppose there were two distinct smooth closed geodesics l_1 and l_2 . If l_1 and l_2 intersect, when lifted to the simply connected covering space they bound a lune L with $\int K dA < 0$ which contradicts Gauss-Bonnet. If l_1 and l_2 do not intersect, there exists a geodesic $\gamma \subset M$ which realizes the minimum distance between l_1 and l_2 . When l_1 , l_2 , and γ are lifted to the simply connected covering space, we obtain a geodesic rectangle R where $\int_R K dA < 0$ which again contradicts Gauss-Bonnet.

REMARK: For $t \neq 0$, $H_1(\tilde{C}_t, \mathbb{Z}) = \mathbb{Z}$ and any element of $H_1(\tilde{C}_t, \mathbb{Z})$, $t \neq 0$ is called a vanishing cycle [2]. Theorem 3 shows that there exist unique smooth geodesic representatives of the vanishing cycles.

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I claim that one can draw the following picture of the curves of constant curvature on $B \cap C_t$, for sufficiently small t , in the case that neither branch of C_0 has a flex at the singular point.



In this case for sufficiently small t , $C_t \cap B \cap C_1$ and $C_t \cap B \cap C_2$ each contain exactly three flexes. The dots denote the six flexes. The nipples denote the areas of positive curvature.

The other assumptions made on drawing this picture are $\exists c_0$ depending on t and the coefficients of the third degree terms of the defining equation such that: (1) for $c_0 < c < 2$ the set $K^{-1}(C)$ has six components each of which is a simple closed curve. (2) If M denotes $\min K$ and then $K^{-1}(M)$ is a simple closed curve in $B \cap C_t$ and (3) if $M < c < c_0$, $K^{-1}(c)$ consists of two disjoint simple closed curves, one on each side of $k \equiv M$.

We outline the analytic proof of the picture in the case that the singular point $P = (0, 0)$ and the tangent lines are the axes. Hence we are assuming as in Section 1 that

$$(3) \quad f(y, z) = yz + ay^3 + by^2z + cyz^2 + dz^3 + \text{higher order terms.}$$

We will denote by $Q(\eta)$ the polycylinder

$$Q(\eta) = \{(y, z): |y| < \eta \text{ and } |z| < \eta\}.$$

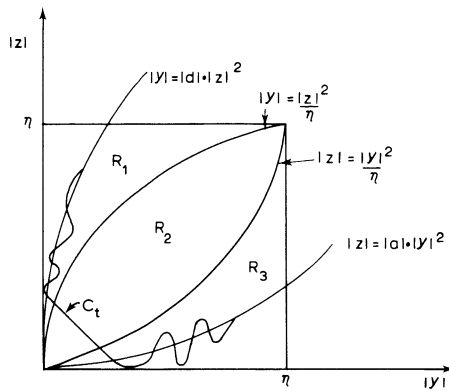
Fix η so small that $Q(\eta) \subset B$ and hence $C_0 \cap Q(\eta)$ is the union of two analytic branches defined by (4) and (5). We consider the following subregions of $Q(\eta)$

$$R_1 = Q(\eta) \cap \left\{ (y, z) : \frac{|y|}{|z|^2} < \frac{1}{\eta} \right\}$$

$$R_2 = Q(\eta) \cap \left\{ (y, z) : \frac{|y|}{|z|^2} > \frac{1}{\eta} \text{ and } \frac{|z|}{|y|^2} > \frac{1}{\eta} \right\}$$

$$R_3 = Q(\eta) \cap \left\{ (y, z) : \frac{|z|}{|y|^2} < \frac{1}{\eta} \right\}$$

Hence we have the following diagram of the projection of $Q(\eta)$ into the $|y|, |z|$ plane.



We now introduce a change of coordinate on R_1 and R_3 . Let $t = \tau^3$.

$$\text{On } R_1 \text{ let } y' = \frac{y}{\tau^2} \quad z' = \frac{z}{\tau}$$

$$\text{On } R_3 \text{ let } y' = \frac{y}{\tau} \quad z' = \frac{z}{\tau^2}$$

R_1 and R_3 are each invariant under these coordinate changes which just expand the regions along the parabolas $y = \gamma z^2$ and $z = \gamma y^2$ respectively, $|\gamma| < 1/\eta$.

LEMMA: With respect to the (y', z') coordinate system in R_1 as $t \rightarrow 0$

- (i) $(C_t \cap R_1)$ the curve $0 = 1 + z'y' + d(z')^3$
- (ii) $K \rightarrow 2 - 4|y'/z'^2|^2$ on $C_t \cap R_1$
- (iii) If $d \neq 0$ the three flexes on $C_t \cap R_1$ approach

$$(y', z') = (0, \sqrt[3]{-1/d})$$

(iv) *The outer boundary of $C_t \cap R_1$ given by*

$$C_t \cap (|y| = \eta) \rightarrow \infty.$$

The convergence in (i) and (ii) is uniform on compact subsets in the (y', z') space. The analogous results hold for $C_t \cap R_3$.

PROOF: Straightforward computation.

By the previous lemma, then, we can obtain a picture of the curves of constant curvature

$$k = 2 - 4\gamma^2 \quad 0 < \gamma < \frac{1}{\eta}$$

on $C_t \cap R_1$ for t very small by considering the curve

$$C: 0 = 1 + yz + dz^3$$

and the subsets $P_\gamma \subset C$ where

$$P_\gamma = \left\{ (y, z) \in C: \left| \frac{y}{z^2} \right| = \gamma \quad 0 < \gamma < \frac{1}{\eta} \right.$$

Note that if t is small enough the image of C_t in the $|y|, |z|$ plane lies in an arbitrarily small neighborhood of the image of C_0 , which if η is small enough lies in an arbitrarily small neighborhood of $(|y| = |d| |z|^2 \cup |z| = |a| |y|^2)$.

LEMMA: *On $R_2 \cap C_t$, for t sufficiently small*

$$K < \left(2 - \frac{2}{\eta^2} \right) + \delta(t)$$

where $0 < \delta(t) < 1/\eta^2$ and $\delta(t) \rightarrow 0$ as $t \rightarrow 0$.

PROOF: Straightforward computation.

LEMMA: *On $C_t \cap R_1$*

(i) *For $\gamma > |d|$, P_γ is a simple closed curve.*

If $d \neq 0$

- (ii) When $\gamma = |d|$, P_γ is a simple closed curve minus one point at ∞ .
- (iii) For $\gamma < |d|$, P_γ consists of three disjoint simple closed curves
- (iv) $\lim_{\gamma \rightarrow 0} P_\gamma = (0, \sqrt[3]{-1/d})$ which are the flexes.

PROOF: Obvious.

An analogous lemma holds on $C_t \cap R_3$.

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(Oblatum 1-XII-1975 & 6-IX-1976)

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