

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 35, n° 1 (1977), p. 91-97

[http://www.numdam.org/item?id=CM\\_1977\\_\\_35\\_1\\_91\\_0](http://www.numdam.org/item?id=CM_1977__35_1_91_0)

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## A FINITENESS THEOREM FOR THE BURNSIDE RING OF A COMPACT LIE GROUP

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Let  $G$  be a compact Lie group and let  $A(G)$  be its Burnside ring [6]. We show that after inverting a finite number of primes the ring  $A(G)$  is generated by idempotent elements. The following result (Theorem 1) about compact Lie groups is basic for our investigations.

Let  $H$  be a subgroup of  $G$  (subgroups will always be closed), let  $NH$  be its normalizer in  $G$  and denote  $NH/H$  by  $WH$ . If  $K$  is a compact Lie group let  $K_0$  be its component of the unit element.

**THEOREM 1:** *There exists an integer  $b$  such that for each closed subgroups  $H$  of  $G$  the index  $|WH : (WH)_0|$  is less than  $b$ .*

Let  $A_c(G)$  be the integral closure of  $A(G)$  in its total quotient ring.

**THEOREM 2:** *There exists an integer  $n > 0$  such that  $nA_c(G) \subset A(G)$ . The minimal such  $n$  is the least common multiple of the numbers  $WH$  where  $H$  runs through all subgroups such that  $WH$  is finite.*

The minimal integer  $n(G)$  provided by Theorem 1 replaces the order of the finite group if one extends the general Artin induction theorem (see Dress [7], Theorem 2, p. 204) to compact Lie groups, whence its importance.

### 1. Normalizers

1.1. We prove in this section Theorem 1. The proof proceeds in three steps: We first reduce to the case that  $WH$  is finite; then we reduce to the case that  $H$  is finite; and finally show that for finite  $H$

with finite  $WH$  the order of  $WH$  is uniformly bounded.

Let  $H$  be a closed subgroup of  $G$  (notation:  $H < G$ ). Let  $\text{Aut}(H)$  be the automorphism group of  $H$  and  $\text{In}(H)$  the closed normal subgroup of inner automorphisms. The group  $\text{Aut } H/\text{In } H$  is discrete. Mapping  $n \in NH$  to the conjugation automorphism  $c(n): h \mapsto nhn^{-1}$  of  $H$  induces a homomorphism  $NH \rightarrow \text{Aut } H/\text{In } H$  with kernel  $ZH \cdot H$ , where  $ZH$  denotes the centralizer of  $H$ . Hence  $NH/ZH \cdot H$  being a compact subgroup of the discrete group  $\text{Aut } H/\text{In } H$  is finite. We conclude

LEMMA 1:  $WH$  is finite if and only if  $ZH/(ZH \cap H)$  is finite.

LEMMA 2: A compact Lie group contains only a finite number of conjugacy classes ( $K$ ) where  $K$  is the centralizer of a closed subgroup.

PROOF: Let  $G$  act on  $M = G$  via conjugation  $G \times M \rightarrow M: (g, m) \mapsto gmg^{-1}$ . If  $H < G$  then the fixed point set  $M^H$  is the centralizer  $ZH$ . A compact differentiable  $G$ -manifold has finite orbit type. Hence there exist finitely many conjugacy classes  $(H_1), \dots, (H_k)$  such that for any closed subgroup  $H$   $M^H = M^K$  and  $(K) = (H_i)$  for a suitable  $i$ .

LEMMA 3: For any  $H < G$  the group  $ZH \cdot H$  has finite index in its normalizer.

PROOF: We have  $Z(ZH \cdot H) < ZH < ZH \cdot H$ ; hence the assertion follows from Lemma 1.

If  $n \in G$  normalizes  $H$  then also  $ZH$  and hence  $ZH \cdot H$ . We therefore have

$$NH/ZH \cdot H < N(ZH \cdot H)/ZH \cdot H.$$

Using Lemma 3 and the existence of an upper bound for the set

$$\{|WH| \mid H < G, WH \text{ finite}\} =: F(G)$$

we obtain

LEMMA 4: There exists an integer  $c$  such that for all  $H < G$  we have

$$|NH/ZH \cdot H| < c.$$

Now we are able to obtain the first reduction of our problem. From the exact sequence  $1 \rightarrow ZH/ZH \cap H \rightarrow WH \rightarrow NH/ZH \cdot H \rightarrow 1$  we see

that  $WH/(WH)_0 \rightarrow NH/ZH \cdot H$  has a kernel which is a quotient of  $ZH/(ZH)_0$ . Lemmas 2 and 4 then show that

$$\{|WH/(WH)_0|H < G\}$$

is bounded. But note that Lemma 4 requires a bound for the set  $F(G)$ .

1.2. We show by induction over  $|G/G_0|$  and  $\dim G$  that  $F(G)$  has an upper bound  $a = a(G/G_0, \dim G)$ . For finite  $G$  we can take  $a = |G|$ .

Suppose that an upper bound  $a(K/K_0, \dim K)$  is given for all  $K$  with  $\dim K < \dim G$ . Let  $\Sigma(G) = \{H < G | WH \text{ finite}\}$ . Suppose  $H \in \Sigma(G)$  is not finite. We consider the projection  $p: NH_0 \rightarrow NH_0/H_0 =: U$ . Let  $V$  be the normalizer of  $H/H_0$  in  $U$ . Then  $WH = V/(H/H_0)$  and therefore  $H/H_0 \in \Sigma(U)$ . Since  $\dim U < \dim G$  we obtain by induction hypothesis

$$|WH| \leq a(U/U_0, \dim U).$$

We show that the possible values for  $|U/U_0|$  are finite in number. The group  $NH_0$  is the normalizer of a connected subgroup. By [8], Ch. VII, Lemma 3.2, there are only a finite number of conjugacy classes of such subgroups. Hence for a given  $G$  the possible  $|U/U_0|$  are bounded, say  $|U/U_0| \leq m(G)$ . We have inequalities

$$\begin{aligned} |U/U_0| &\leq |NH_0/N_0H_0| |N_0H_0/(NH_0)_0| \\ &\leq |G/G_0| m(G), \end{aligned}$$

where  $N_0$  means normalizer in  $G_0$ . By the classification theory of compact connected Lie groups there are only a finite number in each dimension. Hence there exists a bound for  $|U/U_0|$  depending only on  $|G/G_0|$  and  $\dim G$ . This proves the induction step as far as the non-finite  $H$  in  $\Sigma(G)$  are concerned.

1.3. Let  $H \in \Sigma(G)$  be finite. Let  $\sigma(G)$  be the set of finite subgroups of  $G$ . We use the following classical theorem of Jordan.

**LEMMA 5:** *There exists an integer  $j = j(|G/G_0|, \dim G)$  with the following properties: To each  $H \in \sigma(G)$  there exists an abelian normal subgroup  $A_H$  of  $H$  such that  $|H/A_H| \leq j$ . Moreover the  $A_H$  can be chosen such that  $H < K$  implies  $A_H < A_K$ .*

**PROOF:** Boothby and Wang [2]. Wolf [9]. In these references only connected groups are considered. The straightforward extension to non-connected groups we leave to the reader.

If  $H \in \Sigma(G)$  is finite then also  $K := NH$  is finite and by Lemma 1  $K \in \Sigma(G)$ . We choose  $j = j(|G/G_0|, \dim G)$  and  $A_H, A_K$  according to Lemma 5. We have

$$|K/H| \leq |K/A_K| \cdot |A_K/H \cap A_K| \leq j|A_K/H \cap A_K|.$$

Hence it is sufficient to find a bound for  $|A_K/H \cap A_K|$ . Consider the exact sequence  $1 \leftarrow S \leftarrow H \leftarrow A_H \leftarrow 1$ . The conjugation  $c(a)$  with  $a \in A_K$  is trivial on  $A_H$ , because  $A_K > A_H$ , and hence  $c(a)$  induces an automorphism of  $S$ . Since  $|S| \leq j$  this automorphism has order at most  $J = j!$ , i.e.  $c(a^r)$  is the identity on  $S$  and  $A_H$  for a suitable  $r \leq J$ . The group of such automorphisms modulo the subgroups of inner automorphisms by elements of  $A_H$  is isomorphic to  $H^1(S; A_H)$ , with  $S$  acting on  $A_H$  by conjugation. Since this group is annihilated by  $|S|$  we see that  $c(a^s)$  is an inner automorphism by an element of  $A_H$  for a suitable  $s \leq J|S| \leq jJ$ . In other words:  $a^s h^{-1} \in ZH$ . Hence it is sufficient to find a bound for the order of  $A_K \cap ZH/H \cap A_K \cap ZH$ .

Let  $U_1 = A_K \cap ZH$ . By [3], Théorème 1,  $U_1$  is contained in the normalizer  $NT$  of a maximal torus of  $G$ . Put  $U = U_1 \cap T$ . Then  $|U_1/U| \leq |G/G_0| |w(G_0)|$  where  $w(G_0)$  denotes the Weyl group of  $G_0$ . We estimate the order of  $U$ . Since  $U$  is abelian we have  $U < ZU$ . Moreover  $H < ZU$  by definition of  $ZH$ . Since  $U$  is contained in  $ZU$  it is contained in the center  $C := CZU$  of  $ZU$ . The inclusion  $H < ZU$  implies  $C < NH$ . Hence  $C$  is finite.

We proceed to show that for the order of a finite center  $C(G)$  of  $G$  there exists a bound depending only on  $|G/G_0|$  and  $\dim G$ . We let  $G/G_0$  act by conjugation on  $C(G_0)$ . Then  $C(G) \cap G_0$  is the fixed point set of this action. We have  $C(G_0) = A \times T_1$ , where  $A$  is a finite abelian group and  $T_1$  a torus. The group  $A$  is the center of a semisimple group and therefore, by the classification theory of these groups,  $|A|$  is bounded by a constant  $c$  depending only on  $\dim G$ . The exact cohomology sequence associated to the universal covering  $0 \rightarrow \pi_1 T_1 \rightarrow V \rightarrow T_1 \rightarrow 0$  shows, that the fixed point set of the action of  $G/G_0$  on  $T_1 = C(G_0)_0$  is isomorphic to  $H^1(G/G_0, \pi_1 T_1)$ , hence its order is bounded by a constant  $d$  depending only on  $|G/G_0|$  and the rank of  $T_1$ . Hence  $|C(G)| \leq |G/G_0|cd$ .

Finally we have to show that for the possible groups  $ZU$  the order  $|ZU/(ZU)_0|$  is bounded.

$U$  is contained in a maximal torus of  $G$ . Therefore  $ZU$  is a subgroup of maximal rank and  $(ZU)_0$  a connected subgroup of maximal rank. By [4] there exist only finitely many conjugacy classes of connected subgroups of maximal rank. We have

$$|ZU/(ZU)_0| \leq |N(ZU)_0/(ZU)_0| \leq |G/G_0| |N_0(ZU)_0/(ZU)_0|.$$

There are only finitely many possibilities for normalizers  $N_0(ZU)$  in  $G_0$  of  $(ZU)_0$ .

This finishes the proof of Theorem 1.

## 2. The integral closure of $A(G)$

Let  $\phi = \text{Spec}(A(G) \otimes Q)$  be the prime ideal spectrum of  $A(G) \otimes_Z Q =: A_0$  with Zariski topology. By [6], Theorem 4, the prime ideals of  $A_0$  correspond bijectively to kernels of ring homomorphisms  $A_0 \rightarrow Q$ . Therefore  $\phi$  is a totally disconnected compact Hausdorff space ([5], §4. Ex. 16; [1], Ch. 3. Ex. 11).

LEMMA 6: (1) For each  $a \in A_0$  the map  $\varphi_a: \phi \rightarrow Q: \varphi \mapsto \varphi(a)$  is locally constant. If  $a \in A(G)$  then  $\varphi(a) \in Z$ . (2) Let  $C(\phi, Q)$  be the ring of locally constant functions  $\phi \rightarrow Q$ . The ring homomorphism  $\alpha: A_0 \rightarrow C(\phi, Q): a \mapsto \varphi_a$  is an isomorphism. The image  $\alpha A(G)$  is contained in  $C(\phi, Z)$ . (3) The map  $A(G) \rightarrow A_0: a \mapsto a \otimes 1$  is the inclusion of  $A$  into its total quotient ring. The map  $\alpha: A(G) \rightarrow C(\phi, Z)$  is the inclusion of  $A(G)$  into the integral closure of  $A(G)$  in  $A_0 \cong C(\phi, Q)$ .

PROOF: (1) For  $k \in Q$  the set  $\varphi_a^{-1}(k)$  is closed in  $\phi$ , by definition of the Zariski topology. Since  $A_0$  is an absolutely flat ring, this set is also open by [5], §4. Ex. 16.b. Hence  $\varphi_a$  is continuous and there exist only a finite number of non-empty sets  $\varphi_a^{-1}(k)$ , because  $\phi$  is compact. A homomorphism  $A_0 \rightarrow Q$  is induced from a homomorphism  $A(G) \rightarrow Z$ , by [6] Theorem 4.

(2) Since the localizations of  $A_0$  at its prime ideals are canonically isomorphic to  $Q$ , we can identify  $C(\phi, Q)$  with the ring of sections of the structure sheaf of  $A_0$ . Then  $\alpha$  corresponds to the canonical map of  $A_0$  into this ring, hence  $\alpha$  is an isomorphism.

(3) To form the total quotient ring we have to invert the elements which are not zero divisors. Hence  $A_0$  is contained in the total quotient ring. If  $x \in C(\phi, Q) \cong A_0$  is not a zero divisor then it is a locally constant function without zeros, hence a unit. Therefore  $C(\phi, Q)$  is its own total quotient ring. Since a locally constant function  $\phi \rightarrow Z$  takes only finitely many values the ring  $C(\phi, Z)$  is generated by idempotent elements hence integral over any subring. Under the isomorphism  $\alpha$  the ring  $C(\phi, Z)$  corresponds to  $\{a \in A_0 \mid \varphi \in \phi \Rightarrow \varphi(a) \in Z\}$ . If  $x \in A_0$  is integral over  $A(G)$  then  $\varphi(x)$  is integral over  $Z$ , hence  $\varphi(x) \in Z$  and  $\alpha(x) \in C(\phi, Z)$ .

Let  $A_c(G)$  denote the pre-image of  $C(\phi, Z)$  under  $\alpha$ . We recall that  $A(G)$  is additively the free abelian group on homogeneous spaces  $G/H$  where  $H$  runs through a complete system of non-conjugate subgroups  $H$  of  $G$  with finite index in their normalizer ([6], Theorem 1). Let  $\Sigma(G)$  denote the set of conjugacy classes  $(H)$  of subgroups  $H$  of  $G$  with finite  $WH = NH/H$ .

LEMMA 7:  $A_c(G)$  is additively the free abelian group with basis  $x_{(H)} := |WH|^{-1}G/H$ ,  $(H) \in \Sigma(G)$ .

PROOF: The elements  $x_{(H)}$  are contained in  $A_c(G)$ : Suppose  $\varphi \in \phi$  is given. Then there exists  $(K) \in \Sigma(G)$  such that  $\varphi(x \otimes r) = r\chi(x^K)$  where  $\chi$  denotes the Euler characteristic and  $x^K$  the  $K$  fixed point set of any manifold representing  $x$ . But  $WH$  acts freely as a  $G$ -automorphisms group on  $G/H$ , hence also on  $G/H^K$ . Therefore  $\chi(G/H^K)$  is divisible by  $|WH|$  and we see that  $\varphi(x_{(H)}) \in Z$  for all  $\varphi$ . By definition of  $A_c(G)$  this means  $x_{(H)} \in A_c(G)$ .

The elements  $x_{(H)}$  are obviously linearly independent over  $Z$ . We have to show that any  $x \in A_c(G)$  is an integral linear combination of the  $x_{(H)}$ . In any case we have an expression  $x = \sum r_H x_{(H)}$  with rational  $r_H$ . Take  $(L) \in \Sigma(G)$  maximal with respect to inclusion such that  $r_L \neq 0$ . Then  $\sum r_H |WH|^{-1} \chi(G/H^L) \in Z$ . But  $G/H^L = \phi$  for  $(H) \neq (L)$ ,  $r_H \neq 0$ ; and  $\chi(G/L^L) = |WH|$ . Therefore  $r_L \in Z$ . We apply the same argument to  $x - r_L x_{(L)}$  and complete the proof by induction.

Lemma 7 and Theorem 1 give a proof of Theorem 2.

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(Oblatum 26-V-1976)

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