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DECOMPOSABILITY OF EVALUATION FIBRATIONS AND THE BRACE PRODUCT OPERATION OF JAMES

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1. Introduction

Evaluation at the base point in the domain of a mapping space defines a class of Hurewicz fibrations over the target, one for each (path-)component in the mapping space. A study of these fibrations, called evaluation fibrations, was initiated in [5]. In this paper we shall deal exclusively with the evaluation fibration corresponding to the component of homotopically trivial maps.

Before describing the contents of the paper we fix some notation. Throughout, X denotes a connected CW-complex. All spaces are equipped with a base point. We shall assume that X is a simple space, i.e. the action of the fundamental group in the homotopy of X is trivial. Let S^m denote the m-sphere, $m \geq 1$, and denote by $G_0(S^m, X)$ that component in the space of free maps of S^m into X, which consists of the homotopically trivial maps. Denote by $\Omega_0^m X$ the corresponding component in the space of based maps of S^m into X, which contains the constant based map. All mapping spaces are equipped with the compact-open topology. We note that $\Omega_0^m X$ is just the identity component in the H-space of m-loops on X. Evaluation at the base point in S^m defines a Hurewicz fibration $p: G_0(S^m, X) \to X$, which we call the *neutral evaluation fibration* determined by S^m and X and denote by Ev (S^m, X). Since X is m-simple, Ev (S^m, X) has $\Omega_0^m X$ as fibre over the base point in X. Finally, we note that Ev (S^m, X) admits a canonical section $s: X \to G_0(S^m, X)$ defined by $s(x)(y) = x$ for $x \in X$ and $y \in S^m$.

Now a short description of the paper. In Section 2 we investigate how the brace product operation, introduced by James in e.g. [6] and

[7] for fibrations with a section, works in the class of neutral evaluation fibrations. In Theorem 2.1 we shall prove a formula relating brace products in $\mathrm{Ev}(S^m, X)$ to Whitehead products in X.

The motivation behind Section 3 is that $\mathrm{Ev}(S^m, X)$ at first sight looks like a principal fibration with a section so that one might expect it to be fibre homotopically trivial. Surprisingly enough, it turns out that this is very seldom the case. In fact, we shall describe a method, with Theorem 3.2 as the main result, which in many cases can be used to decide that the total space $G_0(S^m, X)$ in a given neutral evaluation fibration $\mathrm{Ev}(S^m, X)$ does not even decompose up to homotopy type as the product $X \times \Omega_0^m X$, i.e. it is not decomposable in the terminology of James [7]. The formula in Theorem 2.1 is used in the proof of Theorem 3.2.

2. Brace products in evaluation fibrations

First we introduce the brace product operation following James in e.g. [6] or [7]. Let

$$F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} B$$

be a Serre fibration which admits a section s. Choose base points in the spaces F, E and B such that they are preserved by the maps i, p and s. The section s induces a splitting of the homotopy sequence for the fibration so that we get short split exact sequences

$$0 \to \pi_*(F) \xrightarrow{i_*} \pi_*(E) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_*(B) \to 0.$$

For any pair of homotopy classes $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$ with $p, q \geq 1$ we can form the Whitehead product $[s_*(\alpha), i_*(\beta)] \in \pi_{p+q-1}(E)$. Since $p_* \circ i_* = 0$, it follows that $p_*([s_*(\alpha), i_*(\beta)]) = 0$. By exactness of the homotopy sequence, and since i_* is a monomorphism, there exists therefore a unique element $\{\alpha, \beta\} \in \pi_{p+q-1}(F)$ such that $i_*(\{\alpha, \beta\}) = [s_*(\alpha), i_*(\beta)]$. This element is called the *brace product* of α and β.

The neutral evaluation fibration $\mathrm{Ev}(S^m, X)$,

$$\Omega_0^m X \xrightarrow{i} G_0(S^m, X) \underset{s}{\overset{p}{\rightleftarrows}} X,$$

with its canonical section s of constant maps, turns out to behave well under the brace product operation. As base point in $\Omega_0^m X$, respectively $G_0(S^m, X)$, we use always the constant map of S^m into X with the base point in X as value. Base points are then clearly preserved by the maps i, p and s.

Before we can state our main theorem on brace products in $\mathrm{Ev}(S^m, X)$, we have to recall the definition of the adjoint isomorphism. For each $i \geq 1$ this is the isomorphism

$$H: \pi_i(\Omega_0^m X) \to \pi_{i+m}(X)$$

defined as the composition of natural isomorphisms

$$\pi_i(\Omega_0^m X) = \pi(S^i, \Omega_0^m X) \underset{\simeq}{\to} \pi(S^i \wedge S^m, X) \underset{\simeq}{\to} \pi(S^{i+m}, X) = \pi_{i+m}(X).$$

Here $\pi(\cdot,\cdot)$ denotes a homotopy set of based maps. $\vee$ and $\wedge$ between based spaces denotes respectively wedge product and smash product.

THEOREM 2.1: *Given homotopy classes $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(\Omega_0^m X)$. Form the brace product $\{\alpha, \beta\} \in \pi_{p+q-1}(\Omega_0^m X)$ in* $\mathrm{Ev}(S^m, X)$ *and the Whitehead product $[\alpha, H(\beta)] \in \pi_{p+q+m-1}(X)$ in X. Then*

$$H(\{\alpha, \beta\}) = [\alpha, H(\beta)].$$

PROOF: Define the Whitehead product map $W_{p,q}: S^{p+q-1} \to S^p \vee S^q$ as follows. Let D^i, respectively ∂D^i, denote the i-cell, respectively its boundary sphere. Then we have obvious identifications

$$S^{p+q-1} = \partial(D^{p+q}) = \partial(D^p \times D^q) = D^p \times \partial D^q \cup \partial D^p \times D^q.$$

By pinching ∂D^p and ∂D^q to a point in both factors we get the canonical projection $W_{p,q}: S^{p+q-1} \to S^p \vee S^q$. Similarly, we can define the Whitehead product map $W_{p,q+m}: S^{p+q+m-1} \to S^p \vee S^{q+m}$.

Denote by b_p, b_q and b_m respectively the base point in S^p, S^q and S^m and let $\mathrm{proj}_2: S^p \times b_q \times S^m \cup b_p \times S^q \times S^m \to S^p \vee S^{q+m}$ be the map, which projects $S^p \times b_q \times S^m$ onto S^p and projects $b_p \times S^q \times S^m$ onto S^{q+m} by identifying $b_p \times b_q \times S^m \cup b_p \times S^q \times b_m$ to a point. Let $\alpha \vee H(\beta): S^p \vee S^{q+m} \to X \vee X$ denote a map obtained by choosing representatives for the homotopy classes involved and let $\nabla: X \vee X \to X$ denote the folding map.

Consider then the following diagram in which proj_1 is a canonical projection and the maps marked $\simeq$ are natural identifications,

$$
\begin{tikzcd}
& S^{p+q-1} \times S^m \arrow[dl, "W_{p,q} \times 1"'] \arrow[dr, "\mathrm{proj}_1"] & \\
(S^p \vee S^q) \times S^m \arrow[d, "\simeq"'] & & S^{p+q-1} \wedge S^m \arrow[d, "\simeq"] \\
S^p \times b_q \times S^m \cup b_p \times S^q \times S^m \arrow[dr, "\mathrm{proj}_2"'] & & S^{p+q+m-1} \arrow[dl, "W_{p,q+m}"] \\
& S^p \vee S^{q+m} \arrow[d, "\alpha \vee H(\beta)"] & \\
& X \vee X \arrow[d, "\nabla"] & \\
& X &
\end{tikzcd}
$$

This diagram is commutative and hence the composition of maps from top to bottom on the left in the diagram is identical with that on the right. The maps of S^{p+q-1} into $G_0(S^m, X)$ induced from these compositions of maps are therefore also identical and hence they represent the same homotopy class in $\pi_{p+q-1}(G_0(S^m, X))$. Now it follows almost by definition that considered as a map of S^{p+q-1} into $G_0(S^m, X)$, the composition on the left represents the homotopy class $i_*(\{\alpha, \beta\}) = [s_*(\alpha), i_*(\beta)]$ and the composition on the right represents the homotopy class $i_*(H^{-1}([\alpha, H(\beta)]))$. Since i_* is a monomorphism we get then $\{\alpha, \beta\} = H^{-1}([\alpha, H(\beta)])$, which proves the theorem.

REMARK 2.2: A weaker result related to Theorem 2.1 was obtained by Federer ([4], p. 356).

3. Decomposability of evaluation fibrations

Let

$$F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} B$$

be a Serre fibration with section s. Following James [7] we say that this fibration is *decomposable* if E and $B \times F$ have the same homotopy type. Clearly a fibration is decomposable if it is fibre homotopically trivial.

The purpose of this section is to investigate the neutral evaluation fibration $\mathrm{Ev}\,(S^m, X)$,

$$\Omega_0^m X \xrightarrow{i} G_0(S^m, X) \underset{s}{\overset{p}{\rightleftarrows}} X,$$

w.r.t. decomposability.

First we observe that the spaces $G_0(S^m, X)$ and $X \times \Omega_0^m X$ have the same homotopy groups. This follows easily, since the section s splits the homotopy sequence for $\mathrm{Ev}(S^m, X)$. We remark that $(m+1)$-simplicity of X is needed here in order to get this statement for π_1, see Abe [1].

Next we mention the following well-known positive result.

THEOREM 3.1: *Suppose that X is an H-space with a strict unit element. Then the neutral evaluation fibration* $\mathrm{Ev}(S^m, X)$ *is fibre homotopically trivial, in particular decomposable.*

PROOF: If we take the unit element for the multiplication on X as base point in X, and if we use the multiplication properly, it is easy to construct a map $\theta: X \times \Omega_0^m X \to G_0(S^m, X)$, which commutes with projections onto X, and which is the identity map on the fibre over the base point in X. Hence θ is a fibre homotopy equivalence by a fundamental theorem of Dold ([3], Theorem 6.3). This proves the theorem.

The following theorem provides a method to obtain non-decomposability results.

THEOREM 3.2: *Suppose that there exist homotopy classes $\alpha \in \pi_n(X)$ and $\beta \in \pi_{m+k}(X)$ with m, n, $k \geq 1$ such that the Whitehead product $[\alpha, \beta] \neq 0$. Suppose also that the set of Whitehead products $[\pi_n(X), \pi_k(X)] = 0$.*

Then the neutral evaluation fibration $\mathrm{Ev}(S^m, X)$ *is not decomposable.*

PROOF: Let $\beta' \in \pi_k(\Omega_0^m X)$ be the unique element such that $\beta = H(\beta')$. By Theorem 2.1 we have then $H(\{\alpha, \beta'\}) = [\alpha, H(\beta')] = [\alpha, \beta] \neq 0$. Hence $\{\alpha, \beta'\} \neq 0$ and then also $[s_*(\alpha), i_*(\beta')] \neq 0$.

Consequently, the set of Whitehead products $[\pi_n(G_0(S^m, X)), \pi_k(G_0(S^m, X))] \neq 0$. In passing we mention that this will also follow from the theorem of Federer ([4], p. 356).

On the other hand, the set of Whitehead products $[\pi_n(X \times \Omega_0^m X), \pi_k(X \times \Omega_0^m X)] = 0$. This follows, since the two sets of Whitehead products $[\pi_n(X), \pi_k(X)]$ and $[\pi_n(\Omega_0^m X), \pi_k(\Omega_0^m X)]$ vanishes, the first one by assumption and the second one since $\Omega_0^m X$ is an H-space.

$G_0(S^m, X)$ and $X \times \Omega_0^m X$ must therefore have different homotopy types and the theorem is proved.

We restrict now our attention to the case $X = S^n$, the n-sphere. First we note the following special case of Theorem 3.2.

THEOREM 3.3: *Denote by* $\iota_n \in \pi_n(S^n)$ *the homotopy class represented by the identity map on* S^n *and suppose that there exists a homotopy class* $\beta \in \pi_{m+k}(S^n)$ *with* $1 \leq k < n$ *such that the Whitehead product* $[\iota_n, \beta] \neq 0$.

Then the neutral evaluation fibration $\mathrm{Ev}(S^m, S^n)$ *is not decomposable.*

Next we give some concrete applications of Theorem 3.3.

COROLLARY 3.4: *Let* $1 \leq m < n$. *Then the neutral evaluation fibration* $\mathrm{Ev}(S^m, S^n)$ *is decomposable if and only if* $n = 3, 7$.

For $n = 3, 7$ *the fibration is even fibre homotopically trivial.*

PROOF: For $n = 3, 7$, S^n is an H-space with a strict unit element and hence $\mathrm{Ev}(S^m, S^n)$ is fibre homotopically trivial by Theorem 3.1.

For $n \neq 3, 7$, the Whitehead product $[\iota_n, \iota_n] \neq 0$ by Adams [2]. Writing $n = (n - m) + m$ we see that ι_n can be used as the element β in Theorem 3.3. Hence $\mathrm{Ev}(S^m, S^n)$ is not decomposable for $n \neq 3, 7$.

COROLLARY 3.5: *Consider the neutral evaluation fibration* $\mathrm{Ev}(S^n, S^n)$ *for* $n \geq 1$.

(1) *For* $n = 1, 3, 7$, $\mathrm{Ev}(S^n, S^n)$ *is fibre homotopically trivial.*

(2) *For* $n \geq 8$, $n \neq 11, 27$ *and* $n \not\equiv 15 \bmod 16$, $\mathrm{Ev}(S^n, S^n)$ *is not decomposable.*

PROOF:

(1) Follows from Theorem 3.1.

(2) For $n \geq 8$, let $\sigma_n \in \pi_{n+7}(S^n)$ be the element represented by suspensions of the Hopf map $S^{15} \to S^8$. By results of Mahowald [9] and Kristensen and Madsen [8], it follows that $[\iota_n, \sigma_n] \neq 0$ for $n \neq 11$, 27 and $n \not\equiv 15 \bmod 16$. Hence for these values of n, the element σ_n can be used as the element β in Theorem 3.3. This proves (2).

REMARK 3.6: Corollary 3.5 is certainly not the best possible result for the evaluation fibrations $\mathrm{Ev}(S^n, S^n)$. The author is indebted to Siegfried Thomeier for showing him his tables over Whitehead products. Using these tables in connection with Theorem 3.3 it is possible to prove that $\mathrm{Ev}(S^n, S^n)$ is not decomposable if $n \neq 1, 2, 3, 7$ and $n \not\equiv 15 \bmod 16$.

Of the remaining cases we mention in particular that it is unknown whether $\mathrm{Ev}(S^2, S^2)$ is decomposable or not. Since the set of Whitehead products $[\pi_3(S^2), \pi_2(S^2)] = 0$, this problem cannot be solved using Theorem 3.3.

Using Theorem 3.3 and all known information on Whitehead products we can prove that $\mathrm{Ev}(S^m, S^n)$ is not decomposable in many special cases. However, we cannot get the complete solution to the following problem using this method.

PROBLEM: Let $1 \leq n \leq m$. Determine when the neutral evaluation fibration $\mathrm{Ev}(S^m, S^n)$ is decomposable.

It is a tempting conjecture to suggest that $\mathrm{Ev}(S^m, S^n)$ is decomposable if and only if $n = 1, 3, 7$.

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