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**A RESULT ON THE INTEGRAL CHOW RING  
OF A GENERIC PRINCIPALLY POLARIZED  
COMPLEX ABELIAN VARIETY OF DIMENSION FOUR**

Charles Barton and C. H. Clemens

**0. Introduction**

In this paper, we wish to show that a certain positive algebraic two-cycle on a generic abelian variety of dimension four is not, in general, represented by an *effective* algebraic subvariety. This problem was suggested by the fact that this cycle is effectively representable if the abelian variety is the Jacobian of a curve or the intermediate Jacobian of a cubic threefold.

The method of proof is via a degeneration argument – we construct (in some detail) the “generic” degeneration of a family of principally polarized abelian varieties of dimension four, then we see what the existence of the effective two-cycle would imply in the limit.

**1. A “generic” degeneration**

Our purpose in this section is to construct a “generic” proper mapping of a holomorphic manifold  $J$  onto the unit disc  $\Delta$

$$(1.1) \quad \pi : J \rightarrow \Delta$$

such that:

- (i) if  $z \neq 0$ ,  $J_z = \pi^{-1}(z)$  is a principally polarized abelian variety [4; Chapter 1] of dimension four;
- (ii)  $J_0$  is non-singular except that it crosses itself transversely along  $M$ , a principally polarized abelian variety of dimension three;
- (iii)  $\tilde{J}_0$ , the normalization of  $J_0$ , is a bundle over  $M$  with fibre  $\mathbb{P}_1$  (complex projective one-space).

To accomplish this, we begin with the set

$$(1.2) \quad H = \left\{ \left( \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}; \sigma, \tau \right) \in \mathbf{C}^3 \times \mathbf{C}^2 : \sigma\tau < 1 \right\}.$$

Let

$$(1.3) \quad A : \Delta \rightarrow GL(3; \mathbf{C})$$

be a holomorphic mapping of the unit disc into the group of invertible  $3 \times 3$  matrices over the complex numbers such that:

- (i)  $A(z)$  is symmetric for each  $z \in \Delta$ ;
- (ii) (imaginary part of  $A(z)$ ) is positive definite for each  $z \in \Delta$ .

Let

$$E_1, E_2, E_3$$

be the standard basis of  $\mathbb{R}_3$  and

$$A_1, A_2, A_3$$

the columns of  $A(z)$ . Let

$$L(z) = \{\Sigma m_j E_j + n_j A_j : m_j, n_j \in \mathbf{Z}\}.$$

Also let

$$(1.4) \quad B : \Delta \rightarrow \mathbf{C}^3$$

be any holomorphic mapping,

$$B(z) = \begin{bmatrix} b_1(z) \\ b_2(z) \\ b_3(z) \end{bmatrix}.$$

Using  $L(z)$  and  $B(z)$  we define an equivalence relation on  $H$  as follows. We put

$$(u; \sigma, \tau) \sim (u'; \sigma', \tau')$$

if

- (i)  $\sigma \cdot \tau = \sigma' \cdot \tau' = z \in \Delta$ ;
- (ii)  $(u - u') = \Sigma (m_j E_j + n_j A_j) \in L(z)$ ;
- (iii)  $\sigma = e^{2\pi i (\Sigma n_j b_j(z))} \cdot \sigma'$  and  $\tau = e^{2\pi i (-\Sigma n_j b_j(z))} \cdot \tau'$ .

Let

$$K = H / \{\sim\}.$$

Then  $K$  is a complex manifold and we have a natural mapping

$$(1.5) \quad \kappa : K \rightarrow \Delta.$$

$$\{(u; \sigma, \tau)\} \mapsto \sigma \cdot \tau.$$

If  $z \neq 0$ ,  $\kappa^{-1}(z)$  is a  $\mathbf{C}^*$ -bundle over a principally polarized abelian variety

$$(1.6) \quad M_z = \mathbf{C}^3/L(z)$$

of dimension three.  $\kappa^{-1}(0)$  is the union of two (mutually dual) line bundles over  $M = \mathbf{C}^3/L(0)$ .

The idea now, of course, is to construct  $J$  as a quotient of  $K$ . On  $K$  then, we define

$$\{(u; \sigma, \tau)\} \sim \{(u'; \sigma', \tau')\}$$

whenever

- (i)  $\sigma \cdot \tau = \sigma' \cdot \tau' = z$ ;
- (ii)  $\sigma' \tau = 1$ ;
- (iii)  $(u - u') = B(z)$ .

Then “ $\sim$ ” generates an equivalence relation and we can define

$$J = K/\{\sim\}$$

Clearly,  $J$  is smooth and the mapping  $\kappa$  in (1.5) induces a proper mapping

$$\pi : J \rightarrow \Delta.$$

Of the assertions (i)–(iii) following (1.1), (ii) and (iii) are clear for the mapping  $\pi$  we have just constructed. Assertion (i) is, in fact, only correct for sufficiently small values of  $z$ .

We will check this last fact by computing the period matrix for  $J_z = \pi^{-1}(z)$ . Let

$$\ell(\sigma) = \frac{1}{2\pi i} \log \sigma.$$

Then we can make a mapping

$$(1.7) \quad \begin{aligned} \kappa^{-1}(z) &\rightarrow \mathbf{C} \times \mathbf{C}^3 \\ \{(u; \sigma, \tau)\} &\mapsto (\ell(\sigma); u) \end{aligned}$$

which is well-defined modulo integral combinations of the vectors

$$(1.8) \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ E_j \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} b_j \\ A_j \end{bmatrix}.$$

(See conditions (ii) and (iii) for the equivalence relation defining  $K$ .) To

pass from  $\kappa^{-1}(z)$  to  $J_z$  we have introduced a second equivalence relation. Since  $z \neq 0$ , our equivalence is generated by the conditions

(i)  $\sigma/\sigma' = z$ ,

(ii)  $(u - u') = B(z)$ ;

in other words, the mapping (1.7) induces a mapping

$$J_z \rightarrow \mathbf{C} \times \mathbf{C}^3$$

which is well-defined modulo integral combinations of the vectors (1.8) and the vector

$$(1.9) \quad \begin{bmatrix} \ell(z) \\ B(z) \end{bmatrix}.$$

So  $J_z$  is simply the quotient of  $\mathbf{C}^4$  by the subgroup generated by the vectors (1.8) and (1.9). If  $z$  is sufficiently small, the vector (1.9) is clearly linearly independent (over  $\mathbf{R}$ ) from the others, so

$$(1.10) \quad J_z = \text{complex torus with period matrix } \Omega(z)$$

where

$$\Omega(z) = \begin{bmatrix} \ell(z) & {}^t B(z) \\ B(z) & A(z) \end{bmatrix}.$$

Also, if  $z \neq 0$  is sufficiently small, the matrix

$$(\text{imaginary part of } \Omega(z))$$

is positive definite. This means that  $J_z$  does indeed have the structure of a principally polarized abelian variety. From here on, we assume that we have adjusted the parameter  $z$  so that this is the case for *all*  $z \in (\Delta - \{0\})$ . We call the family (1.1) a *generic degeneration* since the varieties  $J_0$  constructed as above make up the ‘‘largest component’’ of a natural compactification of the moduli space of principally polarized abelian varieties of dimension four [5].

Finally we will need a family of theta-functions on the varieties  $J_z$ . We define these as functions on  $H$  (see (1.2)), but for  $z \neq 0$  they will just give the usual theta functions of characteristic

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

on  $J_z$ . Let  $N$  be a positive integer and let

$$n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

be a triple of integers such that  $0 \leq n_j < N$  for  $j = 1, 2, 3$ . For  $u \in \mathbb{C}^3$  define

$$(1.11) \quad \theta^{n,N}(u; z) = \sum_{m \in \mathbb{Z}^3} e^{\pi i (Nm+n)[A(z)(m+\frac{1}{2})+2u]}$$

where  $A(z)$  is as in (1.3). Then for  $0 \leq n_0 < N$ , and  $(u; \sigma, \tau) \in H$  (see (1.2)), define

$$(1.12) \quad \theta^{n_0, n, N}(u; \sigma, \tau) = \sum_{m_0 \in \mathbb{Z}} \left[ \sigma^{(N(m_0+1)+n_0)} (\sigma\tau)^{(N(m_0(m_0+1)/2)+n_0(m_0+1))} \cdot \theta^{n,N} \left( u + \left[ m_0 + \frac{1}{2} + \frac{n_0}{N} \right] B(\sigma\tau); \sigma\tau \right) \right]$$

where  $B(z)$  is as in (1.4). From the definition itself, nothing is clear, not even the convergence of the series. Assume absolute convergence uniform on compact subsets of  $H$ . Then on the subset of  $H$  given by  $\sigma\tau = 0$ , the series in (1.12) reduces to

(1.13) (i)

$$\theta^{n,N} \left[ u - \frac{B(0)}{2}; 0 \right] + \sigma^N \theta^{n,N} \left[ u + \frac{B(0)}{2}; 0 \right] + \tau^N \theta^{n,N} \left[ u - \frac{3B(0)}{2}; 0 \right]$$

if  $n_0 = 0$ ;

(ii)

$$\sigma^{n_0} \theta^{n,N} \left[ u + \left[ \frac{n_0}{N} - \frac{1}{2} \right] B(0); 0 \right] + \tau^{(N-n_0)} \theta^{n,N} \left[ u + \left[ \frac{n_0}{N} - \frac{3}{2} \right] B(0); 0 \right]$$

if  $n_0 \neq 0$ .

Now, to check convergence, we use the relations  $\sigma\tau = z$  and  $\sigma = e^{2\pi i u_0}$ , which allow us to rewrite (1.12) as follows:

(1.14)

$$\theta^{n_0, n, N}(\tilde{u}; \sigma, \tau) = \sigma^{N/2} z^{n_0} e^{-\pi i (N(1/2+n_0/N)^2 \ell(z))} \sum_{\tilde{m}} e^{\pi i (N\tilde{m}+\tilde{n})[\Omega(z)(\tilde{m}+N^{-1}\tilde{n})+2\tilde{u}]}$$

where

$$\tilde{m} = \begin{bmatrix} m_0 + 1/2 \\ (m) \end{bmatrix}, m_0 \in \mathbb{Z}, m \in \mathbb{Z}^3,$$

$$\tilde{n} = \begin{bmatrix} n_0 \\ (n) \end{bmatrix} \in \{0, \dots, (N-1)\}^4,$$

and

$$\tilde{u} = \begin{bmatrix} u_0 \\ (u) \end{bmatrix} \in \mathbb{C}^4.$$

Now on a set

$$(1.15) \quad \begin{aligned} |\sigma| = \delta > 0, \quad |\tau| = \epsilon > 0, \\ \tilde{u} \in (\text{compact subset of } \mathbb{C}^4), \end{aligned}$$

the series (1.14) is absolutely and uniformly convergent – this is an immediate corollary of the proof of the uniform and absolute convergence of the Fourier series of  $N$ -th order theta-functions [2; page 96]. So the series (1.12) converges absolutely and uniformly on sets (1.15) and so on any compact subset of  $H$ .

Indeed, in the formation (1.14), the functions

$$\theta^{n_0, n, N}(\tilde{u}; z) = \sigma^{-N/2} \theta^{n_0, n, N}(u; \sigma, \tau)$$

for  $\sigma\tau = z \neq 0$  give a basis for the  $N$ -th order theta-functions on  $J_z$  with characteristic

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Also these functions are invariant under the substitution

$$\ell(z) \mapsto (\ell(z) + 1);$$

thus the zero set of the function (1.12) on

$$(1.16) \quad (H - \{(u; \sigma, \tau) : \sigma\tau = 0\})$$

is invariant with respect to the identifications used to define

$$(J - J_0)$$

as a quotient space of (1.16). Also from the formulas (1.13) it is clear that the zero set of a function (1.12) in  $H$  is simply the closure of its zero set in (1.16). These two facts imply that the zero set of (1.12) in  $H$  is invariant with respect to the identifications used to define  $J$  as a quotient space of  $H$  and so defines a divisor

$$(1.17) \quad \Theta^{n_0, n, N}$$

on  $J$ . The linear system spanned by the divisors (1.17) has projective dimension

$$N^4 - 1.$$

The rest of this section will be devoted to the study of this linear system, which we denote by

$$(1.18) \quad \mathcal{D}_N.$$

First of all, the formulas (1.13) immediately imply that

$$J_z \not\subseteq D$$

for any  $D \in \mathcal{D}_N$  and any  $z \in \Delta$ . Thus the algebraic cycle (with multiplicity)

$$(J_z \cdot D)$$

always makes sense and for any  $z \in \Delta$

$$(1.19) \quad (J_z \cdot D) \equiv (J_0 \cdot D)$$

in  $H_6(J; \mathbb{Z})$ . Also, by [1; §5-6], the semi-group  $[0, 1] \times \mathbb{R}$  acts on  $J$  in such a way that

- (i)  $\pi((r, \theta) \cdot x) = re^{2\pi i \theta} \pi(x)$  for all  $x \in J$  and  $(r, \theta) \in [0, 1] \times \mathbb{R}$ ;
- (ii)  $(r, \theta) \cdot x = x$  whenever  $x \in J_0$ .

So in  $H_6(J; \mathbb{Z}) = H_6(J_0; \mathbb{Z})$ :

$$(D \cdot J_z) \equiv (0, 0) \cdot (D \cdot J_z)$$

and therefore by (1.19)

$$(1.20) \quad (D \cdot J_0) \equiv (0, 0) \cdot (D \cdot J_z).$$

But we can explicitly compute the right-hand-side of (1.20). To do this, notice that the real coordinates

$$(\xi_0, \dots, \xi_3, \eta_0, \dots, \eta_3)$$

give a set of coordinates for  $J_z$  via the mapping

$$(\xi, \eta) \mapsto \sum_{j=0}^3 \xi_j E_j + \sum_{j=0}^3 \eta_j \Omega_j$$

where

$$E_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{etc.},$$

and  $\Omega_j = (j + 1)$ -st column of the period matrix  $\Omega(z)$ . Let  $\gamma_j$  be the element of  $H_1(J_z; \mathbb{Z})$  defined by fixing  $\xi_k$  for  $k \neq j$  and all the  $\eta_k$  and letting  $\xi_j$  run from 0 to 1. Similarly define  $\delta_j \in H_1(J_z; \mathbb{Z})$  by letting  $\eta_j$  run from 0 to 1. Then

$$(1.21) \quad \{\gamma_0, \dots, \gamma_3, \delta_0, \dots, \delta_3\}$$

is a basis for  $H_1(J_z; \mathbb{Z})$ . From the classical theory of theta-functions we have that if  $D \in \mathcal{D}_N$  and  $z \neq 0$ :

$$(1.22) \quad (D \cdot J_z) \equiv N \cdot \sum_{j=0}^3 \gamma_0 \times \delta_0 \times \dots \times \widehat{\gamma_j \times \delta_j} \times \dots \times \gamma_3 \times \delta_3$$



where “ $\times$ ” denotes Pontriagin product in the topological group  $J_z$  and “ $\hat{\phantom{x}}$ ” means “delete.”

Next let

$$\tilde{J}_0 = (\text{normalization of } J_0).$$

We then have a  $\mathbb{P}_1$ -bundle

$$(1.23) \quad \mu : \tilde{J}_0 \rightarrow M = M_0$$

with fibre coordinate  $\sigma$  (see (1.5)–(1.6)). The bundle  $\mu$  has distinguished sections

$$(1.24) \quad \begin{aligned} M^0 \text{ given by } \sigma = 0 \\ M^\infty \text{ given by } \sigma = \infty \end{aligned}$$

which are identified (via translation by  $B(0)$ ) under the normalization mapping

$$(1.25) \quad \nu : \tilde{J}_0 \rightarrow J_0.$$

Their common image, which we will denote simply by  $M$ , is the double variety of  $J_0$ .

Topologically, for  $z \neq 0$

$$J_z \cong \gamma_0 \times \delta_0 \times M$$

and the “collapsing” map

$$\begin{aligned} J_z &\rightarrow J_0 \\ x &\mapsto (0, 0) \cdot x \end{aligned}$$

is given by fixing  $0 \in \delta_0$  and collapsing

$$\gamma_0 \times \{0\} \times M$$

to  $\{u\} \subseteq M$  for each point  $u \in M$ . So using (1.20) and (1.22), we can explicitly describe

$$(1.26) \quad (D \cdot J_0) \in H_6(J_0; \mathbb{Z})$$

as follows. Abusing notation, let

$$(1.27) \quad \{\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3\}$$

denote the standard basis of  $H_1(M; \mathbb{Z})$  with respect to the period matrix  $A(0)$ . Then by (1.22) and our description of the collapsing map, we have that

$$(0, 0) \cdot (D \cdot J_z)$$

is given in  $H_6(J_0; \mathbb{Z})$  by

$$(1.28) \quad N \cdot \left[ \nu(M^0) + \nu \left( \mu^{-1} \left( \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k \right) \right) \right].$$

So by (1.20) the class of  $(D \cdot J_0)$  for  $D \in \mathcal{D}_N$  must be given by the same formula. If  $P$  denotes a fibre of  $\mu$ , then by (1.28) we have

$$(1.29) \quad \nu(P) \cdot D = N$$

which agrees with the formulas (1.13). (In (1.13) we can, for example, set  $\tau = 0$  and use  $\sigma$  as the fibre coordinate of  $\mu: \tilde{J}_0 \rightarrow M$ .)

Now if  $N = 1$ , then  $\mathcal{D}_N$  contains a unique divisor, which we will call

$$(1.30) \quad \Theta.$$

For  $z \neq 0$ ,  $(\Theta \cdot J_z)$  is called the *theta-divisor* of  $J_z$ .

**THEOREM 1.31:** *If the mappings  $A$  and  $B$  in (1.3) and (1.4) are chosen generically,  $\Theta$  is smooth in a neighborhood of its intersection with  $J_0$ . Also for  $z$  near 0,  $\Theta$  meets  $J_z$  transversely.*

**PROOF:** Let

$$\tilde{\Theta}_0 \subseteq \tilde{J}_0$$

be such that  $\nu(\tilde{\Theta}_0) = (\Theta \cdot J_0)$ . By elementary properties of analytic varieties, the theorem will be proved if we can show that  $\tilde{\Theta}_0$  is a smooth subvariety (of multiplicity one) in  $\tilde{J}_0$  which intersects  $M^0$  and  $M^\infty$  transversely. By (1.13) (i),  $(\Theta \cdot J_0)$  is given by the zero set of

$$(1.32) \quad \theta \left( u - \frac{B(0)}{2} \right) + \sigma\theta \left( u + \frac{B(0)}{2} \right) + \tau\theta \left( u - \frac{3B(0)}{2} \right)$$

where  $\theta(u) = \theta^{0,1}(u; 0)$ . So

$$\tilde{\Theta}_0 \cap (\tilde{J}_0 - M^\infty)$$

is given by setting  $\tau = 0$  in (1.32) and looking at the zero set of the resulting function. If  $(u', \sigma')$  is a singular point of this zero set, then

$$(i) \quad \theta \left( u' + \frac{B(0)}{2} \right) = \theta \left( u' - \frac{B(0)}{2} \right) = 0,$$

and for  $j = 1, 2, 3$ :

$$(ii) \quad \sigma' \frac{\partial \theta}{\partial u_j} \left( u' + \frac{B(0)}{2} \right) = -\frac{\partial \theta}{\partial u_j} \left( u' - \frac{B(0)}{2} \right).$$

If  $A(0)$  is chosen generically, there is no common zero of  $\theta(u)$  and  $(\partial\theta/\partial u_j)(u)$ ,  $j = 1, 2, 3$ . Otherwise, for example, the Riemann singularity theorem would imply that every curve of genus three is

hyperelliptic. So, for general  $A(0)$ , the *Gauss map*

$$g : (\text{zero set of } \theta \text{ in } M) \rightarrow \mathbb{P}_2$$

$$u \mapsto \left[ \frac{\partial \theta}{\partial u_j}(u) \right]_{j=1,2,3}$$

is a morphism and is surjective (recall that  $M$  is a Jacobian). But then one computes immediately that

$$\{u' - u'' : g(u') = g(u'')\} \subseteq M$$

is a subvariety of dimension  $\leq 2$ . If we choose  $B(0)$  outside this subvariety (and  $A(0)$  as above) then (i) and (ii) have no common solutions  $(u', \sigma') \in (\tilde{J}_0 - M^\infty)$ . Also by (1.32) (with  $\tau \equiv 0$ ),  $\tilde{\Theta}_0$  meets  $M^0$  transversely whenever  $\theta(u)$  and the  $(\partial\theta/\partial u_j)(u)$  have no common zeros. Putting  $\sigma \equiv 0$  in (1.32), the analogous argument works for  $(\tilde{\Theta}_0 \cap (\tilde{J}_0 - M^0))$ . This proves the first statement of Theorem 1.31. The second statement then follows from the fact that  $(\Theta \cap J_z)$  is given locally by the equation  $\sigma = z$  or by the equation  $\sigma\tau = z$ .

**THEOREM 1.33:** *Suppose  $N \geq 3$  and  $B(0) \neq 0$  in  $M$ . Let*

$$F_N : J \rightarrow \mathbb{P}_{(N^4-1)}$$

*be the mapping defined by the linear system  $\mathcal{D}_N$  in (1.18). The system  $\mathcal{D}_N$  has no basepoints so that  $F_N$  is a regular mapping. In fact, the mapping*

$$G_N : J \rightarrow \mathbb{P}_{(N^4-1)} \times \Delta$$

$$x \rightarrow (F_N(x), \pi(x))$$

*is an embedding.*

**PROOF:** Except along  $J_0$  this is a standard classical theorem. The same classical theorem says that the linear system spanned by the divisors of the functions (1.12) in  $M_z$  gives an embedding of  $M_z$ . Applying this for  $z = 0$  and the formulas (1.13) (i), it is clear that  $F_N$  embeds  $M \subseteq J_0$  in  $\mathbb{P}_{(N^4-1)}$ . To show that  $G_N$  is also an immersion at points of  $M \subseteq J$ , it suffices to note that, given  $u' \in M$ , there exists by (1.13) (ii) a divisor in  $\mathcal{D}_N$  which is smooth and tangent to  $\{(u; \sigma, \tau) : \sigma = 0\}$  at  $u'$  and which contains  $M$ , as well as a divisor which is smooth and tangent to  $\{(u; \sigma, \tau) : \tau = 0\}$  at  $u'$  and which contains  $M$ . (We use again that  $N \geq 3$ .) Next, recall that to study the linear system cut out by  $\mathcal{D}_N$  on  $(J_0 - M)$  we can set  $\tau = 0$  in (1.13) and use  $\sigma$  as the fibre coordinate of the  $\mathbb{C}^*$ -bundle

$$(1.34) \quad \mu : (J_0 - M) \rightarrow M.$$

So, by (1.13) (ii),  $\mathcal{D}_N$  has no fix-points on  $(J_0 - M)$  and

$$F_N((J_0 - M)) \cap F_N(M) = \phi.$$

Given  $(u'; \sigma')$  and  $(u''; \sigma'') \in (J_0 - M)$ , (1.13) (ii) also shows that if  $F_N((u'; \sigma')) = F_N((u''; \sigma''))$ , then

$$u' = u'',$$

and, considering the cases  $n_0 = 2$  and  $n_p = 1$ ,

$$(\sigma'/\sigma'')^2 = (\sigma'/\sigma'')$$

so that

$$\sigma' = \sigma''.$$

Finally, to show that  $G_N$  is an immersion at a point of  $(J_0 - M)$ , it suffices to show that

$$F_N|_{(J_0 - M)}$$

is an immersion. But this follows immediately from (1.13) and the facts:

- (i) the linear system spanned by the divisors of the functions (1.12) (in the case  $z = 0$ ) embeds  $M$ ;
- (ii) given  $(u'; \sigma') \in (J_0 - M)$ , there exists a vector

$$(a_n) \in \mathbb{C}^{N^3}$$

such that

$$\sum_n a_n \left( \theta^{n,N} \left( u' - \frac{B(0)}{2}; 0 \right) + (\sigma')^N \theta^{n,N} \left( u' + \frac{B(0)}{2}; 0 \right) \right) = 0$$

but

$$\sum_n a_n \theta^{n,N} \left( u' + \frac{B(0)}{2}; 0 \right) \neq 0$$

(see (1.13) (i)). Notice that (ii) follows from the fact that  $B(0) \neq 0$  in  $M$  which implies that the vectors

$$\left( \theta^{n,N} \left( u' - \frac{B(0)}{2}; 0 \right) + (\sigma')^N \theta^{n,N} \left( u' + \frac{B(0)}{2}; 0 \right) \right)_n \quad \text{and}$$

$$\left( \theta^{n,N} \left( u' + \frac{B(0)}{2}; 0 \right) \right)_n \quad \text{are not proportional.}$$

Notice that the argument in Theorem 1.31 can be applied inductively to show that a generic principally polarized abelian variety of dimension  $k$  has non-singular theta-divisor. The proof of Theorem 1.33 also applies, of course, in higher dimensions.

## 2. The “generic” Chow ring

On a complex torus  $J_1$  of dimension four, a principal polarization is given by an element

$$(2.1) \quad \Omega_1 \in H^2(J_1; \mathbb{Z}) \cong (\Lambda^2 H_1(J_1; \mathbb{Z}))^*$$

such that

(i)  $\Omega_1$  is a positive form of type  $(1, 1)$  in the Hodge decomposition of  $H^2(J_1; \mathbb{C})$ ;

(ii)  $\Omega_1$  is unimodular as a bilinear form on  $H_1(J_1; \mathbb{Z})$ .

Given  $(J_1, \Omega_1)$ , we can choose a basis (1.21) for  $H_1(J_1; \mathbb{Z})$  which is *symplectic*, that is,

$$\Omega_1(\gamma_j, \gamma_k) = \Omega_1(\delta_j, \delta_k) = 0,$$

$$\Omega_1(\gamma_j, \delta_k) = \text{Kronecker } \delta_{jk}.$$

If

$$(2.2) \quad \omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},$$

where  $\{\omega_i\}_{i=0,\dots,3}$  is a basis for  $H^{1,0}(J_1)$  such that

$$\int_{\gamma_j} \omega = E_j, \quad \int_{\delta_j} \omega = \Omega_j$$

where the  $E_j$  are the standard basis for  $\mathbb{C}^4$ , then the imaginary part of

$$(2.3) \quad \Omega = (\Omega_0 \Omega_1 \Omega_2 \Omega_3)$$

is positive definite and the associated  $N$ -th order theta-functions

$$\theta^{n_\nu, n, N}(\vec{u}) = \sum_{\vec{m}} e^{\pi i (N\vec{m} + \vec{n})(\Omega(\vec{m} + N^{-1}\vec{n}) + 2\vec{a})}$$

(see (1.14)) have zero sets on  $J_1$  whose associated homology class is the Poincare dual of  $N\Omega_1$  (see (1.22)). The question we wish to treat is the following:

(2.4) Which elements of  $H_*(J_1; \mathbb{Z})$  are *always* representable by effective algebraic cycles (i.e. subvarieties) in  $J_1$ ?

From what we have said so far, the duals of

$$\Omega_1, \Omega_1 \wedge \Omega_1, \Omega_1 \wedge \Omega_1 \wedge \Omega_1$$

are all representable by subvarieties. In terms of a symplectic basis

(1.21) for  $H_1(J_1; \mathbb{Z})$ , we can write these homology classes in the form

$$\sum_{0 \leq i < j < k \leq 3} \gamma_i \times \delta_i \times \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

$$(2.5) \quad 2! \cdot \sum_{0 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

$$3! \sum_{j=0}^3 \gamma_j \times \delta_j$$

It is a theorem of Matsusaka [3] and Hoyt that  $\sum \gamma_i \times \delta_i$  is representable by an algebraic curve if and only if  $(J_1, \Omega_1)$  is the Jacobian variety of that (possibly reducible) curve. So since not all principally polarized abelian varieties of dimension four are (products of) Jacobians, the cycle  $\sum_{j=0}^3 \gamma_j \times \delta_j$  is not in general representable by a subvariety.

**LEMMA 2.6 (Mattuck):** *There exist principally polarized abelian varieties  $(J_1, \Omega_1)$  of dimension four such that any element of  $H_*(J_1; \mathbb{Z})$  which is representable by an algebraic subvariety is a positive rational multiple of one of the cycles (2.5).*

**PROOF:** For elements of  $H_6(J_1; \mathbb{Z})$ , the lemma is simply the classical fact that the Picard number of a generic principally polarized abelian variety is one. By duality, therefore, the lemma is also true for elements of  $H_2(J_1; \mathbb{Z})$ . We must only examine  $H_4(J_1; \mathbb{Z})$ . Suppose the lemma is false. Then for each family (1.1) there will exist an element

$$\alpha \in H_4(J_z; \mathbb{Z})$$

for each  $z \neq 0$  and a three-dimensional closed analytic subvariety

$$S \subseteq J$$

such that:

$$(i) \quad S_z = (J_z \cdot S)$$

represents the homology class  $\alpha$ ;

(ii)  $\alpha$  is not an integral multiple of

$$\sum_{j < k} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

(This is because of Theorem 1.33 and the fact that the set of algebraic cycles in  $\mathbb{P}_{(N^4-1)}$  of fixed degree forms a finite union of irreducible algebraic families of cycles.) Then just as in (1.19)–(1.29), we can conclude that there exists a finite set of cycles

$$\alpha_1, \dots, \alpha_r \in H_4(J_0; \mathbb{Z})$$

and positive integers  $p_1, \dots, p_r$  such that:

$$(i) \quad (0, 0) \cdot \alpha = \sum_{i=1}^r p_i \alpha_i;$$

(ii) each  $\alpha_i$  is represented by an irreducible algebraic subvariety

$$S_i \subseteq J_0.$$

Let  $\tilde{S}_i \subseteq \tilde{J}_0$  be such that

$$\nu(\tilde{S}_i) = S_i$$

(see (1.25)). If  $M$  has Picard number 1, then under

$$\mu : \tilde{J}_0 \rightarrow M$$

$\tilde{S}_i$  must go to an algebraic cycle whose homology class is a positive multiple of

$$\sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k \in H_4(M; \mathbb{Z}).$$

We can therefore conclude that

$$\alpha_i = \gamma_0 \times \beta_i + r_i \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

for some  $r_i \geq 0$ , and so by (i) above and the topological description of the degeneration  $\alpha \mapsto (0, 0) \cdot \alpha$  given in (1.19)–(1.29), we have that

$$\alpha = \gamma_0 \times \beta + r \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

for some  $\beta \in H_3(J_z; \mathbb{Z})$  and some  $r \geq 0$ . Now we can arrange so that for some  $z_0 \neq 0$ , the period matrix for  $J_{z_0}$  is given by

$$\Omega_{z_0} = i \cdot (\text{identity matrix}) + \Omega'$$

where each entry in  $\Omega'$  has small absolute value and each entry in

$$(\Omega_{z_0})^{-1} + i \cdot (\text{identity matrix})$$

has small absolute value. Therefore for each  $j = 1, 2, 3$ ,  $J_{z_0}$  fits into a family (1.1) in which  $M$  has Picard number one and  $\gamma_j$  plays the role of  $\gamma_0$ . Therefore by elementary algebra

$$\alpha = r \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

This completes the proof of Lemma 2.6.

The above lemma reduces the search for the answer to the question posed in (2.4) to the homology classes

$$(2.7) \quad (i) \quad r \sum_{j=0}^3 \gamma_j \times \delta_j,$$

$$(ii) \quad s \sum_{0 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

We have seen that, if  $r = 1$ , the cycle (2.7) (i) is, in general, not representable by a subvariety. By an as yet unpublished result of A. Beauville, every principally polarized abelian variety of dimension four is the *Prym variety* associated to a two-sheeted covering of a (possibly singular) algebraic curve. The image of this two-sheeted cover in its Prym variety has homology class (2.7) (i) where  $r = 2$ . Thus the only cycles (2.7) (i) which remain in doubt are those for which  $r$  is odd and greater than 1. Similarly, since  $\Omega_1 \wedge \Omega_1$  has as its dual the cycle (2.7) (ii) with  $s = 2$ , the only cycles (2.7) (ii) which remain in doubt are those for which  $s$  is odd. Our next project is to eliminate the possibility  $s = 1$ .

Suppose

$$(2.8) \quad \Gamma = \sum_{0 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

is representable by a subvariety for all principally polarized abelian varieties of dimension four. Then in general the representing subvariety must be irreducible since no element in fourth homology which is not a positive integral multiple of  $\Gamma$  is generically representable. Therefore, by the general theory of the Chow ring of  $\mathbb{P}_{(N^4-1)}$ , [6], there must exist for each sufficiently general family (1.1) a closed, irreducible, three-dimensional analytic subvariety

$$(2.9) \quad S \subseteq J$$

such that:

(i) if  $z \neq 0$

$$(J_z \cap S) = S_z^{(1)} \cup \cdots \cup S_z^{(s)}$$

where each  $S_z^{(j)}$  represents the homology class  $\Gamma$ ,

(ii) for almost all  $z$ , the varieties  $S_z^{(j)}$  are all distinct and irreducible.

For such a general family (1.1), consider the set

$$S' = \cup \{S_z^{(1)} : z \text{ real, } > 0\}.$$

The topological closure  $\overline{S'}$  of  $S'$  intersects  $J_0$  in a union

$$(2.10) \quad S_0 = S_{(1)} \cup \cdots \cup S_{(r)} \subseteq J_0$$

of irreducible analytic subvarieties of dimension two. Just as in



(1.19)–(1.29), if  $\tilde{S}_{(i)}$  is a subvariety of  $\tilde{J}_0$  such that (counting multiplicities)

$$\nu(\tilde{S}_{(i)}) = S_{(i)}$$

and if  $\tilde{S}_{(i)}$  has homology class  $\alpha_i$ , then

$$(2.11) \quad \sum_{i=1}^r m_i \alpha_i = \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k + \mu^{-1} \left( \sum_{j=1}^3 \gamma_j \times \delta_j \right)$$

for some  $m_i > 0$ . If the double locus  $M$  of  $J_0$  is chosen suitably generally, then for each  $i$ , the homology class of  $(M^0 \cdot \tilde{S}_{(i)})$  is a non-negative multiple of  $\sum_{j=1}^3 \gamma_j \times \delta_j$  and the homology class of  $\mu(\tilde{S}_{(i)})$  is a non-negative multiple of  $\sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$ . Then the only possibilities in (2.11) are:

- (i)  $r = 1$  and  $m_1 = 1$ ;
- (ii)  $r = 2$ ,  $m_1 = m_2 = 1$  and

$$\alpha_1 = \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$$

$$\alpha_2 = \mu^{-1} \left( \sum_{j=1}^3 \gamma_j \times \delta_j \right).$$

Assume that possibility (ii) holds for a general family (1.1). It is impossible that  $S_{(i)} \subseteq M$ , the double locus, because the multiplicity of any component of  $(S \cap J_0)$  which lies in  $M$  must be greater than one. Thus

$$S_{(i)} \subseteq (J_0 - M)$$

and so

$$\tilde{S}_{(i)} \subseteq \tilde{J}_0 - (M^0 \cup M^\infty).$$

This implies that the bundle

$$\mu : \tilde{J}_0 \rightarrow M$$

is trivial when restricted to the theta-divisor  $\Sigma$  of  $M$ , since (up to translation)

$$\mu(\tilde{S}_{(i)}) = \Sigma$$

and

$$((\mu^{-1}(\text{point})) \cdot \tilde{S}_{(i)}) = 1$$

in  $\mu^{-1}(\Sigma)$ . Since the mapping

$$\text{Pic}^0(M) \rightarrow \text{Pic}^0(\Sigma)$$

is an isomorphism for non-singular  $\Sigma$ , possibility (ii) is ruled out unless  $\tilde{J}_0$  is the trivial bundle over  $M$  which is in general not the case. Thus we can conclude that for our general family:

(2.12)  $S_0$  is irreducible and lifts to a cycle  $\tilde{S}_0$  in  $\tilde{J}_0$  with homology class

$$\mu^{-1} \left( \sum_{j=1}^3 \gamma_j \times \delta_j \right) + \sum_{1 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k.$$

Using (2.12), up to translation

$$\mu(\tilde{S}_0) = \Sigma \subseteq M.$$

Also the homology class of

$$(\tilde{S}_0 \cdot M^\infty) \text{ or } (\tilde{S}_0 \cdot M^0)$$

in  $H_2(M; \mathbb{Z})$  is

$$\sum_{j=1}^3 \gamma_j \times \delta_j.$$

Also we can suppose that

$$(M, \Sigma) = (J(C), C^{(2)}),$$

the Jacobian of a non-singular, non-hyperelliptic curve  $C$  of genus three and that  $M$  has endomorphism ring  $\mathbb{Z}$ . Also  $\mu(\tilde{S}_0 \cap M^\infty)$  and  $\mu(\tilde{S}_0 \cap M^0)$  are homologous in the second symmetric product

$$C^{(2)} = \Sigma = \mu(\tilde{S}_0) \subseteq M.$$

With the help of the theorem of Matsusaka mentioned previously, we can therefore conclude that there are only two possibilities:

(i) there exist  $P_0, P_\infty \in C$  such that

$$\mu(\tilde{S}_0 \cap M^0) = \{(P_0, P) \in C^{(2)} : P \in C\}$$

$$\mu(\tilde{S}_0 \cap M^\infty) = \{(P_\infty, P) \in C^{(2)} : P \in C\},$$

(ii) there exist  $P_0, P_\infty \in C^{(2)}$  such that

$$\mu(\tilde{S}_0 \cap M^0) = \{(P_1, P_2) \in C^{(2)} : P + P_0 + P_1 + P_2$$

is a canonical divisor of  $C$  for some  $P \in C\}$

$$\mu(\tilde{S}_0 \cap M^\infty) = \{(P_1, P_2) \in C^{(2)} : P + P_\infty + P_1 + P_2 \text{ is } \dots\}.$$

Furthermore, since

$$\mu^{-1}(\text{point}) \cdot \tilde{S}_0 = 1$$

in  $\mu^{-1}(\Sigma)$ ,  $\tilde{S}_0$  gives a meromorphic section of the line bundle

$$\mu : (\mu^{-1}(\Sigma) - (M^\infty \cap \mu^{-1}(\Sigma))) \rightarrow \Sigma$$

whose associated divisor is

$$(2.13) \quad \mu(\tilde{S}_0 \cap M^0) - \mu(\tilde{S}_0 \cap M^\infty).$$

However since the natural mapping

$$\text{Pic}^0(M) \rightarrow \text{Pic}^0(\Sigma)$$

is bijective, our assumption that the cycle  $\Gamma$  in (2.8) is always representable by a subvariety forces a contradiction. For if we choose  $B(0)$  in (1.4) sufficiently generally, the line bundle

$$\mu : (\tilde{J}_0 - M^\infty) \rightarrow M$$

will *not* restrict over  $\Sigma$  to a line bundle belonging to the two-parameter family of line bundles whose associated divisor has the form (2.13). Thus we have proved the following theorem.

**THEOREM 2.14:** *There exist principally polarized abelian varieties  $(J_1, \Omega_1)$  of dimension four such that the cycle  $\Gamma = \sum_{0 \leq j < k \leq 3} \gamma_j \times \delta_j \times \gamma_k \times \delta_k$  is not representable by a subvariety of  $J_1$ .*

Notice that the two possibilities for the families of divisors (2.13) correspond to the degenerations of  $D_z^{(2)}$  and  $-D_z^{(2)}$  respectively where  $D_z$  is a curve of genus four which acquires a double point as  $z \mapsto 0$  and  $J_z$  is the Jacobian of  $D_z$ .

Left open is the very intriguing question as to the odd values of  $r$  and  $s > 1$  in (2.7) for which the corresponding homology classes are always carried by subvarieties. Of course, if we find a value of  $r$  such that the cycle (2.7) (i) is carried by an algebraic curve  $D$ , the cycle (2.7) (ii) with  $s = r^2$  will be carried by the image of  $D^{(2)}$  in  $J_1$  so the representability of the cycles (2.7) (i) and the cycles (2.7) (ii) are related. If it turns out, for example, that there exists an abelian variety  $J_1$  on which no odd multiple of  $\sum_{j=0}^3 \gamma_j \times \delta_j$  is representable by a subvariety, one would have a new type of counter-example to the (false) Hodge conjecture over  $\mathbb{Z}$ , one that did not involve torsion cycles.

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