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A SHORT PROOF OF DVORETZKY'S THEOREM ON ALMOST SPHERICAL SECTIONS OF CONVEX BODIES

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In this note we present a short proof of the following important result of A. Dvoretzky [1].

THEOREM: *For every integer $k \geq 2$ and every $\epsilon > 0$, there exists an $N = N(k, \epsilon)$ such that every normed space (X, p) with $\dim X \geq N$ contains a k-dimensional subspace that is ϵ-Euclidean.*

Let us recall that a normed space (X, p) is said to be ϵ-Euclidean if there exist an inner-product norm, say $|\cdot|$, and a constant C such that

$$C(1-\epsilon)|x| \leq p(x) \leq C|x|, \qquad \text{for } x \in X.$$

We shall use another measure of the "distance" between p and $|\cdot|$, which will be denoted here by $v(X, p, |\cdot|)$. The theorem will follow, once we have established that:

A. There exists a sequence (c_n) tending to 0 such that, for any n-dimensional normed space (X, p), there exists an inner-product norm $|\cdot|$ on X with $v(X, p, |\cdot|) \leq c_n$.

B. For any $(X, p, |\cdot|)$ and any integer k with $1 < k < \dim X$, there exists a subspace E of X such that $\dim E = k$ and $v(E, p\big|_E, |\cdot|\big|_E) \leq v(X, p, |\cdot|)$.

C. For any k, $\epsilon > 0$, there exists a $\delta > 0$ such that, if $\dim E = k$ and $v(E, p, |\cdot|) < \delta$, then (E, p) is ϵ-Euclidean.

Let us introduce some notation. Given a normed real or complex space (X, p), $2 \leq \dim X < \infty$, with a Euclidean norm $|\cdot|$, we set

$$S_X = \{x \in X : |x| = 1\},$$

$$x^\perp = \{y \in X : |x+y| = |x-y|\}, \qquad \text{for } x \in X,$$

$$\Sigma_X = \{(x, y) \in S_X \times S_X : y \in x^\perp\},$$

$$\lambda_X \text{ (resp. } \sigma_X) = \text{the normalized } |\cdot|\text{-rotation invariant Borel measure on } S_X \text{ (resp. } \Sigma_X),$$

$$v(x, p, |\cdot|) = \int_{\Sigma_X} (Dp(x)(y))^2 p(x)^{-2} d\sigma_X(x, y).$$

The last formula makes sense because the convex function p is differentiable almost everywhere on S_X and Dp is a measurable and bounded function of x.

Our proof of Property A uses the well known Dvoretzky–Rogers lemma, which can be stated as follows (cf. [1]).

(D–R) For every normed space (X, p) with $\dim X = n$, there exists an integer $m > \frac{1}{2}\sqrt{n} - 1$ and linear operators $T : l_2^n \to X$, $U : X \to l_\infty^m$ such that $\|T\| = 1$, $\|U\| \le 2$, T is one-to-one, and $UT((x_1, \ldots, x_n)) = (x_1, \ldots, x_m)$ for $(x_1, \ldots, x_n) \in l_2^n$.

We define the Euclidean norm on X letting $|x| = \|T^{-1}(x)\|_{l_2^n}$. If p is differentiable at an $x \in X$, and $y \in X$, then $|Dp(x)(y)| \le p(y) \le |y|$, therefore we have

$$q(x) \stackrel{\text{def}}{=} \int_{S_X \cap x^\perp} |Dp(x)(y)|^2 d\lambda_{x^\perp}(y)$$
$$= \frac{1}{n-1} \sup\{|Dp(x)(y)|^2 : y \in S_X \cap x^\perp\} \le \frac{1}{n-1}.$$

Writing $v(X, p, |\cdot|)$ as an iterated integral, and using (D–R), we get

$$v(X, p, |\cdot|) = \int_{S_X} q(x) p(x)^{-2} d\lambda_X(x)$$
$$\le \frac{1}{n-1} \int_{S_X} \|U\|^2 \|Ux\|^{-2} d\lambda_X(x) = \frac{\|U\|^2}{n-1} \int_{S_{l_2^n}} \|UTz\|^{-2} d\lambda_{l_2^n}(z)$$
$$\le \frac{4}{n-1} \int_{S_{l_2^n}} \left(\max_{1 \le i < \sqrt{n}/2} |x_i| \right)^{-2} d\lambda_{l_2^n}(z) \stackrel{\text{def}}{=} c_n.$$

It is known (cf. [4]) that $\lim_{n\to\infty} c_n = 0$, we shall prove a more general fact in a lemma below.

Property B follows easily from a well known formula, in which γ denotes the normalized rotation invariant measure on the Grassmann manifold Γ of all k-dimensional linear subspaces E of X,

$$\int_{\Sigma_X} f(x, y) d\sigma_X(x, y) = \int_\Gamma d\gamma(E) \int_{\Sigma_X} f(x, y) d\sigma_E(x, y),$$

after substituting $f(x, y) = (Dp(x)(y))^2 p(x)^{-2}$. The formula is valid for any function f that is σ_X-integrable on Σ_X, since the right hand side also defines a normalized invariant integral on Σ_X.

Finally, if C were false, then there would exist numbers k, ϵ and a sequence (p_n), $n = 1, 2, \ldots$, of norms on $E = l_2^k$ such that $v(E, p_n, |\cdot|) < 1/n$ and p_n fails to be ϵ-Euclidean for $n = 1, 2, \ldots$. Let $S = \{x \in E : |x| = 1\}$. We may assume that $\sup_{x \in S} p_n(x) = 1$ for all n, and hence $\inf_{x \in S} p_n(x) \leq 1 - \epsilon$. By passing to a subsequence (Ascoli's theorem) we may also assume that $p_0(x) = \lim_{n \to \infty} p_n(x)$ exists for $x \in E$. Clearly

$$\sup_{x \in S} p_0(x) = 1 > 1 - \epsilon \geq \inf_{x \in S} p_0(x).$$

Let

$$\begin{aligned} A &= \{x \in S : Dp_n(x) \text{ exists for } n = 0, 1, 2, \ldots\}, \\ B &= \{x \in A : Dp_0(x)(y) = 0 \text{ for } y \in x^{\perp}\}. \end{aligned}$$

Since the p_n's are convex functions, we have $\lambda_E(A) = 1$, and

$$\lim_{n \to \infty} Dp_n(x) = Dp_0(x), \quad \text{for } x \in A.$$

Hence, by Fatou's lemma,

$$\begin{aligned} \int_{\Sigma X} Dp_0(x)(y))^2 d\sigma_E(x, y) &\leq \underline{\lim} \int (Dp_n(x)(y))^2 d\sigma_E(x, y) \\ &\leq \underline{\lim}\, v(E, p_n, |\cdot|) = 0. \end{aligned}$$

It follows that $\lambda(A \backslash B) = 0$. Therefore any two points $x_1, x_2 \in S$ can be connected in S by a rectifiable curve $g(t)$, $a \leq t \leq b$, whose almost all points are in B. Consequently,

$$p_0(x_2) - p_0(x_1) = \int_a^b Dp_0(g(t))(g'(t))dt = \int_a^b 0 \ dt = 0,$$

i.e. p_0 is constant on S. This contradiction completes the proof of C.

For the sake of completeness we include the following lemma (the probabilistic argument has been indicated by D. L. Burkholder; the approach used in [2] can also be adapted).

LEMMA: *Let $m(n)$ be a sequence of positive integers, such that $m(n) \leq n$ and $\lim_{n \to \infty} m(n) = \infty$, and let*

$$\alpha(n) = \frac{1}{n} \int_S \left(\max_{1 \leq i \leq m(n)} |x_i| \right)^{-2} d\lambda(x).$$

where λ is the normalized invariant measure on the unit sphere S of l_2^n. Then $\lim_{n \to \infty} \alpha(n) = 0$.

PROOF: Let $X_1, X_2, \dots$ be a sequence of independent normalized Gaussian random variables on a probability space (Ω, Σ, P). If one considers the complex spaces l_2^n, then also the X_i's should be complex-valued.

Fix n, for the time being, and let $Y_i(\omega) = X_i(\omega)/(\Sigma_{i=1}^n |X_i(\omega)|^2)^{1/2}$. The map $\omega \to (Y_1(\omega), \dots, Y_n(\omega))$ transports the measure P onto a normalized rotation invariant measure on S. Thus we have

$$\alpha(n) = \frac{1}{n}\int_\Omega \left(\max_{1\le i\le m(n)} |Y_i|\right)^{-2} dP$$

$$= \frac{1}{n}\int_\Omega \sum_{i=1}^n |X_i|^2 \left(\max_{i\le m(n)} |X_i|\right)^{-2} dP$$

$$= \frac{m(n)}{n}\int_\Omega |X_1|^2 \left(\max_{i\le m(n)} |X_i|\right)^{-2} dP + \frac{n-m(n)}{n}\int_\Omega \left(\max_{i\le m(n)} |X_i|\right)^{-2} dP.$$

Now we let n tend to infinity. Then the integrals on the right-hand side tend to zero, by the dominated convergence theorem, because they are finite (the second one, if $m(n) \ge 4$) and the X_i's are unbounded almost surely. This completes the proof.

REMARKS: Our introducing of the quantity $v(X, p, |\cdot|)$ has been suggested by Szankowski's [6]. The lemma was motivated by Lemma 9 in [4]. In fact, our Property A is essentially equivalent to Lemma 10 in [4].

It is not difficult to prove quantitative versions of properties A and C, but our estimate for $N(k, \epsilon)$ is not as good as those given in [5] and [6].

Finally, we should mention that there exists a "combinatorial" proof of Dvoretzky's theorem, at least in the real case, (cf. [7] and [3]).

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