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LINEAR FORMS AND SIMULTANEOUS APPROXIMATIONS

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1. Introduction

We shall improve and generalize several results on transcendence and on simultaneous approximations of numbers associated with the exponential function. We deduce these results from the following lower bound of an inhomogeneous linear form in logarithms of algebraic numbers with algebraic coefficients:

THEOREM 1: *Let $n \geq 1$ and $d \geq 1$ be given integers. There exists an effectively computable number C, depending only on n and d, with the following property. Let $l_1, \ldots, l_n$ be complex numbers such that $\alpha_1 = e^{l_1}, \ldots, \alpha_n = e^{l_n}$ are algebraic numbers of degrees at most d; for $1 \leq j \leq n$, let $A_j \geq 6$ be an upper bound for the height of α_j and for $e^{|l_j|}$. Further, let β_0, $\beta_1, \ldots, \beta_n$ be algebraic numbers of degrees at most d and heights at most B (≥ 6); put*

$$\Lambda = \beta_0 + \beta_1 l_1 + \cdots + \beta_n l_n \tag{1}$$

and define

$$\Omega = (\mathrm{Log}\, A_1) \ldots (\mathrm{Log}\, A_n).$$

Then, if $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\{-C\Omega(\mathrm{Log}\, \Omega)(\mathrm{Log}\, B + \mathrm{Log}\, \Omega)\}. \tag{2}$$

From this point on, we shall write $\log \alpha_j$ instead of l_j so that (1) takes the form

$$(3) \qquad \Lambda = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n;$$

then the theorem says, that, if $\Lambda \neq 0$, (2) holds where C is independent of the branches of the logarithms.

We remark, that it is known that the linear form Λ in (3) is non-zero when $\alpha_1, \ldots, \alpha_n$ are non-zero and either $\log \alpha_1, \ldots, \log \alpha_n$ not all zero and $\beta_1, \beta_2, \ldots, \beta_n$ linearly independent over $\mathbb{Q}$, or $\beta_0 \neq 0$.

For earlier results in this direction, see in particular A. Baker's papers [2] and [3]. It was pointed out to us by A.J. van der Poorten that by minor changes we could have replaced the bound $\exp\{-C\Omega\,(\operatorname{Log}\Omega)(\operatorname{Log} B + \operatorname{Log}\Omega)\}$ by

$$(4) \qquad \exp\{-C\Omega(\operatorname{Log}\Omega')(\operatorname{Log} B + \operatorname{Log}\Omega)\},$$

where $\Omega' = (\operatorname{Log} A_1)\ldots(\operatorname{Log} A_{n-1})$. We shall develop this remark at the end of the present paper. We understand that, independently, A. Baker recently obtained a similar bound for the linear form (1), which is somewhat more refined than our bound. However, these refinements are not essential for our applications, and moreover, our proof is simpler.

The actual proof of theorem 1 will follow closely the proof given in [3]. Nevertheless a simplification is introduced here. In earlier similar proofs, the extrapolation is performed by means of an induction, and then a further inductive argument permits one to decrease the number of terms in the auxiliary function. In our proof, there is no need to increase the number of zeros, and we therefore avoid the induction in the extrapolation argument. This simplification allows us to reduce the main part of the proof from a chain of lemmas to a single unified argument. Moreover, in a certain part of the proof where a prime p has to be used which ordinarily must be sufficiently large, we can deal with an arbitrary prime p, even with $p = 2$.

We want to express our gratitude to A. Baker and A. J. van der Poorten for some very useful discussions at the conference on transcendental number theory during early 1976 at Cambridge, where this paper was put into its final form.

2. Simultaneous approximations

We shall apply theorem 1 to give bounds for the simultaneous approximability of certain numbers. First, let $a \in \mathbb{C}$ with $a \neq 0$, $a \neq 1$,

$b \in \mathbb{C}$, $\epsilon > 0$ and $d \in \mathbb{N}$ be given. We consider the number of solutions in algebraic numbers α, β and γ, of degrees at most d, of the inequality

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-(\operatorname{Log} H)^\kappa),$$

where H is a bound for the heights of α, β and γ and where $a^b = e^{b \log a}$ for some fixed value of the logarithm. After some early results of Ricci [10] and Franklin [8], Schneider [11] proved that if $b \notin \mathbb{Q}$, then there are only finitely many solutions (α, β, γ) with $\beta \notin \mathbb{Q}$ when $\kappa > 5$. Smelev [12] improved this to $\kappa > 4$, and Bundschuh [5] reached $\kappa \geq 4$, without the restriction to the irrationality of β. Note, that $b \in \mathbb{Q}$ implies that $\beta = b$ for H large enough and that in this case infinitely many solutions are possible when a is a suitable U-number. The next theorem shows, that there are only finitely many solutions (α, β, γ) with $\beta \notin \mathbb{Q}$ even when $\kappa > 3$.

THEOREM 2: *Let $a \in \mathbb{C}$ with $a \neq 0$, let* $\log a$ *be an arbitrary value of the logarithm of a with* $\log a \neq 0$. *Let $b \in \mathbb{C}$, $\epsilon > 0$ and $d \in \mathbb{N}$ be given. There are only finitely many triples (α, β, γ) with $\beta \notin \mathbb{Q}$ of algebraic numbers with degrees at most d for which*

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp\{-(\operatorname{Log} H)^3(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\}$$

where H is a bound for the heights of α, β and γ and where $a^b = e^{b \log a}$.

PROOF: Let (α, β, γ) be a triple as indicated in the statement of the theorem, and let H be sufficiently large. Since $a \neq 0$ and $a^b \neq 0$, we have $\alpha \neq 0$ and $\gamma \neq 0$. Further, we know that for suitable determinations of the logarithms

$$|\log a - \log \alpha| < \exp\{-(\operatorname{Log} H)^3(\operatorname{Log}\operatorname{Log} H)^{1+(2\epsilon)/3}\}$$

and

$$|b \log a - \log \gamma| < \exp\{-(\operatorname{Log} H)^3(\operatorname{Log}\operatorname{Log} H)^{1+(2\epsilon)/3}\};$$

moreover, $\log \alpha \neq 0$ since $\log a \neq 0$. Combining both inequalities and using that β is close to b, we see that

$$|\beta \log \alpha - \log \gamma| < \exp\{-(\operatorname{Log} H)^3(\operatorname{Log}\operatorname{Log} H)^{1+(\epsilon/3)}\}.$$

However, in both cases $\log \gamma = 0$ and $\log \gamma \neq 0$, it follows from the condition $\beta \notin \mathbb{Q}$ that the linear form $\beta \log \alpha - \log \gamma$ does not vanish. We obtain from theorem 1 the inequality

$$|\beta \log \alpha - \log \gamma| > \exp\{-(\operatorname{Log} H)^3 (\operatorname{Log}\operatorname{Log} H)^{1+(\epsilon/3)}\},$$

so that we have a contradiction for H large enough. Hence, H must be bounded and the theorem follows. □

Wallisser [15] stated a theorem saying essentially that for algebraic $\alpha_1, \ldots, \alpha_n$ and certain numbers $b_1, \ldots, b_n$ which can be approximated very well by algebraic numbers, the number $\alpha_1^{b_1} \ldots \alpha_n^{b_n}$ must be transcendental. Meyer [9] improved this and also considered $e^{b_0}\alpha_1^{b_1} \ldots \alpha_n^{b_n}$; in both cases he achieved essentially the best possible result. Bundschuh [6] recently showed, that for fixed algebraic $\alpha_1, \ldots, \alpha_n$, the numbers $b_1, \ldots, b_n$, $\alpha_1^{b_1} \ldots \alpha_n^{b_n}$ cannot be approximated simultaneously very well (up to $\exp\{-(\operatorname{Log} H)^{n^2+2n+2+\epsilon}\}$) by infinitely many $(n+1)$-tuples of algebraic numbers of bounded degrees; he used only natural hypotheses. The next theorem gives an important improvement of Bundschuh's main theorem for a special case.

In theorem 4 we state an analogous result in which only natural conditions are used.

THEOREM 3: *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers, let $\log \alpha_j$ be an arbitrary value of the logarithm with $\log \alpha_j \neq 0$ $(j = 1, \ldots, n)$, let $b_1, \ldots, b_n$ be complex numbers, $\epsilon > 0$ and $d \in \mathbb{N}$. There are only finitely many $(n+1)$-tuples $(\beta_1, \ldots, \beta_n, \gamma)$ of algebraic numbers with 1, $\beta_1, \ldots, \beta_n$ linearly independent over $\mathbb{Q}$ and of degrees at most d for which*

$$\max(|b_1 - \beta_1|, \ldots, |b_n - \beta_n|, |\alpha_1^{b_1} \ldots \alpha_n^{b_n} - \gamma|) < \exp\{-(\operatorname{Log} H)^2 (\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\},$$

where H denotes a bound for the heights of $\beta_1, \ldots, \beta_n$, γ and where

$$\alpha_j^{b_j} = e^{b_j \log \alpha_j} \quad (j = 1, \ldots, n).$$

PROOF: Use the linear form

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \log \gamma,$$

of which $\alpha_1, \ldots, \alpha_n$, γ and $\log \alpha_1, \ldots, \log \alpha_n$ are non-zero. By the assumption on the linear independence of 1, $\beta_1, \ldots, \beta_n$ this linear form is non-zero. Use theorem 1 and proceed as in the proof of theorem 2. □

THEOREM 4: *Let $\alpha_1, \ldots, \alpha_n$, $\log \alpha_1, \ldots, \log \alpha_n$, $b_1, \ldots, b_n$, ϵ and d be as in theorem 3. Let $b_0 \in \mathbb{C}$ with $b_0 \neq 0$.*

Then there are only finitely many $(n+2)$-tuples $(\beta_0, \beta_1, \ldots, \beta_n, \gamma)$ of algebraic numbers of degrees at most d for which

$$\max(|b_0 - \beta_0|, \ldots, |b_n - \beta_n|, |e^{b_0}\alpha_1^{b_1} \ldots \alpha_n^{b_n} - \gamma|) < \exp\{-(\operatorname{Log} H)^2(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\},$$

where H denotes a bound for the heights of $\beta_0, \beta_1, \ldots, \beta_n, \gamma$ and where $\alpha_j^{b_j} = e^{b_j \log \alpha_j}$.

PROOF: Use the linear form $\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \log \gamma$ which is nonzero for H large enough, since then the condition $b_0 \neq 0$ implies $\beta_0 \neq 0$. □

Remark, that (4) enables us to replace $(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}$ by $(\operatorname{Log}\operatorname{Log} H)^{\epsilon}$ in theorem 3 and theorem 4.

In his paper [6], Bundschuh remarked that in theorem 3 and theorem 4, one should like to replace the α_i by complex numbers a_i which can be approximated very well by algebraic numbers. The following theorems give such results. In theorem 5, of which the second part is quite an extension of theorem 2, we again need a condition on linear independence; in theorem 6 this is not the case.

THEOREM 5: *Let $a_1, \ldots, a_n$ be non-zero complex numbers, let $\log a_j$ denote an arbitrary value of the logarithm of a_j such that $\log a_j \neq 0$. Let $b_1, \ldots, b_n$ be complex numbers, let $\epsilon > 0$ and $d \in \mathbb{N}$.*

Firstly, suppose that there exist infinitely many $(2n)$-tuples $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ of algebraic numbers with $1, \beta_1, \ldots, \beta_n$ linearly independent over $\mathbb{Q}$ and of degrees at most d, for which

$$\max(|a_1 - \alpha_1|, \ldots, |a_n - \alpha_n|, |b_1 - \beta_1|, \ldots, |b_n - \beta_n|) < \exp\{-(\operatorname{Log} H)^{n+1}(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\}$$

where H is a bound for the heights of $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$. Then $a_1^{b_1} \ldots a_n^{b_n}$ (where $a_j^{b_j} = e^{b_j \log a_j}$) is transcendental.

Secondly, there are only finitely many $(2n+1)$-tuples $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma)$ of algebraic numbers with $1, \beta_1, \ldots, \beta_n$ linearly in-

dependent over $\mathbb{Q}$ *and of degrees at most d, for which*

$$\max(|a_1-\alpha_1|,\ldots,|a_n-\alpha_n|,|b_1-\beta_1|,\ldots,|b_n-\beta_n|,|a_1^{b_1}\ldots a_n^{b_n}-\gamma|) < \exp\{-(\operatorname{Log} H)^{n+2}(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\}$$

where H is a bound for the heights of $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$, γ *and where* $a_j^{b_j}=e^{b_j\log a_j}$.

PROOF: Use the linear form $\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n-\log\alpha_n-\log\gamma$ with γ fixed for the first assertion and free for the second one, and proceed as in the proof of theorem 3. □

THEOREM 6: *Let* $a_1,\ldots,a_n$, $\log a_1,\ldots,\log a_n$, $b_1,\ldots,b_n$, ϵ *and d be as in theorem 5 and let* $b_0\in\mathbb{C}$ *with* $b_0\neq 0$.

Firstly, suppose that there exist infinitely many $(2n+1)$*-tuples* $(\alpha_1,\ldots,\alpha_n,\beta_0,\beta_1,\ldots,\beta_n)$ *of algebraic numbers of degrees at most d, for which*

$$\max(|a_1-\alpha_1|,\ldots,|a_n-\alpha_n|,|b_0-\beta_0|,\ldots,|b_n-\beta_n|) < \exp\{-(\operatorname{Log} H)^{n+1}(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\}$$

where H is a bound for the heights of $\alpha_1,\ldots,\alpha_n$, $\beta_0,\ldots,\beta_n$. *Then* $e^{b_0}a_1^{b_1}\ldots a_n^{b_n}$ *(where* $a_j^{b_j}=e^{b_j\log a_j}$*) is transcendental.*

Secondly, there are only finitely many $(2n+2)$*-tuples* $(\alpha_1,\ldots,\alpha_n,\beta_0,\beta_1,\ldots,\beta_n,\gamma)$ *of algebraic numbers of degrees at most d, for which*

$$\max(|a_1-\alpha_n|,\ldots,|a_n-\alpha_n|,|b_0-\beta_0|,\ldots,|b_n-\beta_n|,|e^{b_0}a_1^{b_1}\ldots a_n^{b_n}-\gamma|) < \exp\{-(\operatorname{Log} H)^{n+2}(\operatorname{Log}\operatorname{Log} H)^{1+\epsilon}\}$$

where H is a bound for the heights of $\alpha_1,\ldots,\alpha_n$, β_0, $\beta_1,\ldots,\beta_n$, γ *and where* $a_j^{b_j}=e^{b_j\log a_j}$.

PROOF: As the proof of theorem 5. □

We would like to remark here, that at the present state of development of Baker's method, the dependence on the degrees of the algebraic numbers involved seems to become more interesting (for some old results, see [7]). Among many possibilities, one could try to investigate theorems like the above ones, in which the dependence on the heights as well as on the degrees is explicitly given.

3. Preliminary lemmas

When α is an algebraic number, we denote by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of α, by $H(\alpha)$ the height of α and by $\operatorname{den}(\alpha)$ the denominator of α.

For elements α, β of a number field K of degree D, we have the following inequalities

$$\overline{|\alpha|} \leq H(\alpha)+1$$
$$H(\alpha) \leq (2 \operatorname{den}(\alpha) \max\{1, \overline{|\alpha|}\})^D,$$
$$H(\alpha+\beta) \leq c_1(H(\alpha)H(\beta))^{c_2},$$
$$H(\alpha \cdot \beta) \leq c_3(H(\alpha)H(\beta))^{c_4}$$

where $c_1, \ldots, c_4$ depend only on D; from the first inequality, we deduce, for $\alpha \neq 0$,

$$|\alpha| \geq [H(\alpha)+1]^{-1}.$$

We shall also use the fact that for any algebraic number α and any positive integer q,

$$\overline{|\alpha^{1/q}|} = \overline{|\alpha|}^{1/q}, \operatorname{den}(\alpha^{1/q}) \leq \operatorname{den}(\alpha).$$

We shall use the notation Log for the principle value of the logarithm.

LEMMA 1: *Let K be a number field of degree D over $\mathbb{Q}$. Let $a_{i,j}$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$ be algebraic integers in K, whose conjugates are bounded in absolute value by an integer A. If $n > mD$, there exists a non-trivial solution $(x_1, \ldots, x_n)$ of the system*

$$\sum_{i=1}^{n} a_{ij}x_i = 0, \quad (1 \leq j \leq m),$$

in rational integers x_i, bounded by

$$\max_{1\leq i\leq n} |x_i| \leq (\sqrt{2}nA)^{mD/(n-mD)}.$$

PROOF: This is lemma 1.3.1 of [14]. □

LEMMA 2: *Let f be an analytic function in a disc $|z| \leq R$ of the complex plane. Let E be a finite subset of the disc $|z| \leq r$ with $r \leq R/2$,* *consisting of k points which are lying on a straight line and have a mutual distance at least δ with $\delta \leq \min(r/2, 1)$. Let t be a positive integer. Then*

$$|f|_{2r} \leq 2|f|_R \left(4\frac{r}{R}\right)^{kt} + 5\left(\frac{18r}{\delta k}\right)^{kt} \max_{\substack{x \in E \\ 0 \leq \tau \leq t-1}} \left|\frac{f^{(\tau)}(x)}{\tau!}\right|.$$

PROOF: Without loss of generality, we may assume that $r < R/4$. Consider the circles Γ: $|\zeta| = R$ and Γ_x: $|\zeta - x| = \delta/2$ for $x \in E$, described in the positive sense; define $Q(z) = \Pi_{x \in E}(z-x)^t$. We use Hermite's interpolation formula:

$$\frac{f(z)}{Q(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{Q(\zeta)}\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\sum_{x \in E}\sum_{\tau=0}^{t-1}\frac{f^{(\tau)}(x)}{\tau!}\int_{\Gamma_x}\frac{(\zeta - x)^\tau}{Q(\zeta)}\frac{d\zeta}{\zeta - z},$$

for $|z| = 2r$.

For $|z| = 2r$ and $|\zeta| = R$ we have $|Q(z)| \leq (3r)^{kt}$; further,

$$\left|\frac{f(\zeta)}{Q(\zeta)}\right| \leq \frac{|f|_R}{(R-r)^{kt}}; \quad \frac{R}{|\zeta - z|} \leq \frac{R}{R - 2r} \leq 2.$$

For $|z| = 2r$ and $\zeta \in \Gamma_x$ we have $1/|\zeta - z| \leq 2/r$; we estimate $|Q(\zeta)|$ as follows: Let $\epsilon_0 = x, \epsilon_1, \ldots, \epsilon_{k-1}$ be the points of E, arranged in such a way that $|\zeta - \epsilon_0| \leq \cdots \leq |\zeta - \epsilon_{k-1}|$. Then $|\zeta - \epsilon_0| = \delta/2$ and $|\zeta - \epsilon_\kappa| \geq \kappa(\delta/2)$ for $\kappa = 1, 2, \ldots, k-1$. Hence, for $\zeta \in \Gamma_x$ we obtain

$$|Q(\zeta)| \geq \left(\left(\frac{\delta}{2}\right)^k (k-1)!\right)^t > \left(\frac{\delta}{2}\right)^{kt}\left(\frac{k}{3}\right)^{kt} = \left(\frac{\delta k}{6}\right)^{kt}.$$

Finally, we use

$$\frac{3r}{R-r} \leq 4\frac{r}{R}, \quad \sum_{\tau=0}^{t-1}\left(\frac{\delta}{2}\right)^\tau < 2 \quad \text{and} \quad k \leq \frac{2r}{\delta} + 1 \leq \frac{5}{2}\frac{r}{\delta}.$$ □

LEMMA 3: *For any positive integer k, let $\nu(k)$ be the least common multiple of $1, 2, \ldots, k$. Define, for $z \in \mathbb{C}$,*

$$\Delta(z; k) = (z+1)\ldots(z+k)/k! \quad (k \in \mathbb{Z}, k \geq 1),$$

and

$$\Delta(z; 0) = 1.$$

For any integers $l \geq 0$, $m \geq 0$, $k \geq 0$, and any $z \in \mathbb{C}$,

$$\left| \frac{d^m}{dz^m} (\Delta(z;k))^l \right| \leq m!4^{l(|z|+k)}.$$

Moreover, let q be a positive integer, and let x be a rational number such that qx is a positive integer. Then

$$q^{2kl}(\nu(k))^m \frac{1}{m!} \frac{d^m}{dz^m} (\Delta(z;k))^l_{z=x}$$

is a positive integer, and we have $\nu(k) \leq 4^k$.

PROOF: The second part is lemma T1 of [13]. The first one follows from the following estimates (see [13]):

$$\begin{aligned} \left| \frac{1}{m!} \frac{d^m}{dz^m} (\Delta(z;k))^l \right| &\leq \frac{1}{m!} \frac{d^m}{dx^m} (\Delta(x,k))^l_{x=|z|} \\ &\leq \binom{[|z|]+k+1}{k}^l \binom{kl}{m} \leq 4^{l(|z|+k)}. \end{aligned}$$

□

LEMMA 4: *For any positive integers k, R and L with $k \geq R$, the polynomials $(\Delta(z+r;k))^l$ $(r = 0, 1, \ldots, R-1;\ l = 1, \ldots, L)$, where $\Delta(z;k)$ has been introduced in lemma 3, are linearly independent.*

PROOF: Follows immediately by induction on L from the following result (see lemma 2 of [1]): If P is a polynomial with degree $n > 0$ and with coefficients in a field K, then, for any integer m with $0 \leq m \leq n$, the polynomials $P(x)$, $P(x+1), \ldots, P(x+m)$ and $1, x, \ldots, x^{n-m-1}$ are linearly independent over K. □

LEMMA 5: *Let M and T be non-negative integers, let a, b and E_{mt} $(m = 0, 1, \ldots, M;\ t = 0, 1, \ldots, T)$ be complex numbers.*

If

$$\sum_{m=0}^{M} \sum_{\tau=0}^{t} \binom{t}{\tau} ((a+m)b)^{t-\tau} E_{m\tau} = 0 \tag{5}$$

for $t = 0, 1, \ldots, T$, then

$$\sum_{m=0}^{M} \sum_{\tau=0}^{t} \binom{t}{\tau} (mb)^{t-\tau} E_{m\tau} = 0 \tag{6}$$

for $t = 0, 1, \ldots, T$.

PROOF: First we prove that a certain linear combination of the left hand side of (6) for $t = 0, 1, \ldots, n$ just gives the left hand side of (5) with $t = n$, when $0 \leq n \leq T$. We prove:

$$(7)\quad \sum_{t=0}^{n} \binom{n}{t} (ab)^{n-t} \sum_{m=0}^{M} \sum_{\tau=0}^{t} \binom{t}{\tau} (mb)^{t-\tau} E_{m\tau} = \sum_{m=0}^{M} \sum_{\tau=0}^{n} \binom{n}{\tau} ((a+m)b)^{n-\tau} E_{m\tau}.$$

Namely, the left hand side of (7) is equal to

$$\sum_{m=0}^{M} \sum_{\tau=0}^{n} \sum_{t=\tau}^{n} \frac{n!}{(n-t)!\tau!(t-\tau)!} (ab)^{n-t} (mb)^{t-\tau} E_{m\tau}$$

$$= \sum_{m=0}^{M} \sum_{\tau=0}^{n} \binom{n}{\tau} E_{m\tau} \sum_{t=\tau}^{n} \binom{n-\tau}{t-\tau} (ab)^{n-t} (mb)^{t-\tau}$$

which, by replacing $t - \tau$ by t and applying the binomial formula, is equal to the right hand side of (7).

We now prove the lemma by induction. For $t = 0$, (5) and (6) are the same. Suppose (6) has been proved for $t = 0, 1, \ldots, n-1$ with $n \leq T$. Then from (7) it follows that (6) is also true for $t = n$. □

LEMMA 6: *Let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of an algebraic number field K and let $\alpha_1^{1/p}, \ldots, \alpha_n^{1/p}$ denote fixed p-th roots for some prime p. Further let $K' = K(\alpha_1^{1/p}, \ldots, \alpha_{n-1}^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of K' of degree p or we have*

$$\alpha_n = \alpha_1^{j_1} \ldots \alpha_{n-1}^{j_{n-1}} \gamma^p$$

for some γ in K and some integers $j_1, \ldots, j_{n-1}$ with $0 \leq j_r < p$.

PROOF: This is lemma 3 of [4]. □

In fact, in order to prove theorem 1 we need lemma 6 only for $p = 2$. We remark that for this particular case the arguments of [4] could be considerably simplified.

LEMMA 7: *Suppose that α, β are elements of an algebraic number field with degree D and that for some positive integer q we have $\alpha = \beta^q$. If $a\alpha$ is an algebraic integer for some positive rational integer a and if b is the leading coefficient in the minimal defining polynomial of β, then $b \leq a^{D/q}$.*

PROOF: This is lemma 4 of [4]. □

LEMMA 8: *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers of degrees at most d and of heights at most A; if $\operatorname{Log} \alpha_1, \ldots, \operatorname{Log} \alpha_n$ are $\mathbb{Q}$-linearly dependent, then there exist rational integers $b_1, \ldots, b_n$, not all zero, of absolute values at most*

$$(4^{n^2} d^{2n} \operatorname{Log} A)^{(2n+1)^2}$$

such that

$$b_1 \operatorname{Log} \alpha_1 + \cdots + b_n \operatorname{Log} \alpha_n = 0.$$

PROOF: This is lemma 2 of [3]. □

4. A special case of theorem 1

PROPOSITION 1: *Let $n \geq 1$, $D \geq 1$ and $p_0 \geq 2$ be given integers. There exists an effectively computable number $C_1 > 0$, depending only on n, D and p_0, with the following property. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers; for $1 \leq j \leq n$, let $\log \alpha_j$ be any determination of the logarithm of α_j, and let $A_j \geq 6$ be an upper bound for the height of α_j and for $\exp |\log \alpha_j|$. Further, let $\beta_0, \ldots, \beta_{n-1}$ be algebraic numbers of heights at most B (≥ 6); let K be a number field containing $\alpha_1, \ldots, \alpha_n$, $\beta_0, \ldots, \beta_{n-1}$ and assume that the degree of K over $\mathbb{Q}$ is at most D. Put*

$$\Lambda_0 = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n,$$

and define

$$\Omega = (\operatorname{Log} A_1) \ldots (\operatorname{Log} A_n).$$

Suppose that there exists a prime $p \leq p_0$ such that the degree of the field $K(\alpha_1^{1/p}, \ldots, \alpha_n^{1/p})$ over K is equal to p^n, where $\alpha_j^{1/p} = \exp((1/p) \log \alpha_j)$, $(1 \leq j \leq n)$. Then

$$|\Lambda_0| > \exp\{-C_1 \Omega (\operatorname{Log} \Omega)(\operatorname{Log} B + \operatorname{Log} \Omega)\}.$$

PROOF: First, we give some notations and discuss a certain auxiliary function; then, we give the proof itself in three steps.

The numbers $c_1, \ldots, c_4$ were already introduced. By $c_5, c_6, \ldots, c_{15}$ we shall denote positive numbers that can be specified in terms of n,

D and p_0 only. Let p be a prime satisfying the supposition of proposition 1. Let ν be a sufficiently large integer. Without loss of generality, we may assume that Ω and B are at least C_2, where C_2 is large in comparison to ν (otherwise, we replace Ω by $\max(\Omega, C_2)$, or B by $\max(B, C_2)$). We show that, in this case, C_1 can be chosen as ν^{2n+5}.

We define the integers $L_0, L_1, \ldots, L_n$, S and T by

$$\begin{aligned} L_0 &= [3D\nu^{2n+1}\Omega][\nu(\operatorname{Log} B + \operatorname{Log} \Omega)] - 1; \\ L_j &= [\nu^{2n}\Omega(\operatorname{Log}\Omega)/\operatorname{Log} A_j], \quad (1 \le j \le n); \\ S &= [\nu^2(\operatorname{Log} B + \operatorname{Log}\Omega)]; \\ T &= [\nu^{2n+2}\Omega \operatorname{Log}\Omega]. \end{aligned}$$

For abbreviation, we denote by U the number $\Omega(\operatorname{Log}\Omega)(\operatorname{Log} B + \operatorname{Log}\Omega)$. We suppose that

$$|\Lambda_0| \le \exp(-\nu^{2n+5}U),$$

and we shall arrive at a contradiction.

To introduce our auxiliary function, we choose $R = [\nu(\operatorname{Log} B + \operatorname{Log}\Omega)]$, and we signify by $\Delta_{\lambda_0}(z)$, $(\lambda_0 = 0, \ldots, L_0)$ the functions $(\Delta(z+r; R))^l$, $(r = 0, 1, \ldots, R-1;\ l = 1, 2, \ldots, (L_0+1)/R)$. Remark that it follows from the lemmas 3 and 4 that these functions Δ_{λ_0}, $(\lambda_0 = 0, 1, \ldots, L_0)$ are linearly independent, that

$$(8) \qquad |\Delta_{\lambda_0}^{(t)}(z)| \le \exp(c_5\nu^{2n+3}U)$$

for $|z| \le 6S$ and $0 \le t \le T-1$, and that for all integers q and s with $1 \le q \le T$ and $0 \le s \le qS$, the numbers $\Delta_{\lambda_0}^{(t)}(s/q)$ $(\lambda_0 = 0, 1, \ldots, L_0\ ;\ t = 0, 1, \ldots, T-1)$ are rational numbers with a common denominator which is at most

$$(9) \qquad \exp(c_6\nu^{2n+3}U).$$

We shall use the auxiliary function

$$F(z) = \sum_{\lambda_0=0}^{L_0} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda_0, \ldots, \lambda_n)\Delta_{\lambda_0}(z) \cdot e^{(\lambda_n\beta_0 + \gamma_1 \log\alpha_1 + \cdots + \gamma_{n-1}\log\alpha_{n-1})z}$$

where $\gamma_j = \lambda_j + \lambda_n\beta_j$ $(1 \le j \le n-1)$. Remark that the exponent is equal to

$$(\lambda_n\Lambda_0 + \lambda_1 \log\alpha_1 + \cdots + \lambda_n \log\alpha_n)z.$$

For brevity, we shall write α_j^z for $e^{z\log\alpha_j}$; we use $\Sigma_{(\lambda)}$ instead of $\Sigma_{\lambda_0=0}^{L_0}\dots\Sigma_{\lambda_n=0}^{L_n}$ and $p(\lambda)$ instead of $p(\lambda_0,\dots,\lambda_n)$.

We shall compute the derivatives of F; for this purpose we introduce the polynomials

$$\mathscr{L}_{(\tau)}(\lambda, z) = \Delta_{\lambda_0}^{(\tau_0)}(z)\prod_{j=1}^{n-1}(\lambda_j + \lambda_n\beta_j)^{\tau_j},$$

where $(\tau) = (\tau_0, \dots, \tau_{n-1}) \in \mathbb{Z}^n$ with $0 \le \tau_j \le T-1$ $(j = 0, \dots, n-1)$ and where $0 \le \lambda_j \le L_j$ $(j = 0, \dots, n)$. The derivatives of F satisfy

$$F^{(t)} = \sum_{|\tau|=t}\frac{t!}{\tau_0!\dots\tau_{n-1}!}\left\{\prod_{j=1}^{n-1}(\log\alpha_j)^{\tau_j}\right\}F_{(\tau)}$$

where $|\tau| = t$ stands for $\tau_0 \ge 0, \dots, \tau_{n-1} \ge 0$ and $\tau_0 + \dots + \tau_{n-1} = t$ and where the summation is extended over all tuples $(\tau_0, \dots, \tau_{n-1}) \in \mathbb{Z}^n$ with this property; further,

$$F_{(\tau)}(z) = \sum_{(\lambda)}p(\lambda)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\lambda_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}\mathscr{L}_{(\tau')}(\lambda, z)\cdot e^{\lambda_n\beta_0 z}\alpha_1^{\gamma_1 z}\dots\alpha_{n-1}^{\gamma_{n-1}z}$$

where $(\tau') = (\tau_0', \tau_1, \dots, \tau_{n-1})$.

We shall also consider the entire functions

$$\Phi_{(\tau)}(z) = \sum_{(\lambda)}p(\lambda)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\lambda_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}\mathscr{L}_{(\tau')}(\lambda, z)\alpha_1^{\lambda_1 z}\dots\alpha_n^{\lambda_n z}.$$

Step 1. *Construction of the* $p(\lambda)$

There exist rational integers $p(\lambda)$, *not all zero, bounded in absolute value by* $\exp(c_7\nu^{2n+3}U)$, *such that the corresponding functions* $\Phi_{(\tau)}$ *defined above satisfy* $\Phi_{(\tau)}(s) = 0$ *for all* $(n+1)$*-tuples* $(\tau_0, \dots, \tau_{n-1}, s) \in \mathbb{Z}^{n+1}$ *with* $0 \le |\tau| \le T-1$ *and* $0 \le s \le S-1$.

PROOF: For $0 \le \tau_0 \le T-1$ and $0 \le s \le S-1$, let $d_{\tau_0,s}$, with

$$d_{\tau_0,s} \le \nu(R)^{\tau_0} \le \exp(c_6\nu^{2n+3}U)$$

be a common denominator of the numbers $\Delta_{\lambda_0}^{(\tau_0')}(s)$, $(\lambda_0 = 0, 1, \dots, L_0;$ $\tau_0' = 0, 1, \dots, \tau_0)$. Consider the system

$$d_{\tau_0,s}(\operatorname{den}\beta_0)^{\tau_0}\prod_{j=1}^{n-1}(\operatorname{den}\beta_j)^{\tau_j}\prod_{j=1}^{n}(\operatorname{den}\alpha_j)^{L_j s}\Phi_{(\tau)}(s) = 0$$

of less than

$$T^n S \le \nu^{2n^2+2n+2}\Omega^n(\operatorname{Log}\Omega)^n(\operatorname{Log}B+\operatorname{Log}\Omega)$$

linear homogeneous equations with

$$(L_0+1)(L_1+1)\dots(L_n+1)\ge 2D\nu^{2n^2+2n+2}\Omega^n(\operatorname{Log}\Omega)^n(\operatorname{Log}B+\operatorname{Log}\Omega)$$

unknowns $p(\lambda)$, and with coefficients which are integers of K, of conjugates bounded by

$$\exp\Big\{c_6\nu^{2n+3}U+nT\operatorname{Log}B+\sum_{j=1}^{n}L_jS\operatorname{Log}A_j+T\operatorname{Log}(2L_n(B+1)) + c_5\nu^{2n+3}U+\sum_{j=1}^{n-1}T\operatorname{Log}(2L_jL_n(B+1))+\sum_{j=1}^{n}L_jS\operatorname{Log}(A_j+1)\Big\} < \exp(c_8\nu^{2n+3}U).$$

Using lemma 1, we obtain Step 1.

Step 2. Induction. When J is a non-negative integer satisfying $p^J\le T-1$, we show that there exist rational integers

$$p^{(J)}(\lambda_0,\dots,\lambda_n),\quad 0\le\lambda_j\le L_j^{(J)}\quad(0\le j\le n),$$

not all zero, bounded in absolute value by $\exp(c_7\nu^{2n+3}U)$, *such that the functions*

$$\varphi_{J,(\tau)}(z)=\sum_{(\lambda)}p^{(J)}(\lambda)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\lambda_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}p^{-\tau_0'J}\cdot\mathscr{L}_{(\tau')}\Big(\lambda,\frac{z}{p^J}\Big)\alpha_1^{\lambda_1 z}\dots\alpha_n^{\lambda_n z}$$

for $0\le|\tau|<p^{-J}T$, satisfy $\varphi_{J,(\tau)}(s)=0$ for all $s\in\mathbb{Z}$ with $0\le s\le Sp^J-1$ $(s,p)=1$, where $L_0^{(J)}=L_0$ and $L_j^{(J)}\le p^{-J}L_j$ $(1\le j\le n)$.

Proof: For $J=0$, we choose $p^{(0)}(\lambda)=p(\lambda)$, thanks to Step 1. Assume that the assertion in Step 2 is correct for some integer J with $1\le p^J\le(T-1)/p$. Define

$$f_{J,(\tau)}(z)=\sum_{(\lambda)}p^{(J)}(\lambda)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\lambda_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}p^{-\tau_0'J}\cdot\mathscr{L}_{(\tau')}\Big(\lambda,\frac{z}{p^J}\Big)e^{\lambda_n\beta_0 z}\alpha_1^{\gamma_1 z}\dots\alpha_{n-1}^{\gamma_{n-1}z},$$

where the sum is for $0\le\lambda_j\le L_j^{(J)}$ $(0\le j\le n)$.

The following relation between $f_{J,(\tau)}$ and $\varphi_{J,(\tau)}$ will be an essential tool in the sequel.

LEMMA 9: *For* $0 \leq \tau \leq p^{-J}T$ *and* $|z| \leq p^J S$,

$$|f_{J,(\tau)}(z) - \varphi_{J,(\tau)}(z)| \leq \exp(-\tfrac{1}{2}\nu^{2n+5}U).$$

PROOF OF LEMMA 9: Consider the formula

$$f_{J,(\tau)}(z) - \varphi_{J,(\tau)}(z) = \sum_{(\lambda)} p^{(J)}(\lambda) \sum_{\tau_0'=0}^{\tau_0} \binom{\tau_0}{\tau_0'} \lambda_n^{\tau_0-\tau_0'} \beta_0^{\tau_0-\tau_0'} p^{-\tau_0' J} \cdot \mathcal{L}_{(\tau')}\left(\lambda, \frac{z}{p^J}\right) \alpha_1^{\lambda_1 z} \ldots \alpha_n^{\lambda_n z}(e^{\lambda_n \Lambda_0 z} - 1).$$

Clearly, we have by the inequality $|e^{\omega} - 1| \leq |\omega| e^{|\omega|}$,

$$|e^{\lambda_n \Lambda_0 z} - 1| \leq L_n^{(J)} \cdot p^J S |\Lambda_0| e^{L_n^{(J)} p^J S |\Lambda_0|} \leq (2\nu^{2n+2}U)|\Lambda_0|.$$

On the other hand, we bound the logarithm of the absolute value of

$$\sum_{(\lambda)} p^{(J)}(\lambda) \sum_{\tau_0'=0}^{\tau_0} \binom{\tau_0}{\tau_0'} \lambda_n^{\tau_0-\tau_0'} \beta_0^{\tau_0-\tau_0'} p^{-\tau_0' J} \mathcal{L}_{(\tau')}\left(\lambda, \frac{z}{p^J}\right) \alpha_1^{\lambda_1 z} \ldots \alpha_n^{\lambda_n z},$$

using (8), by

$$\begin{aligned} &\operatorname{Log}\{(L_0^{(J)}+1)\ldots(L_n^{(J)}+1)\} + c_7\nu^{2n+3}U \\ &\quad + T\operatorname{Log}(2L_n^{(J)}(B+1)) + c_5\nu^{2n+3}U + \sum_{j=1}^{n} p^{-J}T\operatorname{Log}(2L_j^{(J)}L_n^{(J)}(B+1)) \\ &\quad + \sum_{j=1}^{n} L_j^{(J)} p^J S \operatorname{Log}(A_j+1) \leq c_9\nu^{2n+3}U. \end{aligned}$$

Consequently

$$\begin{aligned} |f_{J,(\tau)}(z) - \varphi_{J,(\tau)}(z)| &\leq (2\nu^{2n+2}U)\exp\{c_9\nu^{2n+3}U - \nu^{2n+5}U\} \\ &\leq \exp\{-\tfrac{1}{2}\nu^{2n+5}U\}. \end{aligned}$$

This proves lemma 9.

We shall use also the following arithmetic property of the numbers $\varphi_{J,(\tau)}(s/p)$.

LEMMA 10: *For* $0 \le |\tau| \le p^{-J-1}T$ *and* $0 \le s \le p^{J+1}S$, *either* $\varphi_{J,(\tau)}(s/p)$ *is zero, or*

$$\left|\varphi_{J,(\tau)}\left(\frac{s}{p}\right)\right| \ge \exp\left(-c_{10}\nu^{2n+3}U\right).$$

PROOF OF LEMMA 10: The number $\varphi_{J,(\tau)}(s/p)$ is an algebraic number with a denominator which, using (9), is at most

$$(\operatorname{den}\beta_0)^{\tau_0}p^{\tau_0 J}d_{J+1,\tau,s}\prod_{j=1}^{n-1}(\operatorname{den}(\beta_j))^{\tau_j}\prod_{j=1}^{n}(\operatorname{den}(\alpha_j))^{p^{-J}L_j p^{J+1}S} \le \exp\left(c_{11}\nu^{2n+3}U\right);$$

here, $d_{J+1,\tau,s} \le \exp(c_6\nu^{2n+3}U)$ is a common denominator of all numbers

$$\Delta_{\lambda_0}^{(\tau_0')}\left(\frac{s}{p^{J+1}}\right), \quad (\lambda_0 = 0, 1, \ldots, L_0 - 1;\ \tau_0' = 0, 1, \ldots, \tau_0).$$

Further, the conjugates of this number $\varphi_{J,(\tau)}(s/p)$ have absolute values at most $\exp(c_{12}\nu^{2n+3}U)$. Since the degree of $\varphi_{J,(\tau)}(s/p)$ over $\mathbb{Q}$ is not more than $Dp^n \le Dp_0^n$, it follows that the height of it is not more than $\exp(c_{13}\nu^{2n+3}U)$. Hence, either $\varphi_{J,(\tau)}(s/p) = 0$ or

$$\left|\varphi_{J,(\tau)}\left(\frac{s}{p}\right)\right| \ge \exp\left(-c_{10}\nu^{2n+3}U\right),$$

and lemma 10 has been proved.

We now proceed to prove the assertion of Step 2 for $J+1$. By lemma 9, we obtain from the induction hypothesis

$$|f_{J,(\tau)}(s)| \le \exp\left(-\tfrac{1}{2}\nu^{2n+5}U\right), \tag{10}$$

for $0 \le |\tau| \le p^{-J}T$ and $0 \le s \le p^J S - 1$, $(s, p) = 1$. Since

$$f_{J,(\tau)}(z) = (\log\alpha_1)^{-\tau_1}\ldots(\log\alpha_{n-1})^{-\tau_{n-1}}\sum_{(\lambda)}p^{(J)}(\lambda).$$
$$\cdot\left(\Delta_{\lambda_0}\left(\frac{z}{p^J}\right)e^{\lambda_n\beta_0 z}\right)^{(\tau_0)}(\alpha_1^{\gamma_1 z})^{(\tau_1)}\ldots(\alpha_{n-1}^{\gamma_{n-1}z})^{(\tau_{n-1})},$$

the functions $f_{J,(\tau)}$ satisfy the differential equations

$$f_{J,(\tau)}^{(m)} = \sum_{|\mu|=m}\frac{m!}{\mu_0!\ldots\mu_{n-1}!}\left[\prod_{j=1}^{n-1}(\log\alpha_j)^{\mu_j}\right]f_{J,(\tau+\mu)}.$$

It follows from (10) that

$$|f^{(m)}_{J,(\tau)}(s)| < \exp(-\tfrac{1}{3}\nu^{2n+5}U) \tag{11}$$

for all values of (τ), m, s with $0 \le |\tau| < \frac{1}{2}p^{-J}T$, $0 \le m \le \frac{1}{2}p^{-J}T$, $0 \le s \le p^J S - 1$ and $(s,p) = 1$. We apply lemma 2 to the above indicated functions $f_{J,(\tau)}$ with $t = [\frac{1}{2}p^{-J}T]$ and $\frac{1}{2}p^J S \le k \le p^J S$, using $R = 6r$, $r = p^J S$, $\delta = 1$. We have for $|z| = R$

$$\begin{aligned} |e^{\lambda_n \beta_0 z}\alpha_1^{\gamma_1 z}\dots\alpha_{n-1}^{\gamma_{n-1}z}| &\le \exp R|\lambda_1 \log\alpha_1 + \cdots + \lambda_n \log\alpha_n + \lambda_n \Lambda_0| \\ &\le \exp 6p^J S\left(\sum_{j=1}^n p^{-J}L_j \log A_j + p^{-J}\right) \\ &\le \exp(7n\nu^{2n+3}U). \end{aligned}$$

It follows with (8) that

$$|f_{J,(\tau)}|_R < \exp(c_{14}\nu^{2n+3}U).$$

Lemma 2 and inequalities (11) give

$$\begin{aligned} |f_{J,(\tau)}|_{2p^J S} &\le \exp\{(-\tfrac{1}{11}\nu^{2n+4} + c_4\nu^{2n+3})U\} + \exp\{(c_{15}\nu^{2n+4} - \tfrac{1}{3}\nu^{2n+5})U\} \\ &< \exp(-\tfrac{1}{12}\nu^{2n+4}U). \end{aligned}$$

Hence,

$$\left| f_{J,(\tau)}\left(\frac{s}{p}\right)\right| \le \exp(-\tfrac{1}{12}\nu^{2n+4}U)$$

for all $s \in \mathbb{Z}$ with $0 \le s \le p^{J+1}S$ and all (τ) with $0 \le |\tau| < \frac{1}{2}p^{-J}T$. We compare $f_{J,(\tau)}(s/p)$ with $\varphi_{J,(\tau)}(s/p)$; once more lemma 9 gives

$$\left| \varphi_{J,(\tau)}\left(\frac{s}{p}\right)\right| \le \exp(-\tfrac{1}{13}\nu^{2n+4}U)$$

for $0 \le s \le p^{J+1}S$ and $0 \le |\tau| < \frac{1}{2}p^{-J}T$.

By lemma 10, we see that $\varphi_{J,(\tau)}(s/p) = 0$ for $0 \le |\tau| < p^{-J-1}T$, $0 \le s \le p^{J+1}S$. From this point on, we only use these numbers for so far $(s,p) = 1$. Using our assumption $[K(\alpha_1^{1/p},\dots,\alpha_n^{1/p}): K] = p^n$, we can express $\varphi_{J,(\tau)}(s/p)$ on the basis $\{(\alpha_1^{l_1/p}\dots\alpha_n^{l_n/p}),\ (l_1,\dots,l_n) \in \{0,1,\dots,p-1\}^n\}$. We remark that the coefficients in this expression consist of those contributions of $\Sigma_{(\lambda)}$ for which $\lambda_j \equiv \lambda_j^0 \pmod p$, $(j = 1,\dots,n)$. Since each of these coefficients must be zero, we have

$\varphi^0_{J,(\tau)}(s/p)=0$, where $\varphi^0_{J,(\tau)}$ is obtained from $\varphi_{J,(\tau)}$ by replacing the sum $\Sigma_{(\lambda)}$ for $0\le\lambda_j\le L_j^{(J)}$ $(0\le j\le n)$, by the sum over $(\lambda)^{(J)}$ with $0\le\lambda_j\le L_j^{(J)}$ and $\lambda_j\equiv\lambda_j^0 \pmod p$, $(0\le j\le n)$. We choose $(\lambda_1^0,\ldots,\lambda_n^0)$ with $0=\lambda_j^0=p-1$, $(1\le j\le n)$, in such a way that at least one of the numbers $p^{(J)}(\mu_0,\lambda_1^0+\mu_1 p,\ldots,\lambda_n^0+\mu_n p)$ $(\mu_j=0,1,\ldots,L_j^{(J+1)},\ j=0,\ldots,n$, with $L_0^{(J+1)}=L_0$ and $L_j^{(J+1)}=[1/p(L_j^{(J)}-\lambda_j^0)]$, $j=1,\ldots,n)$ is non-zero. This is possible by the induction hypothesis. We denote by $p^{(J+1)}(\mu_0,\mu_1,\ldots,\mu_n)$ the thus obtained numbers. So we have $\psi_{J+1,(\tau)}(s)=0$ for $0\le|\tau|<p^{-J-1}T$, $0\le s\le p^{J+1}S-1$, $(s,p)=1$, where

$$\psi_{J+1,(\tau)}(s)=\sum_{\mu_0=0}^{L_0^{(J+1)}}\cdots\sum_{\mu_n=0}^{L_n^{(J+1)}}p^{(J+1)}(\mu)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\left(\frac{\lambda_n^0}{p}+\mu_n\right)^{\tau_0-\tau_0'}$$
$$\cdot\,\beta_0^{\tau_0-\tau_0'}p^{-\tau_0'(J+1)}\mathscr{L}_{(\tau')}\left(\mu_0,\lambda_1^0+\mu_1 p,\ldots,\lambda_n^0+\mu_n p,\frac{s}{p^{J+1}}\right)\alpha_1^{\mu_1 s}\ldots\alpha_n^{\mu_n s}.$$

For fixed $\tau_1,\ldots,\tau_{n-1}$ and s, write $\psi_{J+1,(\tau)}(s)$ into the form

$$\sum_{\mu_n=0}^{L_n^{(J+1)}}\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\left\{\left(\frac{\lambda_n^0}{p}+\mu_n\right)\beta_0\right\}^{\tau_0-\tau_0'}E_{\mu_n,\tau_0'}.$$

Since this expression is zero for all values of τ_0 with $0\le\tau_0<p^{-J-1}T-(\tau_1+\cdots+\tau_{n-1})$, it follows from lemma 5 that

$$\sum_{\mu_n=0}^{L_n^{(J+1)}}\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\mu_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}E_{\mu_n,\tau_0'}$$

is zero for the same τ_0. Hence, the numbers

$$\xi_{J+1,(\tau)}(s)\ (0\le|\tau|<p^{-J-1}T;0\le s\le p^{J+1}S-1,(s,p)=1)$$

are zero, where

$$\xi_{J+1,(\tau)}(s)=\sum_{(\mu)}p^{(J+1)}(\mu)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\mu_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}p^{-\tau_0'(J+1)}$$
$$\cdot\,\mathscr{L}_{(\tau')}\left(\mu_0,\lambda_1^0+\mu_1 p,\ldots,\lambda_n^0+\mu_n p,\frac{s}{p^{J+1}}\right)\alpha_1^{\mu_1 s}\ldots\alpha_n^{\mu_n s}.$$

We proceed to prove that the numbers

$$\varphi_{J+1,(\tau)}(s)=\sum_{(\mu)}p^{(J+1)}(\mu)\sum_{\tau_0'=0}^{\tau_0}\binom{\tau_0}{\tau_0'}\mu_n^{\tau_0-\tau_0'}\beta_0^{\tau_0-\tau_0'}$$
$$\cdot\,p^{-\tau_0'(J+1)}\mathscr{L}_{\tau'}\left(\mu_0,\mu_1,\ldots,\mu_n,\frac{s}{p^{J+1}}\right)\alpha_1^{\mu_1 s}\ldots\alpha_n^{\mu_n s}$$

are zero for the same values of (τ) and s.

Since

$$\mu_j + \mu_n\beta_j = \frac{1}{p}\{\lambda_j^0 + \mu_j p + (\lambda_n^0 + \mu_n p)\beta_j - (\lambda_j^0 + \lambda_n^0\beta_j)\}$$

we have

$$\mathscr{L}_{(\tau')}(\mu, z) = \left(\prod_{j=1}^{n-1} p^{-\tau_j}\right) \sum_{\kappa_1=0}^{\tau_1} \cdots \sum_{\kappa_{n-1}=0}^{\tau_{n-1}} \left(\prod_{j=1}^{n-1} \binom{\tau_j}{\kappa_j} (-1)^{\tau_j-\kappa_j}(\lambda_j^0 + \lambda_n^0\beta_j)^{\tau_j-\kappa_j}\right).$$
$$\cdot \mathscr{L}_{\tau_0',\kappa_1,\dots,\kappa_{n-1}}(\mu_0, \lambda_1^0 + \mu_1 p, \dots, \lambda_n^0 + \mu_n p, z)$$

so that each number $\varphi_{J+1,(\tau)}(s)$ can be written as a linear combination of the numbers $\xi_{J+1,(\tau)}(s)$; hence, they are zero for the mentioned values of (τ) and s.

Step 3. *Contradiction.* Let J_0 be the integer such that $p^{J_0} \le T - 1 < p^{J_0+1}$. Since

$$\max_{1\le j\le n} L_j < \nu^{2n}\Omega \log \Omega < \frac{1}{p}(T-1),$$

we have $L_j^{(J_0)} = 0$ $(j = 1, \dots, n)$, so that the $p^{(J_0)}(\lambda)$ have the form $p^{(J_0)}(\lambda_0, 0, \dots, 0)$. Hence,

$$\varphi_{J_0,(0)}(z) = \sum_{\lambda_0=0}^{L_0-1} p^{(J_0)}(\lambda_0, 0, \dots, 0)\Delta_{\lambda_0}(zp^{-J_0})$$

becomes a polynomial; it has zeros for $z = s$, $0 \le s \le p^{J_0}S - 1$, $(s, p) = 1$. The number of these zeros exceeds $ST/(2p)$ which is more than the degree of $\varphi_{J_0,(0)}$; so $\varphi_{J_0,(0)}$ must be identically zero. But the polynomials $\Delta_{\lambda_0}(\lambda_0 = 0, 1, \dots, L_0 - 1)$ are linearly independent; hence all $p^{(J_0)}(\lambda_0, 0, \dots, 0)$ are zero, in contradiction to their construction.

5. Proof of theorem 1

We first use the arguments of [3], §4 to prove the following result.

PROPOSITION 2: *Let $n \ge 1$, $D \ge 1$ and $p \ge 2$ be given integers, with p prime. There exists an effectively computable number $C_3 > 0$, depending only on n, D and p, with the following property. Let K be a* *number field, containing the* p^{th} *roots of unity, of degree at most* D *over* $\mathbb{Q}$; *let* $\alpha_1, \ldots, \alpha_n$ *be nonzero elements of* K; *for* $1 \le j \le n$, *let* $\log \alpha_j$ *be any determination of the logarithm of* α_j, *and let* $A_j \ge 6$ *be an upper bound for the height of* α_j *and for* $\exp|\log \alpha_j|$. *We assume that* $\alpha_1, \ldots, \alpha_n$ *are arranged in such a way that* $A_1 \le A_2 \le \cdots \le A_n$. *Further, let* $\beta_1, \ldots, \beta_n$ *be elements of* K *of heights at most* B *with* $B \ge \operatorname{Log} A_n$. *Assume that the linear form*

$$\Lambda_1 = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

does not vanish.

Then there exist an integer ν, $1 \le \nu \le n+1$, *non-zero elements* $\alpha'_1, \ldots, \alpha'_\nu, \beta'_1, \ldots, \beta'_\nu$ *of* K, *and determinations* $\log \alpha'_1, \ldots, \log \alpha'_\nu$ *of the logarithms of* $\alpha'_1, \ldots, \alpha'_\nu$, *such that*

(*a*) Λ_1 *is equal to*

$$\Lambda'_1 = \beta'_1 \log \alpha'_1 + \cdots + \beta'_\nu \log \alpha'_\nu,$$

(*b*) *the field* $K((\alpha'_1)^{1/p}, \ldots, (\alpha'_\nu)^{1/p})$, *where* $(\alpha'_j)^{1/p} = \exp((1/p)\log \alpha'_j)$, $(1 \le j \le \nu)$, *has degree* p^ν *over* K,

(*c*) *the height of* α'_j *and* $\exp|\log \alpha'_j|$ *are at most* $A_{j+n-\nu}^{C_3}$, *where (in case of* $\nu = n+1$) A_0 *should be read as* 6, *and the heights of the numbers* β'_j $(j = 1, \ldots, \nu)$ *are at most* B^{C_3}.

REMARK: When $\beta_1, \ldots, \beta_n$ are rational, we will find $\beta'_1, \ldots, \beta'_\nu$ rational. When $\log \alpha_1, \ldots, \log \alpha_n$ are the principle values of the logarithm, we will find $\log \alpha'_1, \ldots, \log \alpha'_\nu$ principle valued too.

PROOF: The numbers $c_{16}, \ldots, c_{23}$ will depend only on n, D and p. Let $h \ge 1$ be the greatest integer such that $\exp(p^{-h} i\pi) \in K$; then $p^h < c_{16}$. Note that $\exp(p^{-h} i\pi)$ is an algebraic number of height 1; note also that $\exp(p^{-(h+1)} i\pi)$ generates an extension of K of degree p since K contains the p^{th} roots of unity. It clearly suffices to prove the proposition for linear forms Λ_1 in which $\alpha_1 = \exp(p^{-h} i\pi)$ and $\log \alpha_1 = p^{-h} i\pi$, using $A_1 = 6$, when we show that $\nu \le n$ instead of $\nu \le n+1$ and (in case of $\nu = n$) that the height of α'_1 and $\exp|\log \alpha'_1|$ are at most 6^{C_3}.

We shall prove this by induction. For $n = 1$, our statement is satisfied by our choice of α_1. Let $n \ge 2$ and suppose that the statement is true for all linear forms of the indicated type having $n-1$ logarithms (we shall refer to this assumption as "the induction hypothesis on n"). Let m be the greatest integer with the property that, for

$\mu = 1, \ldots, m$, $\alpha_\mu^{1/p}$ generates an extension of $K'_\mu = K(\alpha_1^{1/p}, \ldots, \alpha_{\mu-1}^{1/p})$ of degree p, where K'_1 should be read as K; remark that $m \geq 1$ by our choice of α_1. For $m = n$, the statement holds trivially. Consequently, we may assume that $m < n$, and moreover, we may suppose that the statement is true for all linear forms of the indicated type, having n logarithms, and with m replaced by m' with $m < m' \leq n$ (we shall refer to this assumption as "the induction hypothesis on m"). By lemma 6, we have

$$\alpha_{m+1} = \alpha_1^{r_1} \ldots \alpha_m^{r_m} \gamma^p,$$

for some $\gamma \in K$ and some integers $r_1, \ldots, r_m$ with $0 \leq r_j < p$ $(1 \leq j \leq m)$. We construct, as far as possible, a sequence $\gamma_1 = \gamma, \gamma_2, \gamma_3, \ldots$ of elements in K such that

$$\gamma_l = \alpha_1^{r_{l,1}} \ldots \alpha_m^{r_{l,m}} \gamma_{l+1}^p,$$

where the integers $r_{l,j}$ satisfy $0 \leq r_{l,j} < p$. Clearly, we have

$$\alpha_{m+1} = \alpha_1^{s_{l,1}} \ldots \alpha_m^{s_{l,m}} \gamma_l^{p^l}, \tag{12}$$

where the $s_{l,j}$ are integers with $0 \leq s_{l,j} < p^l$. The denominator of $\alpha_{m+1}\alpha_1^{-s_{l,1}} \ldots \alpha_m^{-s_{l,m}} = \gamma_l^{p^l}$ is at most

$$A_{m+1} \cdot A_m^{m(p^l-1)} < A_{m+1}^{np^l},$$

so that, by lemma 7, the leading coefficient in the minimal polynomial of γ_l is at most A_{m+1}^{nD}. Since each conjugate of

$$\gamma_l = \alpha_{m+1}^{1/p^l} \alpha_1^{-s_{l,1}/p^l} \ldots \alpha_m^{-s_{l,m}/p^l}$$

has absolute value at most

$$(A_{m+1} + 1)^{1/p^l} \cdot (A_m + 1)^{m(p^l-1)/p^l} < (A_{m+1} + 1)^n,$$

we obtain that the height of γ_l does not exceed

$$[2A_{m+1}^{nD}(A_{m+1} + 1)^n]^D < A_{m+1}^{c_{17}}.$$

Remark, that it follows that

$$\exp|\mathrm{Log}\, \gamma_l| \leq (A_{m+1}^{c_{17}} + 1)e^{2\pi} \leq A_{m+1}^{c_{18}}.$$

We deduce from (12) the existence of some $s'_{l,1} \in \mathbb{Z}$ such that

$$s'_{l,1} \log \alpha_1 + s_{l,2} \log \alpha_2 + \cdots + s_{l,m} \log \alpha_m - \log \alpha_{m+1} + p^l \,\mathrm{Log}\, \gamma_l = 0. \tag{13}$$

We have from (13):

$$|s'_{l,1}| \leq p^l[m \text{ Log } A_{m+1} + |\text{Log } \gamma_l|]p^h \leq c_{19}p^l \text{ Log } A_{m+1};$$

in particular $|s'_{l,1}| < c_{19}p^lB$.

Define

$$H = [4^{(n+1)^2}D^{2n+2}c_{17} \text{ Log } A_{m+1}]^{(2n+3)^2}.$$

We distinguish two cases, according as the sequence $\gamma_1, \gamma_2, \ldots$ terminates at γ_l for some l with $p^l \leq H$ (case 1), or it does not (case 2).

Case 1. Assume that $\gamma_1, \gamma_2, \ldots$ terminates at γ_l with $p^l \leq H$. From (13), Λ_1 is equal to a linear form Λ_2 in $\log \alpha_1, \ldots, \log \alpha_n$, where $\log \alpha_{m+1}$ is replaced by $\text{Log } \gamma_l$, and where the coefficients are algebraic numbers in K having heights at most $B^{c_{20}}$; namely

$$\Lambda_2 = \sum_{j=1}^{m} \zeta_j \log \alpha_j + \beta_{m+1}p^l \text{ Log } \gamma_l + \sum_{j=m+2}^{n} \beta_j \log \alpha_j,$$

where $\zeta_1 = \beta_1 + \beta_{m+1}s'_{l,1}$ and $\zeta_j = \beta_j + \beta_{m+1}s_{l,j}$ $(2 \leq j \leq m)$. By the supposition that the sequence terminates at γ_l, we deduce from lemma 6 that $\gamma_l^{1/p}$ generates an extension of $K(\alpha_1^{1/p}, \ldots, \alpha_m^{1/p})$ of degree p. We recall that the height of γ_l and $\exp |\text{Log } \gamma_l|$ are at most $A_{m+1}^{c_{18}}$. Hence, replacing A_{m+1} by $A_{m+1}^{c_{18}}$, A_j by $\max(A_j, A_{m+1}^{c_{18}})$ $(m+1 \leq j \leq n)$ and B by $\max(B^{c_{20}}, c_{18} \text{ Log } A_{m+1})$, the induction hypothesis on m gives the desired result (remark, that α_1 remains unchanged, and that the induction on m is finite, since $1 \leq m \leq n$).

Case 2. Let l be the least integer with $p^l > H$. Since $\log \alpha_1, \ldots, \log \alpha_{m+1}$, $\text{Log } \gamma_l$ are linearly dependent by (13), we know that also $\text{Log } \alpha_1, \ldots, \text{Log } \alpha_{m+1}$, $\text{Log } \gamma_l$ are linearly dependent. We deduce from lemma 8 that there are rational integers b_0, b_1^0, $b_2, \ldots, b_{m+1}$, not all zero and of absolute values at most H, such that

$$b_1^0 \text{ Log } \alpha_1 + b_2 \text{ Log } \alpha_2 + \cdots + b_{m+1} \text{ Log } \alpha_{m+1} + b_0 \text{ Log } \gamma_l = 0.$$

It follows that there exists a rational integer b_1 with

$$|b_1| < H(m \text{ Log } A_{m+1} + |\text{Log } \gamma_l|)p^h < (\text{Log } A_{m+1})^{c_{21}} \leq B^{c_{21}}$$

for which

$$(14) \qquad b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_{m+1} \log \alpha_{m+1} + b_0 \text{ Log } \gamma_l = 0.$$

Eliminating $\operatorname{Log} \gamma_l$ from (13) and (14), we find

$$b_1' \log \alpha_1 + b_2' \log \alpha_2 + \cdots + b_{m+1}' \log \alpha_{m+1} = 0, \tag{15}$$

where $b_1' = p^l b_1 - b_0 s_{l,1}'$; $b_j' = p^l b_j - b_0 s_{l,j} (2 \leq j \leq m)$; $b_{m+1}' = p^l b_{m+1} + b_0$. We see that

$$|b_j'| < B^{c_{22}}, \quad (1 \leq j \leq m+1).$$

We claim that at least one b_j', $(2 \leq j \leq m+1)$ is not zero; in fact, if $b_{m+1}' = 0$, we deduce from its definition that (because $p^l > H \geq |b_0|$) $b_{m+1} = b_0 = 0$; in this case clearly $b_j' = p^l b_j$, $(1 \leq j \leq m)$, and the claim follows by the observation that now $b_2 = 0, \ldots, b_m = 0$ imply $b_1^0 = 0$. Let j_0, with $2 \leq j_0 \leq m+1$, be such that $b_{j_0}' \neq 0$. Using (15), we express the linear form Λ_1 as

$$\beta_1' \log \alpha_1 + \cdots + \beta_n' \log \alpha_n,$$

where $\beta_j' = \beta_j - (\beta_{j_0}/b_{j_0}') b_j'$, $(1 \leq j \leq m+1)$ and $\beta_j' = \beta_j$ $(m+2 \leq j \leq n)$. Since $\beta_{j_0}' = 0$, and since the height of the numbers β_j', $(1 \leq j \leq n)$ is not more than $B^{c_{23}}$, the statement follows by the induction hypothesis on n. Remark, that α_1 remains unchanged and that the order remains intact. In both cases, our statement has been proved. This completes the proof of proposition 2.

Finally, we give the proof of Theorem 1. We may assume without loss of generality that $A_1 \leq A_2 \leq \cdots \leq A_n \leq e^B$, and that $\Lambda \neq \beta_0$. The numbers $c_{24}, \ldots, c_{28}$ will depend only on n and d. Let p be any fixed prime, say $p = 2$, and take $K = \mathbb{Q}(\beta_0, \beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n, \zeta)$ where $\zeta \neq 1$ is a p^{th} root of unity. Then K has degree $D \leq d^{2n+1} p \leq c_{24}$. We first use proposition 2 with the linear form $\Lambda_1 = \Lambda - \beta_0$, to see that for some ν with $1 \leq \nu \leq n+1$ there exist non-zero elements $\alpha_1', \ldots, \alpha_\nu'$, $\beta_1', \ldots, \beta_\nu'$ in K, and determinations of $\log \alpha_1', \ldots, \log \alpha_\nu'$, such that

$$\Lambda = \beta_0 + \beta_1' \log \alpha_1' + \cdots + \beta_\nu' \log \alpha_\nu'$$

and such that the field $K((\alpha_1')^{1/p}, \ldots, (\alpha_\nu')^{1/p})$ has degree p^ν over K; moreover, the height of α_j' and $\exp |\log \alpha_j'|$ are at most $A_{j+n-\nu}^{c_{25}}$ where (in case of $\nu = n+1$) A_0 should be read as 6, and the heights of the numbers $\beta_0, \beta_1', \ldots, \beta_\nu'$ are at most $B^{c_{26}}$. Remark, that the product of the logarithms of the bounds $A_{j+n-\nu}^{c_{25}}$ $(j = 1, \ldots, \nu)$ is at most $c_{27} \Omega$.

Subsequently, we apply proposition 1 to the linear form

$$-(\beta'_\nu)^{-1}\beta_0-(\beta'_\nu)^{-1}\beta'_1\log\alpha'_1-\cdots-(\beta'_\nu)^{-1}\beta'_{\nu-1}\log\alpha'_{\nu-1}-\log\alpha'_\nu$$

in which the coefficients have heights at most $B^{c_{28}}$. This gives a lower bound of the desired form for $|-(\beta'_\nu)^{-1}\Lambda|$. As β'_ν is non-zero, we have $|\beta'_\nu|>(B^{c_{28}}+1)^{-1}$, so that inequality (2) can be deduced from the above lower bound by multiplying by $|\beta'_\nu|$.

6. A final remark

As we said already in the introduction, it is possible to replace the factor Log Ω by Log Ω' in the conclusion of theorem 1. To do so, one only has to make some changes in proposition 1 and its proof (§4). Namely, use the new parameters:

$$L'_j=[\nu^{2n}\Omega\ \mathrm{Log}\ \Omega'/\mathrm{Log}\ A_j],\quad (1\le j\le n);$$
$$T'=[\nu^{2n+2}\Omega\ \mathrm{Log}\ \Omega'];$$
$$U'=\Omega(\mathrm{Log}\ \Omega')(\mathrm{Log}\ B+\mathrm{Log}\ \Omega).$$

Further, in the induction of step 2, the number J should satisfy $p^J\le L'_n+1$. At the end of the induction, we arrive at functions $\varphi_{J_0,(\tau)}$ for which $L_n^{(J_0)}=0$. We replace step 3 by usual arguments (see for example [2], beginning of §4).

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Added in proof

As pointed out to us by A. Bijlsma and confirmed by Bundschuh, there is a gap in the proof of Satz 2a of [5], which also reflects to [6]. See a forthcoming paper by Bijlsma.