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ON THE SIMULTANEOUS APPROXIMATION OF a, b AND a^b

A. Bijlsma

1. Introduction

Let a and b be complex numbers with $a \neq 0$, $a \neq 1$ and $b \notin \mathbb{Q}$. Let a^b denote $\exp(b \log a)$ for some fixed branch of the logarithm. The problem is to determine whether it is possible that all three numbers a , b and a^b can be well approximated by algebraic numbers of bounded degree. We take d fixed and consider triples (α, β, γ) of algebraic numbers of degree at most d ; H will denote the maximum of the heights of α , β and γ .

After initial results of Ricci [8] and Franklin [5], Schneider [9] stated that for any $\epsilon > 0$, there exist only finitely many triples (α, β, γ) with

$$\max(|a - \alpha|, |b - \beta|, |a^b - \gamma|) < \exp(-\log^{5+\epsilon} H).$$

Bundschuh [2] remarked that in Schneider's proof a condition like $\beta \notin \mathbb{Q}$ is needed and tried to prove a theorem without such a restriction. His assertion is that for any $\epsilon > 0$, there are only finitely many triples (α, β, γ) with

$$\max(|a - \alpha|, |b - \beta|, |a^b - \gamma|) < \exp(-\log^4 H \log_2^{-2+\epsilon} H),$$

where $\log_2$ means $\log \log$. However, there is an error at the beginning of the proof of his Satz 2a, so that his result, too, is only valid if one makes some extra assumption.

Earlier Šmelev [10] had proved that only finitely many triples (α, β, γ) with $\beta \notin \mathbb{Q}$ have the property

$$\max(|a - \alpha|, |b - \beta|, |a^b - \gamma|) < \exp(-\log^4 H \log_2^3 H).$$

Cijsouw and Waldschmidt [4] recently improved upon the above results by showing that for any $\epsilon > 0$, there are only finitely many triples (α, β, γ) with $\beta \notin \mathbb{Q}$ and

$$\max(|a - \alpha|, |b - \beta|, |a^b - \gamma|) < \exp(-\log^3 H \log_2^{1+\epsilon} H).$$

The main purpose of this paper is to show that from all these theorems, the condition $\beta \notin \mathbb{Q}$ cannot be omitted. More precisely, the following will be proved:

THEOREM 1: *For any fixed natural number κ , there exist irrational numbers $a, b \in (0, 1)$ such that for infinitely many triples (α, β, γ) of rational numbers*

$$\max(|a - \alpha|, |b - \beta|, |a^b - \gamma|) < \exp(-\log^\kappa H),$$

where H denotes the maximum of the heights of α, β and γ .

In the theorems cited above, it would seem more natural to place a restriction upon the given number b than upon β . For instance, it is quite easy to see that the estimate of Cijsouw and Waldschmidt holds for arbitrary triples (α, β, γ) if one assumes that, for real b , the convergents p_n/q_n of the continued fraction expansion of b satisfy

$$q_{n+1} \ll \exp(\log^3 q_n), \quad n \rightarrow \infty.$$

(Note that the real numbers b for which this condition is not fulfilled, are U^* -numbers (see [9], III §3) and thus form a set of Lebesgue measure zero.) A sharper result in the same direction is given by the next theorem.

THEOREM 2: *Suppose $\epsilon > 0$, $d \in \mathbb{N}$, $a, b \in \mathbb{C}$, $a \neq 0$, $b \notin \mathbb{Q}$, l a branch of the logarithm with $l(a) \neq 0$. If $b \notin \mathbb{R}$, or if $b \in \mathbb{R}$ such that the convergents p_n/q_n of the continued fraction expansion of b satisfy*

$$q_{n+1} \ll \exp(q_n^3), \quad n \rightarrow \infty,$$

there are only finitely many triples (α, β, γ) of algebraic numbers of degree at most d with

$$\max(|a - \alpha|, |b - \beta|, |a^b - \gamma|) < \exp(-\log^3 H \log_2^{1+\epsilon} H),$$

where H denotes the maximum of the heights of α , β and γ and $a^b = \exp(bl(a))$.

In the proofs of Theorems 1 and 2, the following notations will be employed: if α is an algebraic number, $|\overline{\alpha}|$ denotes the maximum of the absolute values of the conjugates of α , $h(\alpha)$ the height of α , $\deg(\alpha)$ the degree of α and $\text{den}(\alpha)$ the denominator of α . We shall make use of the relations $\text{den}(\alpha) \leq h(\alpha)$ and $|\overline{\alpha}| \leq h(\alpha) + 1$.

2. Two sequences of rational numbers

LEMMA 1: Let $\kappa, \lambda \in \mathbb{N}$ be given. There is a sequence $(\beta_n)_{n=1}^\infty$ of rational numbers in $(0, 1)$ such that for all $n \in \mathbb{N}$ the following inequalities hold:

- (1) $|\beta_n - \beta_{n+1}| < \exp(-h^{\kappa+2}(\beta_n)),$
- (2) $h(\beta_{n+1}) > \exp(h^{\kappa+2}(\beta_n)),$
- (3) $h(\beta_n) > \lambda.$

PROOF: Let β_1 be any rational number in $(0, 1)$ with $h(\beta_1) > \lambda$. The sequence $(\beta_n)_{n=1}^\infty$ will be defined inductively; suppose β_n already chosen. Clearly there are infinitely many rational numbers $\beta^* \in (0, 1)$ with the property

$$|\beta_n - \beta^*| \leq \frac{1}{h(\beta^*)}.$$

Only finitely many rational numbers have heights bounded by $\exp(h^{\kappa+2}(\beta_n))$, so there exists a β_{n+1} with both $h(\beta_{n+1}) > \exp(h^{\kappa+2}(\beta_n))$ and

$$|\beta_n - \beta_{n+1}| \leq \frac{1}{h(\beta_{n+1})} < \exp(-h^{\kappa+2}(\beta_n)). \quad \square$$

LEMMA 2: Let $\kappa, \lambda \in \mathbb{N}$ be given and let $(\beta_n)_{n=1}^\infty$ be a sequence of rational numbers in $(0, 1)$ such that (1), (2) and (3) are satisfied for all $n \in \mathbb{N}$. Put $\beta_n = v_n/w_n$, where $v_n, w_n \in \mathbb{N}$, $(v_n, w_n) = 1$. If λ is sufficiently large, there is a sequence $(\alpha_n)_{n=1}^\infty$ of rational numbers in $(0, 1)$, such that for all $n \in \mathbb{N}$ the following assertions hold:

$$(4) \quad |\alpha_n - \alpha_{n+1}| < \exp(-\log^{\kappa+1} h(\alpha_n)),$$

$$(5) \quad h(\alpha_{n+1}) > h^2(\alpha_n),$$

$$(6) \quad h(\alpha_n) < (2w_n)^{2w_n},$$

$$(7) \quad \alpha_n^{1/w_n} \in \mathbb{Q}.$$

PROOF: Choose $\alpha_1 := 2^{-w_1}$. The sequence $(\alpha_n)_{n=1}^\infty$ will be defined inductively; suppose $\alpha_1, \dots, \alpha_n$ have already been chosen and possess the desired properties. By Bertrand's Postulate ([6], Theorem 418), there is a prime number u_{n+1} with

$$w_{n+1}^2 \leq u_{n+1} \leq 2w_{n+1}^2.$$

Notice that, if λ is sufficiently large,

$$(8) \quad \frac{u_{n+1}}{w_{n+1}} \geq w_{n+1} \geq \exp(w_n^{\kappa+2}) > \exp((2w_n)^{\kappa+1} \log^{\kappa+1}(2w_n)) \\ > \exp(\log^{\kappa+1} h(\alpha_n)).$$

Consider the partition

$$D = \left(0, \frac{1}{u_{n+1}^{w_{n+1}}}, \frac{2^{w_{n+1}}}{u_{n+1}^{w_{n+1}}}, \dots, \frac{(u_{n+1}-1)^{w_{n+1}}}{u_{n+1}^{w_{n+1}}}, 1\right)$$

of the interval $(0, 1)$. Take $t \in \{0, \dots, u_{n+1}-1\}$. Then

$$\frac{(t+1)^{w_{n+1}}}{u_{n+1}^{w_{n+1}}} - \frac{t^{w_{n+1}}}{u_{n+1}^{w_{n+1}}} \leq \frac{w_{n+1}(t+1)^{w_{n+1}-1}}{u_{n+1}^{w_{n+1}}} \leq \frac{w_{n+1}}{u_{n+1}},$$

therefore the width of the partition D does not exceed w_{n+1}/u_{n+1} . By (8) the interval $\{x \in (0, 1): |\alpha_n - x| < \exp(-\log^{\kappa+1} h(\alpha_n))\}$ has a length greater than w_{n+1}/u_{n+1} , so that this interval contains at least one of the points of D . This proves the existence of a $t_{n+1} \in \{1, \dots, u_{n+1}-1\}$ with

$$\left| \alpha_n - \frac{t_{n+1}^{w_{n+1}}}{u_{n+1}^{w_{n+1}}} \right| < \exp(-\log^{\kappa+1} h(\alpha_n)).$$

If one defines $\alpha_{n+1} := (t_{n+1}/u_{n+1})^{w_{n+1}}$, (4) is satisfied, and furthermore $\alpha_{n+1}^{1/w_{n+1}} \in \mathbb{Q}$. Finally,

$$h(\alpha_{n+1}) = u_{n+1}^{w_{n+1}} \leq (2w_{n+1}^2)^{w_{n+1}} < (2w_{n+1})^{2w_{n+1}},$$

and

$$\begin{aligned} h(\alpha_{n+1}) &= u_{n+1}^{w_{n+1}} \geq (w_{n+1}^2)^{w_{n+1}} = \exp(2w_{n+1} \log w_{n+1}) \geq \exp(w_{n+1}) \\ &> \exp(\exp(w_n^{\kappa+2})) > \exp(4w_n \log(2w_n)) > h^2(\alpha_n). \quad \square \end{aligned}$$

3. Proof of Theorem 1

I. Take $\lambda \in \mathbb{N}$ and let $(\beta_n)_{n=1}^\infty$ be a sequence of rational numbers in $(0, 1)$ such that (1)–(3) are satisfied. Put $\beta_n = v_n/w_n$, where $v_n, w_n \in \mathbb{N}$, $(v_n, w_n) = 1$. We may suppose that λ is sufficiently large in the sense of Lemma 2; let $(\alpha_n)_{n=1}^\infty$ be a sequence of rational numbers in $(0, 1)$ such that (4)–(7) are satisfied. Put $\alpha_n = (t_n/u_n)^{w_n}$, where $t_n, u_n \in \mathbb{N}$, $(t_n, u_n) = 1$. Define $\gamma_n := \alpha_n^{\beta_n}$; we have

$$\gamma_n = \alpha_n^{\beta_n} = \left(\frac{t_n^{w_n}}{u_n^{w_n}} \right)^{v_n/w_n} = \frac{t_n^{v_n}}{u_n^{w_n}},$$

so $\gamma_n \in \mathbb{Q}$ and $h(\gamma_n) = u_n^{v_n} < u_n^{w_n} = h(\alpha_n)$. Therefore

$$H_n = \max(h(\alpha_n), h(\beta_n), h(\gamma_n)) = h(\alpha_n).$$

II. The sequence $(\alpha_n)_{n=1}^\infty$ has the property

$$(9) \quad \forall m > n: |\alpha_m - \alpha_n| < \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_n)).$$

For put $I_k := \{x \in \mathbb{R}: |\alpha_k - x| < \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_k))\}$. Then (9) can be written as $\forall m > n: \alpha_m \in I_n$. By (4), $\alpha_m \in I_{m-1}$, so it is sufficient to prove $\forall k > n: I_k \subset I_{k-1}$. Take $x \in I_k$, which means $|\alpha_k - x| < \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_k))$. Then, by (4) and (5),

$$\begin{aligned} |\alpha_{k-1} - x| &\leq |\alpha_k - x| + |\alpha_{k-1} - \alpha_k| \\ &< \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_k)) + \exp(-\log^{\kappa+1} h(\alpha_{k-1})) \\ &< \exp(-2^\kappa \log^{\kappa+1} h(\alpha_{k-1})) + \exp(-\log^{\kappa+1} h(\alpha_{k-1})) \\ &< 2 \exp(-\log^{\kappa+1} h(\alpha_{k-1})) < \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_{k-1})) \end{aligned}$$

if w_1 is sufficiently large, so $x \in I_{k-1}$.

III. From (9) we see that $(\alpha_n)_{n=1}^\infty$ is a Cauchy sequence; it converges to a limit, which we shall call a . Then

$$\forall n: |a - \alpha_n| \leq \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_n)) = \exp(-\tfrac{1}{2} \log^{\kappa+1} H_n),$$

which, by Theorem 186 of [6], implies that a is irrational.

In the same way one can prove that $(\beta_n)_{n=1}^{\infty}$ is a Cauchy sequence and that its limit b satisfies

$$\begin{aligned} \forall n: |b - \beta_n| &\leq \exp(-\tfrac{1}{2}h^{\kappa+2}(\beta_n)) = \exp(-\tfrac{1}{2}w_n^{\kappa+2}) \\ &\leq \exp(-\tfrac{1}{2}(2w_n)^{\kappa+1} \log^{\kappa+1}(2w_n)) \leq \exp(-\tfrac{1}{2} \log^{\kappa+1} h(\alpha_n)) \\ &= \exp(-\tfrac{1}{2} \log^{\kappa+1} H_n) \end{aligned}$$

so that b , too, is irrational.

IV. The function x^y is continuously differentiable on every compact subset K of $(0, 1) \times (0, 1)$, so that a constant C_K , only depending on K , can be found with

$$|x^y - \xi^\eta| < C_K \max(|x - \xi|, |y - \eta|) \text{ for } (x, y), (\xi, \eta) \in K.$$

From this follows the existence of a constant $C_{a,b}$, depending only on a and b , such that

$$\begin{aligned} |a^b - \alpha_n^{\beta_n}| &< C_{a,b} \max(|a - \alpha_n|, |b - \beta_n|) \\ &\leq C_{a,b} \exp(-\tfrac{1}{2} \log^{\kappa+1} H_n) < \exp(-\log^{\kappa} H_n), \end{aligned}$$

for sufficiently large n . $\square$

It may be of interest to note that a p -adic analogue of Theorem 1 can be proved with considerably less difficulty. Indeed, it suffices to construct a sequence $(\beta_n)_{n=1}^{\infty}$ of natural numbers with the properties $|\beta_n - \beta_{n+1}|_p < \exp(-\beta_n^{\kappa+1})$ and $|\beta_n|_p = p^{-2}$. If b is the p -adic limit of this sequence and $a = b + 1$, infinitely many triples (α, β, γ) of natural numbers satisfy

$$\max(|a - \alpha|_p, |b - \beta|_p, |a^b - \gamma|_p) < \exp(-\log^{\kappa} H),$$

where $H = \max(\alpha, \beta, \gamma)$ and a^b is defined by means of the p -adic logarithm and exponential function.

4. A result on vanishing linear forms

LEMMA 3: Suppose $d \in \mathbb{N}$, K a compact subset of the complex plane not containing 0, l_1 and l_2 branches of the logarithm, defined on K , such that l_1 does not take the value 0. Then only finitely many pairs $(\alpha, \gamma) \in K \times K$ of algebraic numbers of degree at most d have the

property that a $\beta \in \mathbb{Q}$ exists with

$$\beta l_1(\alpha) - l_2(\gamma) = 0$$

and

$$h(\beta) \geq \log H,$$

where $H = \max(h(\alpha), h(\gamma))$.

PROOF: I. Suppose $\alpha, \gamma \in K, \beta \in \mathbb{Q}$, such that the conditions of the lemma are fulfilled. By $c_1, c_2, \dots$ we shall denote natural numbers depending only on d, K, l_1 and l_2 ; we suppose that H is greater than such a number, which will lead to a contradiction.

Put $B := h(\beta)$; then

$$(10) \quad \log H \leq B.$$

Define $L := [2dB \log^{-1/3} B] - 1$. We introduce the auxiliary function

$$\Phi(z) = \sum_{\lambda_1=0}^L \sum_{\lambda_2=0}^L p(\lambda_1, \lambda_2) \alpha^{\lambda_1 z} \gamma^{\lambda_2 z}, \quad z \in \mathbb{C},$$

where $\alpha^{\lambda_1 z} = \exp(\lambda_1 z l_1(\alpha))$, $\gamma^{\lambda_2 z} = \exp(\lambda_2 z l_2(\gamma))$ and where $p(\lambda_1, \lambda_2)$ are rational integers to be determined later. We have

$$\begin{aligned} \Phi^{(t)}(z) &= l_1^t(\alpha) \sum_{\lambda_1=0}^L \sum_{\lambda_2=0}^L p(\lambda_1, \lambda_2) (\lambda_1 + \lambda_2 \beta)^t \alpha^{\lambda_1 z} \gamma^{\lambda_2 z}, \\ z &\in \mathbb{C}, t \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Now put $a := \text{den}(\alpha)$, $b := \text{den}(\beta)$, $c := \text{den}(\gamma)$, $S := [\log^{1/3} B]$, $T := [B^2 \log^{-1} B]$ and consider the system of linear equations

$$(ac)^{sL} b^t l_1^{-t}(\alpha) \Phi^{(t)}(s) = 0, \quad s = 0, \dots, S-1, t = 0, \dots, T-1.$$

These are ST equations in the $(L+1)^2$ unknowns $p(\lambda_1, \lambda_2)$; the coefficients are algebraic integers in the number field $\mathbb{Q}(\alpha, \gamma)$ of degree at most d^2 . The absolute values of the conjugates of the coefficients are less than or equal to

$$\begin{aligned} (ac)^{LS} b^T \max(1, |\lambda_1 + \lambda_2 \beta|^T) \max(1, |\alpha|^{LS}) \max(1, |\gamma|^{LS}) \\ \leq H^{4LS} B^T c_1^{LS+T} L^T \leq \exp(c_2 B^2) \end{aligned}$$

(here (10) is used).

As $(L+1)^2 \geq d^2 B^2 \log^{-2/3} B \geq d^2 ST$, Lemme 1.3.1 of [11] states that there is a non-trivial choice for the $p(\lambda_1, \lambda_2)$, such that

$$(11) \quad \Phi^{(t)}(s) = 0, s = 0, \dots, S-1, t = 0, \dots, T-1,$$

while

$$P := \max_{\substack{\lambda_1=0, \dots, L \\ \lambda_2=0, \dots, L}} |p(\lambda_1, \lambda_2)| \leq (c_3 L^2 \exp(c_2 B^2))^{d^2 ST / ((L+1)^2 - d^2 ST)} \leq \exp(c_4 B^2).$$

II. For $k \in \mathbb{N} \cup \{0\}$ we put $T_k := 2^k T$; suppose $2^k \leq \log^{1/6} B$. Then, for our special choice of the $p(\lambda_1, \lambda_2)$, we have

$$(12) \quad \Phi^{(t)}(s) = 0, s = 0, \dots, S-1, t = 0, \dots, T_k - 1.$$

This is proved by induction; for $k = 0$ the assertion is precisely (11). Now suppose that (12) holds for some k , while $2^{k+1} \leq \log^{1/6} B$. By Lemma 7 of [3] we have

$$(13) \quad \max_{|z| \leq 2S} |\Phi(z)| \leq 2 \max_{|z| \leq 2BS} |\Phi(z)| \left(\frac{2}{B} \right)^{ST_k}.$$

Here

$$(14) \quad \max_{|z| \leq 2BS} |\Phi(z)| \leq (L+1)^2 P c_5^{2BLS} \leq \exp(c_6 B^2)$$

and

$$\left(\frac{2}{B} \right)^{ST_k} \leq \exp \left(-\frac{1}{c_7} B^2 \log^{1/3} B \right).$$

Substitution in (13) gives

$$\max_{|z| \leq 2S} |\Phi(z)| \leq \exp \left(-\frac{1}{c_8} B^2 \log^{1/3} B \right).$$

For $s = 0, \dots, S-1, t = 0, \dots, T_{k+1} - 1$ we have

$$\Phi^{(t)}(s) = \frac{t!}{2\pi i} \int_{|z-s|=1} \frac{\Phi(z)}{(z-s)^{t+1}} dz,$$

so

$$\begin{aligned} |\Phi^{(t)}(s)| &\leq \exp \left(t \log t - \frac{1}{c_8} B^2 \log^{1/3} B \right) \\ &\leq \exp \left(c_9 B^2 \log^{1/6} B - \frac{1}{c_8} B^2 \log^{1/3} B \right), \end{aligned}$$

from which we conclude

$$(15) |\Phi^{(t)}(s)| \leq \exp \left(-\frac{1}{c_{10}} B^2 \log^{1/3} B \right), s = 0, \dots, S-1, t = 0, \dots, T_{k+1}-1.$$

However, $l_1^{-t}(\alpha)\Phi^{(t)}(s)$ is algebraic and formula (1.2.3) of [11] states that every non-zero algebraic number ξ has the property

$$(16) \quad |\xi| \geq \exp(-2 \operatorname{dg}(\xi) \max(\log |\bar{\xi}|, \log \operatorname{den}(\xi))).$$

Now for $s = 0, \dots, S-1, t = 0, \dots, T_{k+1}-1$ we have

$$\begin{aligned} \operatorname{dg}(l_1^{-t}(\alpha)\Phi^{(t)}(s)) &\leq d^2, \\ \operatorname{den}(l_1^{-t}(\alpha)\Phi^{(t)}(s)) &\leq (ac)^{LS} b^{T_{k+1}} \leq H^{2LS} B^{T_{k+1}} \leq \exp(c_{11} B^2 \log^{1/6} B), \\ |\overline{l_1^{-t}(\alpha)\Phi^{(t)}(s)}| &\leq P c_{12}^{LS+T_{k+1}} L^{T_{k+1}+2} H^{2LS} \leq \exp(c_{13} B^2 \log^{1/6} B), \end{aligned}$$

so either $\Phi^{(t)}(s) = 0$ or

$$|l_1^{-t}(\alpha)\Phi^{(t)}(s)| \geq \exp(-c_{14} B^2 \log^{1/6} B);$$

in the latter case

$$(17) \quad |\Phi^{(t)}(s)| \geq \exp(-c_{15} B^2 \log^{1/6} B).$$

Combining (15) and (17) gives $\Phi^{(t)}(s) = 0$ for $s = 0, \dots, S-1, t = 0, \dots, T_{k+1}-1$. This completes the proof of (12).

III. Now let k be the *largest* natural number with $2^k \leq \log^{1/6} B$. From (12) it follows that

$$\Phi^{(t)}(s) = 0, s = 0, \dots, S-1, t = 0, \dots, T_k - 1.$$

Once more apply Lemma 7 of [3]; this gives (13) again and (14) also remains unchanged, but from the maximality of k we now get

$$\left(\frac{2}{B}\right)^{ST_k} \leq \exp\left(-\frac{1}{c_{16}} B^2 \log^{1/2} B\right),$$

so

$$\max_{|z| \leq 2S} |\Phi(z)| \leq \exp \left(-\frac{1}{c_{17}} B^2 \log^{1/2} B \right).$$

For $t = 0, 1, \dots, (L+1)^2 - 1$ we have

$$\Phi^{(t)}(0) = \frac{t!}{2\pi i} \int_{|z|=1} \frac{\Phi(z)}{z^{t+1}} dz,$$

whence

$$\begin{aligned} |\Phi^{(t)}(0)| &\leq \exp \left(t \log t - \frac{1}{c_{17}} B^2 \log^{1/2} B \right) \\ &\leq \exp \left(c_{18} B^2 \log^{1/3} B - \frac{1}{c_{17}} B^2 \log^{1/2} B \right). \end{aligned}$$

Conclusion:

$$(18) \quad |\Phi^{(t)}(0)| \leq \exp \left(-\frac{1}{c_{19}} B^2 \log^{1/2} B \right), t = 0, \dots, (L+1)^2 - 1.$$

For these values of t we have

$$\begin{aligned} \text{dg}(l_1^{-t}(\alpha)\Phi^{(t)}(0)) &\leq d^2, \\ \text{den}(l_1^{-t}(\alpha)\Phi^{(t)}(0)) &\leq B^{(L+1)^2} \leq \exp(c_{20} B^2 \log^{1/3} B), \\ \overline{|l_1^{-t}(\alpha)\Phi^{(t)}(0)|} &\leq P(c_{21} L)^{c_{22}(L+1)^2} \leq \exp(c_{23} B^2 \log^{1/3} B), \end{aligned}$$

so according to (16) either $\Phi^{(t)}(0) = 0$ or

$$|l_1^{-t}(\alpha)\Phi^{(t)}(0)| \geq \exp(-c_{24} B^2 \log^{1/3} B);$$

in the latter case

$$(19) \quad |\Phi^{(t)}(0)| \geq \exp(-c_{25} B^2 \log^{1/3} B).$$

Combining (18) and (19) gives

$$\Phi^{(t)}(0) = 0, t = 0, \dots, (L+1)^2 - 1.$$

IV. For $t = 0, \dots, (L+1)^2 - 1$ we now have

$$(20) \quad \sum_{\lambda_1=0}^L \sum_{\lambda_2=0}^L p(\lambda_1, \lambda_2)(\lambda_1 + \lambda_2 \beta)^t = 0.$$

As the $p(\lambda_1, \lambda_2)$ are not all zero, it follows that the coefficient matrix of the system (20), which is of the Vandermonde type, must be singular. From this we deduce the existence of $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \{0, \dots, L\}$ with $\lambda_1 + \lambda_2\beta = \lambda'_1 + \lambda'_2\beta$, or

$$\beta = \frac{\lambda'_1 - \lambda_1}{\lambda_2 - \lambda'_2}.$$

This gives

$$B = h(\beta) \leq L = [2dB \log^{-1/3} B] - 1,$$

so we get a contradiction for sufficiently large H (and B). $\square$

5. Proof of Theorem 2

I. The case $b \notin \mathbb{R}$ is trivial; we shall therefore suppose that b is real and that its continued fraction expansion has the property described in the theorem. Let (α, β, γ) be a triple fulfilling the conditions of the theorem; we suppose H to be greater than a certain bound depending only on ϵ, d, a, b and l . This will lead to a contradiction.

As $a \neq 0$ and $a^b \neq 0$, we may assume $\alpha \neq 0$ and $\gamma \neq 0$. For suitably chosen branches l_1 and l_2 of the logarithm we have

$$(21) \quad |l(a) - l_1(\alpha)| < \exp(-\log^3 H \log_2^{1+2\epsilon/3} H),$$

$$(22) \quad |bl(a) - l_2(\gamma)| < \exp(-\log^3 H \log_2^{1+2\epsilon/3} H);$$

from $l(a) \neq 0$ we thus get $l_1(\alpha) \neq 0$. As a consequence of (21), (22) and

$$|b - \beta| < \exp(-\log^3 H \log_2^{1+\epsilon} H)$$

we have

$$|\beta l_1(\alpha) - l_2(\gamma)| < \exp(-\log^3 H \log_2^{1+\epsilon/3} H).$$

If it were the case that $\beta l_1(\alpha) - l_2(\gamma) \neq 0$, Theorem 1 of [4] would imply

$$|\beta l_1(\alpha) - l_2(\gamma)| > \exp(-\log^3 H \log_2^{1+\epsilon/3} H),$$

which is a contradiction. Therefore $\beta l_1(\alpha) - l_2(\gamma) = 0$.

II. We have just proved that $l_1(\alpha)$ and $l_2(\gamma)$ are linearly dependent

over the field of all algebraic numbers; using Theorem 1 of [1] we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi, \eta \in \mathbb{Q}$, not both zero, such that $\xi l_1(\alpha) + \eta l_2(\gamma) = 0$. Here $\eta \neq 0$ because $l_1(\alpha) \neq 0$, so

$$\beta = \frac{l_2(\gamma)}{l_1(\alpha)} = -\frac{\xi}{\eta} \in \mathbb{Q};$$

using Lemma 3 above we see that $h(\beta) < \log H$.

Put $q := \text{den}(\beta)$; then $q < \log H$, so

$$(23) \quad |b - \beta| < \exp(-\log^3 H \log_2^{1+\epsilon} H) < \exp(-q^3 \log^{1+\epsilon} q).$$

As q must tend to infinity with H , we may assume

$$|b - \beta| < \frac{1}{2q^2},$$

and thus, by Satz 2.11 of [7], β is a convergent of b , say $\beta = p_n/q_n$. By (12) in §13 of [7], we have, for some constant c ,

$$|b - \beta| > \frac{1}{q_n(q_n + q_{n+1})} \geq \frac{1}{q_n(q_n + c \exp(q_n^3))} > \exp(-q^3 \log^{1+\epsilon} q),$$

which contradicts (23). $\square$

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