COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 33, nº 3 (1976), p. 227-259

<http://www.numdam.org/item?id=CM_1976__33_3_227_0>

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COMPOSITIO MATHEMATICA, Vol. 33, Fasc. 3, 1976, pag. 227–259
Noordhoff International Publishing
Printed in the Netherlands

THE LAPLACIAN OPERATOR ON A RIEMANN SURFACE, III

S. J. Patterson

1. Introduction

In the first paper of this series [9] we sketched a method of obtaining the spectral decomposition of the family of Laplace operators Δ_k acting on automorphic forms of weight k and multiplier χ belonging to some Fuchsian group G. In the second paper [10] we introduced the Eisenstein series $E_\xi(z, s)$ and demonstrated various properties of these functions. On $\operatorname{Re}(s) = 1/2$ they may be considered as generalised eigenfunctions. Our first object will be to make this statement exact, and to prove in our case a theorem of Elstrodt [2] (in his case it is only valid when $\delta(G) < 1/2$).

The chief object of this paper is to prove a completeness theorem—to the effect that the functions above, together with a discrete set of eigenfunctions, suffices for the spectral resolution of Δ_k. Unfortunately our proof of this fact is only complete when $k \in \boldsymbol{Z}$, and we shall limit ourselves to the case $k = 0$ in our discussion of the completeness theorem. This is no loss of generality, thanks to the theory of Maaß operators, ([3], [13]). In the case $k \notin \boldsymbol{Z}$ we show that the whole of the continuous spectrum of $-\Delta_k$ is contained in $[1/4, \infty[$, and that the eigenvalues (which lie in $[\delta(G)(1-\delta(G)), 1/4[$, cf. [9]) are, to some extent, regularly distributed. When $\delta(G) < 1/2$ the completeness theorem was proved in [9]. The plan of the proofs of the facts outlined above was sketched in [9]. As a consequence it will follow that $E_\xi(z, s)$ and $\Phi_k(s, \chi)$ have analytic continuations to meromorphic functions on $\boldsymbol{C}$, with the possible exception of one or three points.

The notations and conventions of this work are as in [10]. Mostly they shall be used without any further explanation. The Fuchsian groups considered here will all be finitely generated and contain no parabolic elements.

2. Estimates

In [10; §6] we deduced estimates for the function $Q_k^-(\theta, X, s)$, and, by implication, also for its partial derivatives. These estimates are not uniform in s in unbounded sets, or, more importantly, in θ in the neighbourhood of 0. However, at the cost of not obtaining the optimal behaviour in X we can obtain uniform bounds of the same type.

Let us choose $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < \pi$. Suppose that $\theta \in]0, \alpha_1]$. Recall the integral representation for $Q_k^-(\theta, X, s)$ ([10; Eq. (59)])

(1)

$$Q_k^-(\theta, X, s) = e(k)(\sin\theta)^s \int_{-\infty}^{\infty} e(-Xz)e^{zs}(e^{z+i\theta}+1)^{-s-k}(e^{z-i\theta}+1)^{-s+k}dz$$

Suppose $X > 0$. We move the line of integration to $\operatorname{Im}(z) = -\alpha_2$ and write $z = x - i\alpha_2$. Then we split the integral into two parts, corresponding to $x > 0$ and $x < 0$. If, in the latter, we replace x by $-x$ we obtain

$$\begin{aligned} Q_k^-(\theta, X, s) = (\sin\theta)^s e(k) e^{-2\pi\alpha_2 X}\Big(e^{i(\alpha_2 s - 2k\theta)} &\int_0^\infty e^{(s-2\pi i X)x} \\ &\times (e^x + e^{i(\alpha_2-\theta)})^{-s-k}(e^x + e^{i(\alpha_2+\theta)})^{-s+k}dx \\ &+ e^{-i\alpha_2 s}\int_0^\infty e^{(s+2\pi i X)x}(e^x + e^{-i(\alpha_2-\theta)})^{-s-k} \\ &\times (e^x + e^{-i(\alpha_2+\theta)})^{-s+k}dx\Big). \end{aligned}$$

As e^x lies on the line $[1, \infty[$ it follows after straightforward estimates that

$$\cos((\alpha_1+\alpha_2)/2)(e^x+1) \le |e^x + e^{i(\pm\alpha_2\pm\theta)}| \le e^x + 1$$

and

$$|\arg(e^x + e^{i(\pm\alpha_2\pm\theta)})| \le \pi/2.$$

Let $s = \sigma + it$. Then substituting these estimates in the equation above, we obtain, for $\sigma > 0$,

$$|Q_k^-(\theta, X, s)\cdot \sin^{-s}\theta| \le c_1(\sigma, \alpha_1+\alpha_2)e^{-2\pi\alpha_2|X| + |t|(\alpha_2+\pi)}, \tag{2}$$

where

$$c_1(\sigma, \alpha) = 2(\cos(\alpha/2))^{-2\sigma-|k|}\int_0^\infty e^{x\sigma}(e^x+1)^{-2\sigma}dx.$$

We clearly obtain the same result when $X < 0$ and then by continuity we have that (2) is valid for all X.

Next we note that the same method can be applied to

$$(\partial/\partial\theta)(\sin^{-s}\theta Q_k^-(\theta, X, s)),$$

and we see that this is bounded, in $\sigma > 0$, by

$$c_2(\sigma, \alpha_1 + \alpha_2)e^{-2\pi\alpha_2|X|+|t|(\alpha_2+\pi)}$$

for a suitable $c_2(\sigma, \alpha)$. Integrating this from 0 to θ, we see that, for $\sigma > 0$,

$$|Q_k^-(\theta, X, s) - q_k^-(s, X)\sin^s\theta| \leq c_2(\sigma, \alpha_1 + \alpha_2)e^{-2\pi\alpha_2|X|+|t|(\alpha_2+\pi)}\theta^{\sigma+1}. \tag{3}$$

Next, using [10; (35), (37)] we have

$$\begin{aligned} Q_k^+(\theta, X, s) &= D_k(s)q_k^-(1-s, X)^{-1}Q_k^-(\theta, X, s) \\ &\quad + q_k^+(s, X)q_k^-(s, X)^{-1}Q_k^-(\theta, X, s). \end{aligned} \tag{4}$$

Thus, writing $\tau = \operatorname{Min}(\sigma, 1-\sigma)$, and assuming that $0 < \sigma < 1$, and $|s - 1/2| \geq 1/3$ we find from (3), (4) and Stirling's formula

$$\begin{aligned} &|Q_k^+(\theta, X, s) - D_k(s)\sin^{1-s}\theta - q_k^+(s, X)\sin^s\theta| \\ &\qquad \leq c_3(\sigma, \alpha_1 + \alpha_2)\theta^{\tau+1}e^{2\pi(\pi-\alpha_2)|X|+(\alpha_2+\pi)|t|} \end{aligned} \tag{5}$$

$c_2(\sigma, \alpha)$, $c_3(\sigma, \alpha)$ can be taken to be continuous in σ. The expression on the left-hand side of (5) is entire in $|s - 1/2| \leq 1/3$ and so, by the maximum modulus principle (5) holds also in the interior of the circle (and so in the strip $0 < \sigma < 1$). Note that in deriving these inequalities we have used the arbitrariness of α_2 given α_1 to absorb powers of $|s|$, but unfortunately at the cost of explicit forms for $c_2(\sigma, \alpha)$, $c_3(\sigma, \alpha)$.

Observe that (3), (5) exhibit the nature of the singularity of $Q_k^\epsilon(\theta, X, s)$ as $\theta \to 0$. Using this with [10; Th. 5] we obtain an integral representation of $R_k^{\epsilon\eta}(\theta, s, t, X)$ valid in $\operatorname{Re}(s) > 0$, $\operatorname{Re}(t) > 0$. The formulae are somewhat lengthy, but not complicated, and we shall not write them down. However, we do need the asymptotic behaviour of $R_k^{\epsilon\eta}(\theta, s, t, X)$ which can be deduced from this and (3), (5). Letting $\delta(+) = 1$, $\delta(-) = 0$ we obtain

$$\begin{aligned} &\Big|(s(1-s) - t(1-t))^{-1}R_k^{\epsilon\eta}(\theta, s, t, X) + D_k(s)\overline{D_k(\bar{t})}\delta(\epsilon)\delta(\eta)\frac{(\sin\theta)^{1-s-t}}{1-s-t} \\ &\quad + \delta(\eta)\overline{D_k(\bar{t})}q_k^\epsilon(s, X)\sin^{s-t}\theta/(s-t) + \delta(\epsilon)D_k(s)\overline{q_k^\eta(\bar{t}, X)}\frac{\sin^{t-s}\theta}{t-s} \\ &\quad + q_k^\epsilon(s, X)\overline{q_k^\eta(\bar{t}, X)}\sin^{s+t-1}\theta/(s+t-1)\Big| \\ &\leq c_4\theta^\tau \exp(2\pi(\pi(\delta(\epsilon) + \delta(\eta)) - 2\alpha_2)|X| + (\alpha_2 + \pi)(|\operatorname{Im}(s)| + |\operatorname{Im}(t)|) \end{aligned} \tag{6}$$

where $c_4 = c_4(\operatorname{Re}(s), \operatorname{Re}(t), \alpha_1, \alpha_2)$,

$$\tau = \mathrm{Min}\,(\mathrm{Re}\,(s+t), \mathrm{Re}\,(1-s+t), \mathrm{Re}\,(1-t+s), \mathrm{Re}\,(2-s-t)).$$

We shall close this section with some special results in the case $k=0$. Note that, from (1),

$$Q_0^-(\pi/2, X, s) = \int_{-\infty}^{\infty} e^{(s-2\pi i X)x}(e^{2x}+1)^{-s}dx.$$

This is an Euler beta-function; we have

$$(7) \qquad Q_0^-(\pi/2, X, s) = (1/2)\Gamma(s/2+\pi i X)\Gamma(s/2-\pi i X)/\Gamma(s).$$

Next, as in [9], set

$$\sigma(z,w) = |z-\bar{w}|^2/4\,\mathrm{Im}\,(z)\,\mathrm{Im}\,(w),$$

and define the point-pair invariant

$$K_R(z,w) = 1 \qquad (\sigma(z,w) \leq R),$$
$$= 0 \qquad (\sigma(z,w) > R).$$

Let $h_R(s)$ be the Selberg transform of this kernel ([9], [14]). From the formulae given by Selberg we see that

$$(8) \qquad h_R(1/2+ir) = 2\int_{-x}^{+x} e^{iru}(e^x+e^{-x}-e^u-e^{-u})^{1/2}du$$

where x is the positive solution of

$$e^x + e^{-x} = 4R-2.$$

Suppose now that $r>0$. We deform the path of integration to the polygonal path $-x, -x+i/4, x+i/4, x$. For large r the contribution from the line $-x+i/4, x+i/4$ is $O(e^{-r/4})$, uniformly for x in compact subsets. As $u \to x$

$$(e^x+e^{-x}-e^u-e^{-u})^{1/2} = (x-u)^{1/2}(e^x-e^{-x})^{1/2} + O((x-u)^{3/2}),$$

and, as $u \to -x$,

$$(e^x+e^{-x}-e^u-e^{-u})^{1/2} = (u+x)^{1/2}(e^x-e^{-x})^{1/2} + O((x+u)^{3/2}),$$

uniformly for x lying in compact sets. It now follows by Watson's lemma that

$$h_R(1/2+ir) = \sqrt{\pi}|r|^{-3/2}(e^{irx}+e^{-irx})(e^x-e^{-x})^{1/2}+O(|r|^{-5/2})$$

and this estimate holds uniformly for x lying in compact subsets of $]0,\infty[$. If we let $I_1=[2\pi,4\pi]$ then, for each $r>1$ there exists $x\in I_1$ so that

$$|e^{irx}+e^{-irx}|>1.$$

Using the argument of [10; §7] with the kernel K_R where $x=x(R)\in I_1$ it follows that there exists a φ so that, if $\mathrm{Re}(s)=1/2$, then, for $z\in D^{\pi/2}$,

$$E_\zeta(z,s)\le c_5|s|^{3/2}\left(\int_{D^\varphi}\mathrm{Tr}\,(E_\zeta(z,s)^*E_\zeta(z,s))d\sigma(z)\right)^{1/2}. \tag{9}$$

This makes explicit an estimate in [10; §7]. The fact that K_R is not continuous does not matter; in fact it can be approximated by smooth functions in the L^1 sense and, since $E_\zeta(z,s)$ is bounded on the compact set D^φ this suffices.

3. The inner product formula

Consider a function $f(s,\zeta)$, taking values in V, and defined for s with $\mathrm{Re}\,(s)=1/2$, and $\zeta\in\Omega(G)$. We shall also assume that

$$|g'(\zeta)|^{1-s}f(s,g(\zeta))=\chi(g)\epsilon(g(\infty),g(\zeta))^{-1}f(s,\zeta). \tag{10}$$

$\epsilon(\zeta,\eta)$ was defined in [9]. Hence $f(s,\zeta)$ has a Fourier expansion of the form ($\zeta\in\Omega_{\alpha(j)}$)

$$Y_jf(s,\zeta)|A_j'(\zeta_j)|^{1-s}|\zeta_j|^{1-s}=\sum_{\alpha\in Z_j}\mathfrak{E}_\alpha(\zeta_j)f_\alpha(s). \tag{11}$$

We shall assume that only a finite set of the $f_\alpha(s)$ ($\alpha\in Z^*$) are not identically zero as functions of s and that all the $f_\alpha(s)$ are smooth with compact support on $\{\mathrm{Re}\,(s)=1/2,\ s\ne 1/2\}$. These conditions are very restrictive but shall be weakened later. Let $L=\{\mathrm{Re}\,(s)=1/2\}$ and

$$v_f(z)=\int_L\int_B E_\zeta(z,s)f(s,\zeta)d\zeta|ds|. \tag{12}$$

We shall show that v_f is square integrable and calculate its norm. Let

$(\cdot, \cdot)_V$ be the Hermitian form on V. Then

$$\|v_f\|^2 = \lim_{\varphi\to 0} \int_{D^\varphi} \int_B \int_B \int_L \int_L (E_\eta(z, t)f(t, \eta), E_\zeta((z, s)f(s, \zeta))_V$$
$$\times |ds||dt|d\zeta d\eta d\sigma(z)$$
$$= \lim_{\varphi\to 0} \int_L \int_L \sum_{\alpha,\beta\in Z^*} \int_{D^\varphi} (E^!_\alpha(z, t)f_\alpha(t), E^!_\beta(z, s)f_\beta(s))_V$$
$$\times d\sigma(z)|ds||dt|.$$

Since $E_\zeta(z, s)$ is smooth in ζ and z ([10]) it follows that there is no difficulty in interchanging orders of integration and summation. The inner sum can be evaluated by [10; Th. 4]. The result is

$$\|v_f\|^2 = \lim_{\varphi\to 0} \int_L \int_L \Big(\sum_\alpha (f_\alpha(t), \mathfrak{R}_k^{++}(\varphi_{i(\alpha)}, s, \bar t, \alpha)f_\alpha(s))_V\kappa(\alpha)$$
$$+ \sum_{\alpha,\beta} e(k)(\sigma_{\alpha\beta}(t)f_\beta(t), \mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, \bar t, \alpha)f_\alpha(s))_V$$
$$+ \sum_{\alpha,\beta} e(-k)(f_\beta(t), \mathfrak{R}_k^{-+}(\varphi_{i(\beta)}, s, \bar t, \beta)\sigma_{\beta\alpha}(s)f_\alpha(s))_V$$
$$+ \sum_{\alpha,\beta,\gamma} \kappa(\gamma)^{-1}(\sigma_{\gamma\beta}(t)f_\beta(t), \mathfrak{R}_k^{--}(\varphi_{i(\gamma)}, s, \bar t, \gamma)\sigma_{\gamma\alpha}(s)f_\alpha(s))\Big)$$
$$\times (s(1-s) - t(1-t))^{-1}|ds||dt|.$$

In section 2 we found approximations to

$$(s(1-s) - t(1-t))^{-1}\mathfrak{R}_k^{\epsilon\eta}(\varphi, s, t, \alpha)$$

valid when φ is small, with error terms $O(\varphi)$. It follows by (6), the conditions imposed on f, and the estimate on the $\sigma_{\alpha\beta}(s)$ in [10; Th. 8 (+ Th. 6)] that the error terms give a contribution $O(\varphi)$ to the integral above. Hence

$$\|v_f\|^2 = -\sum_{j=1}^{p} \lim_{\varphi_j\to 0} \int_L \int_L (I_{1,j}(s, t)(\sin^{s+\bar t-1}\varphi_j)/(s+\bar t-1)$$
$$+ I_{2,j}(s, t)(\sin^{s-\bar t}\varphi_j)/(s-\bar t) + I_{3,j}(s, t)(\sin^{\bar t-s}\varphi_j)/(\bar t - s)$$
$$+ I_{4,j}(s, t)(\sin^{1-s-\bar t}\varphi_j)/(1-s-\bar t))|ds||dt|,$$

where

$$I_{1,j}(s, t) = \sum_{\alpha\in Z_j} (\mathfrak{q}_k^+(t, \alpha)f_\alpha(t), \mathfrak{q}_k^+(s, \alpha)f_\alpha(s))_V\kappa(j)$$

$$+\sum_{\alpha\in Z_j}\sum_{\beta\in Z^*} e(k)(\mathfrak{q}_k^-(t,\alpha)\sigma_{\alpha\beta}(t)f_\beta(t),\mathfrak{q}_k^+(s,\alpha)f_\alpha(s))_V$$

$$+\sum_{\beta\in Z_j}\sum_{\alpha\in Z^*} e(-k)(\mathfrak{q}_k^+(t,\beta)f_\beta(t),\mathfrak{q}_k^-(s,\beta)\sigma_{\beta\alpha}(s)f_\alpha(s))_V$$

$$+\sum_{\gamma\in Z_j}\sum_{\alpha,\beta\in Z^*} \kappa(\gamma)^{-1}(\mathfrak{q}_k^-(t,\gamma)\sigma_{\gamma\beta}(t)f_\beta(t),\mathfrak{q}_k^-(s,\gamma)\sigma_{\gamma\alpha}(s)f_\alpha(s))_V$$

$$I_{2,j}(s,t)=\sum_{\alpha\in Z_j}(D_k(t)f_\alpha(t),\mathfrak{q}_k^+(s,\alpha)f_\alpha(s))_V\kappa(j)$$

$$+\sum_{\beta\in Z_j}\sum_{\alpha\in Z^*} e(-k)(D_k(t)f_\beta(t),\mathfrak{q}_k^-(s,\beta)\sigma_{\beta\alpha}(s)f_\alpha(s))_V,$$

$$I_{3,j}(s,t)=\sum_{\alpha\in Z_j}(\mathfrak{q}_k^+(t,\alpha)f_\alpha(t),D_k(s)f_\alpha(s))_V\kappa(j)$$

$$+\sum_{\alpha\in Z_j}\sum_{\beta\in Z^*} e(k)(\mathfrak{q}_k^-(t,\alpha)\sigma_{\alpha\beta}(t)f_\beta(t),D_k(s)f_\alpha(s))_V,$$

$$I_{4,j}(s,t)=\sum_{\alpha\in Z_j}(D_k(t)f_\alpha(t),D_k(s)f_\alpha(s))_V\kappa(j).$$

By the estimates for $\sigma_{\alpha\beta}(s)$ mentioned above $I_{a,j}(s,t)$ is a smooth function of compact support on $(L\setminus\{1/2\})\times(L\setminus\{1/2\})$. Moreover, the functional equation [10; Eq. (40)] gives

$$I_{1,j}(s,s)=I_{4,j}(s,s)=\sum_{\alpha\in Z_j}(f_\alpha(s),f_\alpha(s))_V\kappa(j)|D_k(s)|^2,$$

and

$$\begin{aligned}I_{2,j}(s,1-s)&=I_{3,j}(s,1-s)\\&=\sum_{\alpha\in Z_j}\sum_{\beta\in Z^*}(f_\alpha(1-s),\sigma^*_{\alpha\beta}(s)f_\beta(s))\kappa(j)e(-k)D_k(s),\end{aligned}$$

where $\sigma^*_{\alpha\beta}(s)$ is defined in [10; §4].

We consider the first and fourth terms in the above integral together. We have to evaluate

$$\lim_{\varphi_j\to 0}\int_L\int_L (I_{1,j}(s,t)\sin^{s-t}\varphi_j - I_{4,j}(s,t)\sin^{t-s}\varphi_j)/(s-t)|ds||dt|.$$

We introduce the new variables

$$x=(s-t)/i,\quad y=(s+t)/2,\quad e^{-R}=\sin\varphi_j,$$

and setting

$$A_1(x,y)=(I_{1,j}(s,t)-I_{4,j}(s,t))/2(s-t)$$
$$A_2(x,y)=(I_{1,j}(s,t)+I_{4,j}(s,t))/2,$$

which are smooth functions of compact support on $\boldsymbol{R} \times L$, we see that we must evaluate the integral

$$\lim_{R\to\infty} 2\int_{\boldsymbol{R}} \cos Rx \left(\int_L A_1(x, y)|dy|\right) - x^{-1} \sin Rx \left(\int_L A_2(x, y)|dy|\right) dx.$$

By the Riemann–Lebesgue lemma the first term is zero. The second term is a Dirichlet integral and this gives

$$2\pi \int_L A_2(0, y)|dy| = 2\pi \int_L \sum_{\alpha \in Z_j} (f_\alpha(s), f_\alpha(s))\kappa(j)|D_k(s)|^2|ds|,$$

the equality following from the evaluation of $I_{1,j}(s, s)$, $I_{4,j}(s, s)$ above. By an analogous argument the second and third terms give rise to

$$-2\pi \int_L \sum_{\alpha \in Z_j} \sum_{\beta \in Z^*} (f_\alpha(1-s), \sigma^*_{\alpha\beta}(s) f_\beta(s)) e(k)\kappa(j) D_k(s) ds.$$

In particular, if the support of $f(s, \zeta)$, as a function of s lies in $L_+ = \{\mathrm{Re}\,(s) = 1/2, \mathrm{Im}\,(s) > 0\}$ then this term vanishes.

We form the Hilbert space $\mathfrak{H}_E$ of functions $f(s, \zeta)$ on $L_+ \times \Omega(G)$ satisfying (7) and with the norm

$$\|f\|_E^2 = 2\pi \int_{L^+} \int_B (f(s, \zeta), f(s, \zeta))_V d\zeta |D_k(s)|^2 |ds|.$$

Furthermore, as in [9], let $L(G, V)$ be the Hilbert space of square-integrable automorphic forms of weight k and multiplier χ, and norm

$$\|f\|^2 = \int_D (f(z), f(z))_V d\sigma(z).$$

Then the $f(s, \zeta)$ which we have considered, and which have support on L_+ belong to $\mathfrak{H}_E$, and we have shown that for these functions

$$\|v_f\| = \|f\|_E.$$

Let $L(G, V)_c$ be the closure in $L(G, V)$ of v_f with f as above. Then the following theorem follows by standard Hilbert-space methods:

THEOREM 1: *The map*

$$f \mapsto v_f(z) = \iint E_\zeta(z, s) f(s, \zeta)|ds|\,d\zeta$$

is defined on a dense subspace of $\mathfrak{H}_E$ *and takes values in* $L(G, V)$. *This map extends to an isometry of* $\mathfrak{H}_E$ *onto a closed subspace* $L(G, V)_c$ *of* $L(G, V)$.

This theorem was first proved for (not necessarily finitely generated) G with $\delta(G) < 1/2$ by Elstrodt [2]. His proof is rather more direct than ours although the crucial point in both proofs is the appearance of a Dirichlet integral.

4. Bounds on the Eisenstein series and eigenfunctions

Assume for the moment that $k = 0$. Let

$$A_\varphi(s)^2 = \int_B \int_{D^\varphi} \operatorname{Tr}(E_\zeta(z, s)^* E_\zeta(z, s)) d\sigma(z) d\zeta.$$

If $\operatorname{Re}(s) = 1/2$ then (9), [10; Th. 1] and Cauchy's inequality show that, for some φ

$$|\mathfrak{Q}_0^+(\pi/2, \alpha, s)\delta_{\alpha\beta} + \mathfrak{Q}_0^-(\pi/2, \alpha, s)\sigma_{\alpha\beta}(s)| \leq c_1 A_\varphi(s)|s|^{3/2}.$$

However $\mathfrak{Q}_0^+(\pi/2, \alpha, s) = \mathfrak{Q}_0^-(\pi/2, \alpha, s)$ and so by (7) and Stirling's formula

$$\begin{aligned} |\sigma_{\alpha\beta}(s)| &\leq c_2(1 + A_\varphi(s))|s|^2 e^{\pi \operatorname{Max}(0, \pi|\alpha| - |s|/2)} \\ &\leq c_2(1 + A_\varphi(s))|s|^2 e^{\pi^2|\alpha|}. \end{aligned} \tag{13}$$

Again Stirling's formula with the first of these two gives

$$|\mathfrak{q}_0^-(s, \alpha)\sigma_{\alpha\beta}(s)| \leq c_3(1 + A_\varphi(s))|s|^2 e^{-\pi \operatorname{Max}(0, \pi|\alpha| - |s|/2)}.$$

Thus, from [10; Th. 6] follows

$$|\mathfrak{q}_0^-(s, \alpha)\sigma_{\alpha\beta}(s)| \leq c_3(1 + A_\varphi(s))|s|^2 e^{\pi|s|/2} e^{-\pi^2 \operatorname{Max}(|\alpha|, |\beta|)}. \tag{14}$$

In the same way we also obtain that

$$|\mathfrak{q}_0^-(s, \alpha)^{-1}\sigma_{\alpha\beta}(s)| \leq c_4(1 + A_\varphi(s))|s|^2 e^{3\pi^2|\alpha|}. \tag{15}$$

Next we require some information about $A_\varphi(s)$ itself. Let $h(s)$ be a function satisfying (i), (ii), (iii) of [9; §3], and let $q_h(z, w)$ be the associated point-pair invariant. We shall assume that h is positive on the 'spectrum' of $-\Delta_0$, which Elstrodt [3] has shown to be contained in $\{s : s(1-s) > 0\}$. Now let

$$Q_h(z, w) = \sum_{g \in G} \chi(g)^{-1} q_h(gz, w),$$

which therefore is the kernel of a positive integral operator on $L(G, V)$. It follows now from §3 and the definition of $h(s)$ that the contribution of the space $L(G, V)_c$ to the kernel is

$$Q_h^c(z, w) = (2\pi)^{-1} \int_{L_+} \int_B E_\zeta(z, s) E_\zeta(w, s)^* d\zeta |D_0(s)|^{-2} h(s) |ds|.$$

Mercer's theorem (or at least the argument leading to it in [1]) shows that this integral is absolutely convergent and that

$$\operatorname{Tr}(Q_h^c(z, z)) \leq \operatorname{Tr}(Q_h(z, z)).$$

However, the series defining $Q_h(z, z)$ converges uniformly on compact sets ([11], [12]) and hence

$$\int_{D^\varphi} \operatorname{Tr}(Q_h^c(z, z)) d\sigma(z) < \infty.$$

Taking the special case $h(s) = (a + s(1-s))^{-b}$ $(a > 0, b > 1)$ this gives, for any $b > 1$,

$$\int_{L_+} (\operatorname{Im}(s) A_\varphi(s))^2 |s|^{-1-2b} |ds| < \infty. \tag{16}$$

Cauchy's inequality shows that, for any $b > 1$

$$\int_{L_+} \operatorname{Im}(s) A_\varphi(s) |s|^{-b-1} |ds| < \infty. \tag{17}$$

Let us now consider eigenfunctions $f_\mu(z) \in L(G, V)$ with parameter $s_\mu (\operatorname{Re}(s_\mu) \geq 1/2)$. k can be arbitrary. From [10; §6] it follows that $\operatorname{Re}(s_\mu) > 1/2$, and the Fourier expansion of f_μ has the form

$$j(A_j, z_j)^k Y_j f_\mu(z) = \sum_{\alpha \in Z_j} \mathfrak{E}_\alpha(r_j(z)) \mathfrak{Q}_k^-(\theta_j(z), \alpha, s_\mu) c(\alpha, \mu). \tag{18}$$

The set of all s_μ lies in the compact interval $[1/2, 1]$. It follows from [10; §6] that for s in this interval and $\epsilon > 0$,

$$\int_{\theta_1}^{\theta_2} |Q_k^-(\theta, X, s)|^2 \sin^{-2}\theta d\theta \geq c_5 |\Gamma(s - |k|)|^{-2} e^{-2\pi(\pi - \theta_2 + \epsilon)|X|}.$$

Let

$$\|f\|_\varphi^2 = \int_{D^\varphi} (f, f)_V d\sigma.$$

Suppose $\varphi = (\varphi_1, \ldots, \varphi_p)$, $\varphi_j < \pi/2$. Take $\theta_2 = \pi/2$, $\theta_1 = \varphi_j$ and on evaluating the norm of f_μ integrated over the subset of D^φ defined by $1 \leq r_j(z) < e^{\kappa(j)}$, $\theta_1 \leq \theta_j(z) \leq \theta_2$ we see that, if $\epsilon > 0$ there is $c_6 > 0$ so that

$$\|c(\alpha,\mu)\|_V^2 \le c_6 |\Gamma(s_\mu - |k|)|^2 e^{(\pi^2+\epsilon)|\alpha|} \|f_\mu\|_\varphi^2.$$

From (3) we see that

$$\int_0^{\varphi_j} |Q_k^-(\theta, X, s)|^2 \sin^{-2}\theta d\theta \le (2s-1)^{-1} e^{-2\pi(\pi-\varphi_j-\epsilon)|X|}.$$

This shows that

$$1 = \|f_\mu\|^2 \le c_7 (2s_\mu - 1)^{-1} |\Gamma(s_\mu - |k|)|^2 \|f_\mu\|_\varphi^2.$$

Considering, as above, the contribution of the $\{f_\mu\}$ to the kernel $Q_h(z, w)$ we obtain that

$$\sum_\mu (2s_\mu - 1)/\Gamma(s_\mu - |k|)^2 < \infty.$$

Let k_0 be the unique number so that

$$k_0 = |k| - m \qquad (m \in \boldsymbol{Z})$$
$$1/2 \le k_0 \le \delta(G),$$

noting, however that k_0 might not exist. The 1/2 and k_0 (if it exists) are the only possible accumulation points of $\{s_\mu\}$. Further

$$\sum (s_\mu - 1/2), \sum (s_\mu - k_0)^2$$

are convergent, unless $k_0 = 1/2$, when

$$\sum (s_\mu - 1/2)^3$$

converges.

We now show that there is no continuous spectrum in $]1/2, \delta(G)]$. Suppose $I \subset]1/2, \delta(G)]$, $k_0 \notin I$, I compact and that there exists a family of generalised eigenfunctions $e(z, s)$ with respect to a measure μ. μ is a measure supported on I so that $\mu(\{s\}) = 0$. The map

$$f \mapsto \int_I e(z, s) f(s) d\mu(s)$$

is an isometry of $L^2(I, \mu)$ onto a closed subspace of $L(G, V)$. As Δ_k is an essentially self-adjoint elliptic operator the $e(z, s)$ are, for μ-almost all s, C^∞-eigenfunctions of Δ_k. All these facts can be gleaned from [7; Ch. XVIII §§8, 9, Ch. XI §1], [13; §§3, 5].

In particular, $e(z, s)$ has a Fourier expansion of the form

$$j(A_j, z_j)^k Y_j e(z, s) = \sum_{\alpha \in Z_j} \mathfrak{E}_\alpha(r_j(z))(\mathfrak{Q}_k^+(\theta_j(z), \alpha, s)c_\alpha^+(s) + \mathfrak{Q}_k^-(\theta_j(z), \alpha, s)c_\alpha^-(s)).$$

We have also, by an unnumbered equation between [7; Ch. XVIII §9, Eqs. (9.2) and (9.3)], that if

$$B_\varphi(s)^2 = \int_{D^\varphi} (e(z, s), e(z, s))_V d\sigma(z)$$

$$\int_I B_\varphi(s)^2 d\mu(s) < \infty.$$

Applying the method of point-pair invariants again shows that, if φ is given there is φ' so that

$$|\mathfrak{Q}_k^+(\varphi, \alpha, s)c_\alpha^+(s) + \mathfrak{Q}_k^-(\varphi, \alpha, s)c_\alpha^-(s)| < c_8 B_{\varphi'}(s).$$

Taking two suitable values of φ and using [10; §6] shows that, if $\epsilon > 0$ is given there is $b_\epsilon \in L^2(I, \mu)$ so that

$$\|c_\alpha^\eta(s)\| < b_\epsilon(s) e^{2\pi\epsilon|\alpha|} \quad (\eta = \pm).$$

It follows now from (3) that, if $f \in L^2(I, \mu)$

$$\int_I \sum_{\alpha \in Z_j} \mathfrak{E}_\alpha(r_j(z))\mathfrak{Q}_k^-(\theta_j(z), \alpha, s)c_\alpha^-(s)f(s)d\mu(s)$$

is square-integrable on $D_j = \{z : 1 \le r_j(z) < e^{\kappa(j)}, 0 < \theta_j(z) \le \pi/2\}$. From the definition of $e(z, s)$ it follows that so is

$$\int_I \sum_{\alpha \in Z_j} \mathfrak{E}_\alpha(r_j(z))\mathfrak{Q}_k^+(\theta_j(z), \alpha, s)c_\alpha^+(s)f(s)d\mu(s).$$

This function is, however, by the estimates above and (3) square-integrable on $D_j^* = \{z : 1 \le r_j(z) < e^{\kappa(j)}, \pi/2 \le \theta_j(z) < \pi\}$. Thus it is square-integrable on $D_j \cup D_j^*$, which is a fundamental domain for $G_{\alpha(j)}$. It is then an automorphic form of weight k, multiplier χ whose spectral support with respect to $-\Delta_k$ lies in $[0, 1/4]$. The spectrum lies in $[1/4, \infty[$ ([3], [9]) and thus we have a contradiction unless $c_\mu^+(s) \equiv 0$. But then $e(z, s)$ is a square-integrable eigenfunction. As $\mu(\{s\}) = 0$ and as there are at most a countable set of eigenfunctions it follows that $e(z, s)$ does not exist. Hence:

THEOREM 2: *Suppose k arbitrary. If* $\delta(G) < 1/2$ *then*

$$L(G, V)_c = L(G, V).$$

If $\delta(G) \geq 1/2$, *the continuous spectrum of* $-\Delta_k$ *lies in* $[1/4, \infty[$, *and the point spectrum lies in* $[\delta(G)(1-\delta(G)), 1/4[$. *If the eigenvalues are* $\lambda_\mu = s_\mu(1-s_\mu)$, $s_\mu > 1/2$, *and if* $k_0 \in [1/2, \delta(G)]$, $k_0 - |k| \in \boldsymbol{Z}$ *then*

$$\sum (s_\mu - 1/2)(s_\mu - k_0)^2 < \infty.$$

(The second term is to be omitted if k_0 *does not exist.)*

The only outstanding statement, the first, appears in [9].

COROLLARY: $E_\zeta(z, s)$ *can be continued as a meromorphic function in s to* $\boldsymbol{C}\backslash\{1/2, k_0, 1-k_0\}$, *smooth in z and* ζ. *The only poles in* $\operatorname{Re}(s) \geq 1/2$ *are at the points* s_μ *and these are exactly first order.*

PROOF: The method of [10; §8] shows that if $s_0 \in]1/2, \delta(G)]$, and if $\operatorname{Re}(s_n) \neq 0$, $s_n \to s_0$ then there is a subsequence s'_m, and a sequence w_m so that

$$w_m^{-1} E_\zeta(z, s'_m)$$

converges, uniformly on compact subsets, to a smooth eigenfunction of Δ_k in z, smooth also in ζ. Furthermore, $w_m^{-1} E^{!}_\alpha(z, s'_m)$ converges, for infinitely many $\alpha \in Z^*$, to a non-zero function. If $w_m \to \infty$ then this is an L^2 function (from the Fourier expansion). Thus, when $s_0 \notin \{k_0, s_\mu\}$, $w_m \not\to \infty$ and we can take w_m to be constant. Also, the limit of $E_\zeta(z, s)$, as $s \to s_0$ ($\notin \{k_0, s_\mu\}$) must be unique. This gives, by the reflection principle, the analytic continuation to $\operatorname{Re}(s) > 1/2$, $s \notin \{k_0, s_\mu\}$.

Next in [10; Eq. (51)] we set $s = t = \sigma + ir$, sum over α and obtain

$$\begin{aligned} -i(2\sigma - 1)r \int_B \int_{D^\varphi} & \operatorname{Tr}(E_\zeta(z, s)^* E_\zeta(z, s)) m_{s,\bar{s}}(\zeta) d\sigma(z) d\zeta \\ &= \sum_{\alpha \in Z^*} \operatorname{Tr}(\Re_k^{++}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha) \\ &+ \sum_{\alpha \in Z^*} e(k) \operatorname{Tr}(\sigma_{\alpha\alpha}(s)^* \Re_k^{+-}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha))/\kappa(\alpha) \\ &+ \sum_{\alpha \in Z^*} e(-k) \operatorname{Tr}(\Re_k^{-+}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha)\sigma_{\alpha\alpha}(s))/\kappa(\alpha) \\ &+ \sum_{\alpha,\beta \in Z^*} \kappa(\alpha)^{-1}\kappa(\beta)^{-1} \operatorname{Tr}(\sigma_{\beta\alpha}(s)^* \Re_k^{--}(\varphi_{i(\beta)}, s, \bar{s}, \beta)\sigma_{\beta\alpha}(s)), \end{aligned}$$

where, if $\zeta \in \Omega_{\alpha(j)}$, as in [10; §7],

$$m_{s,t}(\zeta) = |A_j'(\zeta_j)|^{-1+s+t}|\zeta_j|^{s+t}.$$

Let $A_\varphi(s)$ be as defined at the beginning of this section but now for arbitrary k. The method of point-pair invariants shows that the estimates (13), (14), (15) of this section hold for arbitrary k when s is restricted to a compact set. Thus, if $r \neq 0$, $|r| < 1$ we show that the first three terms on the right-hand side above are $O(A_\varphi(s)+1)$, uniformly, when σ lies in compact subintervals of $]1/2, 1]$. By [10; Th. 5]

$$(-i(2\sigma-1)r)^{-1} \sum_\beta \mathrm{Tr}\,(\sigma_{\beta\alpha}(s)^* \mathfrak{R}_k^{--}(\varphi_{i(\beta)}, s, \bar{s}, \beta)\sigma_{\beta\alpha}(s)) < 0.$$

Thus, for σ lying in a compact subinterval of $]1/2, 1]$

$$rA_\varphi(s)^2 = O(A_\varphi(s)+1)$$

uniformly. In other words

$$A_\varphi(s) = O(r^{-1}).$$

We now bring in the elementary lemma:

LEMMA: *Suppose $f(z)$ is analytic in a deleted neighbourhood of* 0, *and that, if* $|\mathrm{Re}\,(z)| \leq 1$, $|\mathrm{Im}\,(z)| \leq 1$,

$$|f(z)| \leq |\mathrm{Im}\,(z)|^{-1}.$$

Then $f(z)$ can be continued to a meromorphic function in the full neighbourhood of 0, *with at most a first-order pole at* 0.

PROOF: Let C_1 be the square $\epsilon/2-3\epsilon i$, $\epsilon/2+3\epsilon i$, $2\epsilon+3\epsilon i$, $2\epsilon-3\epsilon i$, and let L be the line segment $\epsilon+iy$ ($|y| \leq \epsilon$). L lies inside C_1.

$$(z-\epsilon/2)(z-2\epsilon)f(z)$$

is bounded by $\sqrt{(3/2)^2+6^2}\epsilon < 7\epsilon$ on the vertical edges of C_1, and by $((3/2)^2+3^2)^{1/2}\epsilon/(3\epsilon) < 7\epsilon$ on the horizontal edges. Thus, by the maximum-modulus principle, for $z \in L$

$$|(z-\epsilon/2)(z-2\epsilon)f(z)| \leq 7\epsilon.$$

However, for $z \in L$

$$|(z-\epsilon/2)(z-2\epsilon)| > \epsilon^2/2.$$

Hence

$$|f(z)| \leq 14\epsilon^{-1}.$$

From this it follows that

$$|zf(z)| < 20$$

on any square with vertices $\pm\epsilon \pm \epsilon i$. Hence, by Riemann's theorem on removable singularities this function can be continued to an entire function at 0. This proves the lemma.

From the estimates above it follows that $E_\zeta(z, s)$ satisfies the conditions of the lemma and can therefore be continued to each of the points s_μ with at most a first-order pole there.

We show finally that $E_\zeta(z, s)$ has a pole at $s = s_\mu$. Let $f_\mu(z)$ be the corresponding eigenfunction, and let

$$j(A_j, z_j)^k Y_j f_\mu(z) = \sum_{\alpha \in Z_j} \mathfrak{E}_\alpha(r_j(z)) \mathfrak{Q}_k^-(\theta_j(z), \alpha, s_\mu) c(\alpha, \mu),$$

be the Fourier expansion. Then, as in [10; §5]

$$-(s(1-s) - s_\mu(1-s_\mu)) \int_{D^\varphi} E_\alpha^!(z, \bar{s})^* f_\mu(z) d\sigma(z)$$
$$= e(-k)\kappa(\alpha)^2 \mathfrak{R}_k^{-+}(\varphi_{i(\alpha)}, s_\mu, s, \alpha) c(\alpha, \mu)$$
$$+ \sum_{\beta \in Z^*} \kappa(\beta) \sigma_{\beta\alpha}(\bar{s})^* \mathfrak{R}_k^{--}(\varphi_{i(\beta)}, s_\mu, s, \beta) c(\beta, \mu).$$

Suppose $E_\zeta(z, s)$ to be regular at $s = s_\mu$. Then the left-hand side and the second term on the right-hand side of the above equation vanish identically ([10; Cor. to Th. 5]). But $R_k^{-+}(\varphi_{i(\alpha)}, s_\mu, s_\mu, \alpha) \neq 0$ (ibid.). Thus we have a contradiction when there is an α so that $c(\alpha, \mu) \neq 0$ (and there is always one). This completes the proof in $\mathrm{Re}(s) \geq 1/2$. The rest follows using the functional equation.

Note that, it follows that $\Phi_k(s, \chi)$ can be continued to $\mathrm{Re}(s) \geq 1/2$, $s \notin \{k_0, s_\mu\}$. One can easily check that the residue of $E_\zeta(z, s)$ is a finite linear combination of eigenvectors and that therefore $\Phi_k(s)$ has a pole of order at most the dimension of the eigenspace with parameter s_μ—these facts we shall not need.

This now completes the investigation in the case of arbitrary k and henceforth we shall only deal with the case $k = 0$. We shall however prove some of the intermediary results for all k but it is only by means of the estimates obtained at the beginning of this section that we can use these effectively.

It is worth noting that the eigenvalue $k_0(1-k_0)$ of $-\Delta_k$ can be of infinite multiplicity (cf. [3; §12]). Whether or not 1/2 or k_0 can be an accumulation point of the s_μ remains an open question.

5. The local trace formula

We recall the following Maaß–Selberg relation ([10; Th. 4, §7])

$$(s(1-s)-t(1-t))\int_B\int_{D^\varphi}\operatorname{Tr}(E_\zeta(z,\bar t)^* E_\zeta(z,s))m_{s,t}(\zeta)d\sigma(z)d\zeta$$

$$=\sum_{\alpha\in Z^*}\operatorname{Tr}(R_k^{++}(\varphi_{i(\alpha)},s,t,\alpha)$$

$$+\sum_{\alpha\in Z^*}e(k)\operatorname{Tr}(\sigma_{\alpha\alpha}(\bar t)^* R_k^{+-}(\varphi_{i(\alpha)},s,t,\alpha))/\kappa(\alpha)$$

$$+\sum_{\alpha\in Z^*}e(-k)\operatorname{Tr}(R_k^{-+}(\varphi_{i(\alpha)},s,t,\alpha)\sigma_{\alpha\alpha}(s))/\kappa(\alpha)$$

$$(19)\qquad +\sum_{\alpha,\beta\in Z^*}\operatorname{Tr}(\sigma_{\alpha\beta}(\bar t)^* R_k^{--}(\varphi_{i(\alpha)},s,t,\alpha)\sigma_{\alpha\beta}(s))/(\kappa(\alpha)\kappa(\beta)).$$

From [10; Th. 8] it follows that this is valid for all s, t in $C\backslash[1-\delta(G),\delta(G)]$ (C if $\delta(G)<1/2$), with the natural interpretation at the poles. The vanishing of the right-hand side when $t=1-s$ is, as we noted in [10], essentially the functional equation for $(\sigma_{\alpha\beta}(s))$. We shall now calculate

$$A(s,\varphi)=\int_B\int_{D^\varphi}\operatorname{Tr}(E_\zeta(z,\overline{1-s})^* E_\zeta(z,s))d\sigma(z)d\zeta,$$

as a limiting version of (19). Recall that if f_n is a sequence of analytic functions converging uniformly on compact subsets to a function f, then f_n' converges in the same sense to f'. This justifies the passage to the limit in (19).

Unfortunately we must now introduce some more special functions in order to express our results. Define

$$(20)\qquad L_k^{\epsilon\eta}(\varphi,s,a)=\lim_{t\to 1-s}(R_k^{\epsilon\eta}(\varphi,s,t,a)-R_k^{\epsilon\eta}(\varphi,s,t,a))/(s+t-1)$$

and

$$(21)\qquad M_k(\varphi,s,a)=\lim_{t\to s}R_k^{--}(\varphi,s,t,a)/(s-t).$$

We shall also write, if $\alpha=(a_1,\dots,a_n)$,

$$\mathfrak{L}_k^{\epsilon\eta}(\varphi,s,\alpha)=\operatorname{diag}(L_k^{\epsilon\eta}(\varphi,s,a_j)),$$

and

$$\mathfrak{M}_k(\varphi,s,\alpha)=\operatorname{diag}(M_k(\varphi,s,a_j)).$$

Finally, the matrix over W ([10; §4]) with entries $\mathfrak{L}^{\epsilon\eta}(\varphi, s, \alpha)\delta_{\alpha\beta}$ (resp. $\mathfrak{M}_k(\varphi, s, \alpha)\delta_{\alpha\beta}$) will be denoted by $\mathscr{L}_k^{\epsilon\eta}(\varphi, s)$ (resp. $\mathscr{M}_k(\varphi, s)$).

With these notations the passage to the limit gives

$$\begin{aligned}(1-2s)A(s,\varphi) = &\sum_{\alpha\in Z^*} \mathrm{Tr}\,(\mathfrak{L}_k^{++}(\varphi_{i(\alpha)}, s, \alpha))\\ &+\sum_{\alpha\in Z^*} e(k)(\mathrm{Tr}\,(\sigma'_{\alpha\alpha}\overline{(1-s)}^*\mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, 1-s, \alpha))\\ &+\mathrm{Tr}\,(\sigma_{\alpha\alpha}\overline{(1-s)})^*\mathfrak{L}_k^{+-}(\varphi_{i(\alpha)}, s, \alpha))/\kappa(\alpha)\\ &+\sum_{\alpha\in Z^*} e(-k)\,\mathrm{Tr}\,(\mathfrak{L}_k^{-+}(\varphi_{i(\alpha)}, s, \alpha)\sigma_{\alpha\alpha}(s)))/\kappa(\alpha)\\ &+\sum_{\alpha\in Z^*} (\mathrm{Tr}\,(\sigma'_{\alpha\beta}\overline{(1-s)}^*\mathfrak{R}_k^{--}(\varphi_{i(\alpha)}, s, 1-s, \alpha)\sigma_{\alpha\beta}(s))\\ &+\mathrm{Tr}\,(\sigma_{\alpha\beta}\overline{(1-s)})^*\mathfrak{L}_k^{--}(\varphi_{i(\alpha)}, s, \alpha)\sigma_{\alpha\beta}(s))/(\kappa(\alpha)\kappa(\beta)).\end{aligned}$$

As long as we do not separate each of the infinite summations this is uniformly convergent on compact sets. Let s_0 be a regular point of $E_\zeta(z, s)$. Then there is a compact neighbourhood K of s_0 and a constant $c > 0$, so that

$$|\mathfrak{q}_k^-(s,\alpha)\sigma_{\alpha\beta}(s)| < ce^{-2\pi\theta\,\mathrm{Max}(|\alpha|,|\beta|)}$$

(θ given in advance), for $s \in K$. If $K_1 \subset\subset K$ there is a constant $c_1 = c_1(K_1, \theta)$ so that (by Cauchy's theorem)

$$|(d/ds)\mathfrak{q}_k^-(s,\alpha)\sigma_{\alpha\beta}(s)| < c'e^{-2\pi\theta\cdot\mathrm{Max}(|\alpha|,|\beta|)}$$

for $s \in K_1$. This suffices to allow us to split the terms. It is also possible, using the estimates of §2 to find bounds on the $L_k^{\epsilon\eta}(\varphi, s, a)$ which also prove that the series above can be split apart. All problems of convergence arising here can be treated in this way and we shall not deal with them further.

The formula above can be very considerably simplified by means of the functional equation and [10; Th. 6], i.e.

$$(22)\qquad (\mathfrak{q}_k^-\overline{(1-s}, \alpha)\sigma_{\alpha\beta}\overline{(1-s)})^* = e(-3k)\mathfrak{q}_k^-(1-s,\beta)\sigma_{\beta\alpha}(1-s).$$

In the calculations it is advantageous to use matrices over W. Let $I_k(s)$ be the matrix with entries $\mathfrak{q}_k^-(s,\alpha)\delta_{\alpha\beta}$. With the notations above, and those in [10], we obtain

$$\begin{aligned}(1-2s)A(s,\varphi) = {} & \mathrm{Tr}_W\,(\mathscr{L}_k^{++}(\varphi,s)) \\ & + e(-k)(2s-1)\,\mathrm{Tr}_W\,((\Sigma_0 I_k^{-1})'(1-s)J_k(s)I_k(1-s)) \\ & + e(k)\,\mathrm{Tr}_W\,((\Sigma_0 I_k^{-1})(1-s)\mathscr{L}_k^{+-}(\varphi,s)) \\ & + \mathrm{Tr}_W\,(\mathscr{L}_k^{-+}(\varphi,s)I_k(s)^{-1}\Sigma_0(s)) \\ & + (2s-1)e(-k)\,\mathrm{Tr}_W\,((\Sigma_0 I_k^{-1})'(1-s)I_k(1-s)\Sigma_0(s)) \\ & + e(k)\,\mathrm{Tr}_W\,(\Sigma_0(1-s)I_k(1-s)^{-1}\mathscr{L}_k^{--}(\varphi,s)I_k(s)^{-1}\Sigma_0(s)).\end{aligned}$$

The functional equation can be written in the form

$$\begin{aligned}\Sigma_0(s)\Sigma_0(1-s) = {} & -J_k(s)\Sigma_0(1-s) - J_k(1-s)\Sigma_0(s) \\ & + e(-k)I_k(s)I_k(1-s)\end{aligned} \tag{23}$$

(using [10; Eq. (41), (36)]). Thus we have

$$\begin{aligned}(1-2s)A(s,\varphi) = {} & \mathrm{Tr}_W\,(\mathscr{L}_k^{++}(\varphi,s) + \mathscr{L}_k^{--}(\varphi,s)) \\ & + \mathrm{Tr}_W\,(\Sigma_0(1-s)(e(k)I_k(1-s)^{-1}\mathscr{L}_k^{+-}(\varphi,s) \\ & - e(k)I_k(1-s)^{-1}\mathscr{L}_k^{--}(\varphi,s)I_k(s)^{-1}J_k(s))) \\ & + \mathrm{Tr}_W\,(\Sigma_0(s)(\mathscr{L}_k^{-+}(\varphi,s)I_k(s)^{-1} \\ & - e(k)J_k(1-s)I_k(1-s)^{-1}\mathscr{L}_k^{--}(\varphi,s)I_k(s))) \\ & + \mathrm{Tr}_W\,((\Sigma_0 I_k^{-1})'(1-s)I_k(1-s)(J_k(s)+\Sigma_0(s))) \\ & \times (2s-1)e(-k).\end{aligned}$$

From [10; Th. 3, Cor. to Th. 3] we have

$$\begin{aligned}& Q_k^{+}(\theta,a,s)q_k^{-}(s,a) - Q_k^{-}(\theta,a,s)q_k^{+}(s,a) \\ & \qquad = D_k(s)q_k^{-}(s,a)q_k^{-}(1-s,a)^{-1}Q_k^{-}(\theta,a,1-s).\end{aligned}$$

Hence

$$\begin{aligned}& R_k^{+-}(\theta,s,t,a)q_k^{-}(s,a) - R_k^{--}(\theta,s,t,a)q_k^{+}(s,a) \\ & \qquad = D_k(s)q_k^{-}(s,a)q_k^{-}(1-s,a)^{-1}R_k^{--}(\theta,1-s,t,a), \\ & e(-2k)R_k^{-+}(\theta,s,t,a)q_k^{-}(t,a) - e(-k)R_k^{--}(\theta,s,t,a)q_k^{+}(t,a) \\ & \qquad = e(-k)D_k(t)q_k^{-}(t,a)q_k^{-}(1-t,a)^{-1}R_k^{--}(\theta,s,1-t,a).\end{aligned}$$

Using now (20), (21) and [10; Th. 5 + Cor.] we find

$$\begin{aligned}& L_k^{+-}(\theta,s,a)q_k^{-}(s,a) - L_k^{--}(\theta,s,a)q_k^{+}(s,a) \\ & \qquad = D_k(s)q_k^{-}(s,a)q_k^{-}(1-s,a)^{-1}M_k(\theta,1-s,a),\end{aligned}$$

$$L_k^{-+}(\theta, s, a)q_k^-(1-s, a) - e(k)L_k^{--}(\theta, s, a)q_k^+(1-s, a)$$
$$= -e(k)D_k(1-s)q_k^-(1-s, a)q_k^-(s, a)^{-1}M_k(\theta, s, a)$$
$$+ (2s-1)e(-k)q_k^-(s, a)q_k^-(1-s, a)q_k^{+\prime}(1-s, a)$$
$$-(2s-1)e(-k)q_k^+(s, a)q_k^-(1-s, a)q_k^{-\prime}(1-s, a).$$

Using these identities we obtain

$$(1-2s)A(s, \varphi) = \mathrm{Tr}_W\,(\mathcal{L}_k^{++}(\varphi, s) + \mathcal{L}_k^{--}(\varphi, s))$$
$$+ \mathrm{Tr}_W\,(\Sigma_0(1-s)I_k(1-s)^{-2}\mathcal{M}_k(\varphi, 1-s))D_k(s)e(k)$$
$$- \mathrm{Tr}_W\,(\Sigma_0(s)I_k(s)^{-2}\mathcal{M}_k(\varphi, s))D_k(1-s)e(k)$$
$$+ (2s-1)e(-k)\,\mathrm{Tr}_W\,((\Sigma_0 I_k^{-1})'(1-s)I_k(1-s)$$
$$\times (J_k(s) + \Sigma_0(s)) - \Sigma_0(s)I_k'(1-s)I_k(1-s)^{-1}J_k(1-s)$$
$$+ \Sigma_0(s)J_k'(1-s)).$$

The reorganisation of the terms can be easily justified. Let, as in [10],

$$\Phi(s) = \det\,(I + J_k(s)^{-1}\Sigma_0(s)).$$

From this, and the functional equation

$$\Phi'(s)/\Phi(s) = \mathrm{Tr}_W\,((J_k^{-1}\Sigma_0)'(s)(I + J_k(s)^{-1}\Sigma_0(s))^{-1})$$
$$= \mathrm{Tr}_W\,((J_k^{-1}\Sigma_0)'(s)(J_k(1-s) + \Sigma_0(1-s))J_k(s))$$
$$\times (D_k(s)D_k(1-s))^{-1}.$$

Expanding and applying the functional equation again

$$\Phi'(1-s)/\Phi(1-s) = \mathrm{Tr}_W\,(\Sigma_0'(1-s)(J_k(s) + \Sigma_0(s)) + J_k'(1-s)\Sigma_0(s)$$
$$- J_k'(1-s)J_k(1-s)^{-1}I_k(s)I_k(1-s)e(-k))$$
$$\times (D_k(s)D_k(1-s))^{-1}.$$

Define now

$$\psi_1(s) = e(-k)\,\mathrm{Tr}_W\,((I_k/J_k)'(s)I_k(1-s)J_k(s))(D_k(s)D_k(1-s))^{-1}.$$

Then we finally obtain

$$A(s, \varphi)e(k)(D_k(s)D_k(1-s))^{-1} = \mathrm{Tr}_W\,(\mathcal{L}_k^{++}(\varphi, s) + \mathcal{L}_k^{--}(\varphi, s))e(k)$$
$$\times (D_k(s)D_k(1-s))^{-1}(1-2s)^{-1}$$
$$+ e(2k)(\mathrm{Tr}_W\,(\Sigma_0(1-s)I_k(1-s)^{-2}$$
$$\times \mathcal{M}_k(\varphi, 1-s))D_k(1-s)^{-1}(1-2s)^{-1}$$
$$+ \mathrm{Tr}_W\,(\Sigma_0(s)I_k(s)^{-2}\mathcal{M}_k(\varphi, s))D_k(s)^{-1}$$
$$\times (2s-1)^{-1}) - \Phi'(1-s)/\Phi(1-s)$$
$$+ \psi_1(1-s). \tag{24}$$

If

$$\psi(s) = \det(D_k(s)D_k(1-s)J_k(s)^{-1}J_k(1-s)^{-1})$$

then

$$\Phi(s)\Phi(1-s) = \psi(s).$$

A simple calculation shows that

$$\psi_1(s) - \psi_1(1-s) = \psi'(s)/\psi(s)$$

and hence $\Phi'(1-s)/\Phi(1-s) - \psi_1(1-s)$ is invariant under $s \mapsto 1-s$.

If we let $E_\zeta^{(j)}(z,s)$ be the corresponding Eisenstein series for $G_{\alpha(j)}$ and

$$\begin{aligned} D_j &= \{z : 1 \le r_j(z) < e^{\kappa(j)}, 0 < \theta_j(z) \le \pi/2\}, \\ D_j^\varphi &= \{z : 1 \le r_j(z) < e^{\kappa(j)}, \varphi_j < \theta_j(z) \le \pi/2\}, \\ B_j &= \{\zeta : \zeta_j \in [1, e^{\kappa(j)}[\cup] - e^{\kappa(j)}, -1]\}, \text{ and} \\ B(z) &= B_j \quad (z \in D_j) \\ &= \emptyset \quad (\text{otherwise}). \end{aligned}$$

Then one finds, in just the same way as above

$$\sum_{j=1}^{p} \int_{B_j} \int_{D_j} \operatorname{Tr}(E_\zeta^{(j)}(z, 1-\bar{s})^* E_\zeta^{(j)}(z,s)\, d\sigma(z)\, d\zeta\, e(k)(D_k(s)D_k(1-s))^{-1}$$

$$= \operatorname{Tr}_W(\mathscr{L}_k^{++}(\varphi,s) + \mathscr{L}_k^{--}(\varphi,s))e(k)(D_k(s)D_k(1-s))^{-1}. \tag{25}$$

The terms in (24) offering the greatest difficulty are those involving $\mathscr{M}_k(\varphi, s)$ and we shall now investigate these. From the integral representation for $R_k^{--}(\theta, s, t, a)/(s(1-s) - t(1-t))$ used in §2 we obtain also an integral representation for $\mathscr{M}_k(\varphi, s)$. Thus the limiting case of (6) gives estimates for $\mathscr{M}_k(\varphi, s)$. This can easily be verified directly. We find, when $\operatorname{Re}(s) = 1/2$,

$$|M_k(\varphi, s, a)(2s-1)^{-1} - q_k^-(s,a)\overline{q_k^-(\bar{s},a)}(2s-1)^{-1}\sin^{2s-1}\varphi|$$

$$\le c_1 \varphi e^{-4\pi\alpha_2|a| + (\alpha_2+\pi)2|\operatorname{Im}(s)|}.$$

Let $h(s)$ be a continuous function on $L = \{\operatorname{Re}(s) = 1/2\}$. From now on assume that $k = 0$. We shall consider

$$\lim_{\varphi \to 0} \int \operatorname{Tr}_W(\Sigma_0(s)I_k(s)^{-2}\mathscr{M}_k(\varphi, s))D_k(s)^{-1}(2s-1)^{-1}h(s)|ds|.$$

We estimate first the 'error term'. The diagonal matrix elements of

$I_k(s)^{-2}\Sigma_0(s)$ are, up to a constant factor,

$$q_k^-(s,\alpha)^{-1}\sigma_{\alpha\alpha}(s) = O((1 + A_{\varphi 0}(s))|s|^2 e^{3\pi^2|\alpha|})$$

by (15). Choose

$$\alpha_1 = \pi/10, \quad \alpha_2 = 7\pi/8.$$

Then we see that the error term is

$$\varphi \int_L O(A_{\varphi_0}(s) + 1)|s|^2 e^{(15\pi/4)|\mathrm{Im}(s)|} h(s)ds.$$

If h satisfies the conditions that

$$h(s) = O(e^{-4\pi|\mathrm{Im}(s)|}) \tag{26}$$

as $s \to \infty$, and

$$h(s) = O(s - 1/2) \tag{27}$$

as $s \to 1/2$. Then it follows from (17) that the integral converges and hence the error term is $O(\varphi) = o(1)$.

The principal term is, up to a constant factor,

$$\lim_{\varphi \to 0} \int_L \mathrm{Tr}\,(\Sigma_0(s)) \sin^{2s-1} \varphi D_k(s)^{-1}(2s-1)^{-1} h(s)ds.$$

By (14),

$$\mathrm{Tr}\,(\Sigma_0(s)) = O((A_\varphi(s) + 1)|s|^2 e^{\pi|s|})$$

and this, with (26), (27) and the Riemann–Lebesgue lemma show that this limit exists and is 0.

When $\mathrm{Re}\,(s) = 1/2$ $A_\varphi(s) = A(s,\varphi)$ and then (26) and (27) show that

$$\int_L A(s,\varphi) D_k(s)^{-2} h(s)ds$$

exists. It then follows from (24) that

$$\int_L (\Phi'(s)/\Phi(s) - \psi_1(s))h(s)ds$$

converges absolutely.

Define

$$\begin{aligned} E_\zeta^H(z,s) &= E_\zeta^{(j)}(z,s) \quad (z \in D_j) \\ &= 0 \qquad\qquad\; (\text{otherwise}). \end{aligned}$$

Hence

$$\lim_{\varphi\to 0}\int_{D^\varphi}\int_L\Big(\int_B \operatorname{Tr}(E_\zeta(z,s)^*E_\zeta(z,s))d\zeta - \int_{B(z)}\operatorname{Tr}(E_\zeta^H(z,s)^*E_\zeta^H(z,s))d\zeta\Big) \times |D_k(s)|^{-2}h(s)|ds|\,d\sigma(z)$$

$$= -\int_L (\Phi'(s)/\Phi(s)-\psi_1(s))h(s)|ds| \qquad (28)$$

This formula is the object of this section and from it we can deduce the completeness theorem without much additional work. If $\delta(G)<1/2$ then $A_\varphi(s)$ is bounded on L and we can therefore omit the condition (27).

6. Completeness

Let $h(s)$ be a function satisfying the following conditions:

(i) for some $\epsilon>0$ $h(s)$ can be extended to an analytic function in $|\mathrm{Re}\,(s)-1/2|\le 1/2+\epsilon$,
(ii) $h(s)=h(1-s)$,
(iii) for some $c>0$, $|h(s)|\le c(1+|s|^2)^{-1-\epsilon}$ in the strip defined in (i),
(iv) $h(1/2)=0$,
(v) for some $c'>0$, $|h(s)|<c'e^{-4\pi|\mathrm{Im}(s)|}$, on $\{\mathrm{Re}\,(s)=1/2\}$,
(vi) $h(s)\ge 0$ on $\{s: s(1-s)>0\}$.

Here $\epsilon>0$ is a fixed constant. That such functions exist can easily be shown by examples. Form, as in [9], the associated point-pair invariant $q_h(z,w)$, and

$$Q_h(z,w)=\sum_{g\in G}\chi(g)q_h(z,gw).$$

Let

$$Q_h^c(z,w)=(2\pi)^{-1}\int_{L_+}E_\zeta(z,s)E_\zeta(w,s)^*d\zeta\,|D_0(z)|^{-2}h(s)|ds|,$$

which represents the projection of $Q_h(z,w)$ to an operator on $L(G,V)_c$. By (i), (ii), (iii) it follows (cf. [9]) that Q_h is the kernel of an L^2 operator, and the $h(s)$ give the diagonal elements in a common diagonalisation of all such operators (which form a commuting set of normal operators). The operator

$$Q_h(z,w)-Q_h^c(z,w)$$

has finite trace—as we shall show presently. It is also positive and consequently it has a pure point spectrum (which will then be that appearing in theorem 2).

Let

$$Q_h^H(z, w) = \sum_{g \in G_{\alpha(j)}} \chi(g) q_h(z, gw) \quad (z, w \in D_j)$$

$$= 0 \qquad \text{(otherwise)}.$$

From [9], [12] it follows that

$$q_h(z, w) = O(\sigma(z, w)^{-\alpha}),$$

where $\alpha > 1$, and $\sigma(z, w)$ is as before (cf. §2). It follows without difficulty that, for $z \in D_j$,

$$Q_h(z, z) - Q_h^H(z, z) = O(\theta_j(z)^\alpha).$$

Hence

$$I_1 = \int_D \mathrm{Tr}\,(Q_h(z, z) - Q_h^H(z, z)) d\sigma(z)$$

converges absolutely. On the other hand, from [9] we have the spectral resolution for hyperbolic groups which gives

$$Q_h^H(z, z) = (2\pi)^{-1} \int_{L_+} \int_{B(z)} E_\zeta^H(z, s) E_\zeta^H(z, s)^* |D_0(s)|^{-2} d\zeta h(s) |ds|.$$

Hence, from (28) we obtain

$$I_2 = \int_D \mathrm{Tr}\,(Q_h^c(z, z) - Q_h^H(z, z)) d\sigma(z)$$

$$= -(2\pi)^{-1} \int_{L_+} (\Phi'(s)/\Phi(s) - \psi_1(s)) h(s) ds$$

and the right-hand side converges. Thus

$$\int_D \mathrm{Tr}\,(Q_h(z, z) - Q_h^c(z, z)) d\sigma(z)$$

converges. This proves our assertion. Note also that

$$\sum h(s_\mu)$$

converges but that this is weaker than theorem 2 since (ii), (iv), (vi) imply that h has a zero of order 4 at 1/2.

THEOREM 3: *$L(G, V)_c$ spans the continuous spectrum of* $-\Delta_0$.

This theorem is the chief result of this paper. However we note that I_2 has already been calculated, I_1 can be calculated by the transformations of Selberg [14], and

$$\int_D \operatorname{Tr}(Q_h(z,z) - Q_h^c(z,z))d\sigma(z) = \sum h(s_\mu).$$

In fact, note that I_1, I_2 exist if we drop condition (vi) above, and that $Q_h - Q_h^c$ is still trace-class by theorem 2. Thus, for $h(s)$ satisfying (i), . . ., (v) (such that h we call *admissible*),

$$I_1 - I_2 = \sum h(s_\mu).$$

We now calculate I_1, but to avoid inessential difficulties *we shall assume that G has no elliptic elements.*

Let $\{G\}$ be a set of representatives of conjugacy classes in G, and let N_g be the normaliser of g in G. Let $\{G\}_0$ be the subset of G consisting of those elements not conjugate to any element of any $G_{\alpha(j)}$. Let

$$\begin{aligned} \epsilon(z) &= 1 \quad (z \in D^{\pi/2}) \\ &= 0 \quad (\text{otherwise}) \end{aligned}$$

and

$$\begin{aligned} \epsilon_j(z) &= 1 \quad (z \in D \backslash D_j) \\ &= 0 \quad (z \in D_j). \end{aligned}$$

Then it follows that

$$\begin{aligned} Q_h(z,z) - Q_h^H(z,z) &= I \cdot q_h(z,z) \cdot \epsilon(z) \\ &\quad + \sum_{g \in \{G\}_0} \sum_{\gamma \in N_g \backslash G} \chi(g) q_h(\gamma z, g\gamma z) \\ &\quad + \sum_{j=1}^{p} \sum_{g \in G_{\alpha(j)} \backslash \{I\}} \chi(g) q_h(\gamma z, g\gamma z)\epsilon_j(\gamma z). \end{aligned}$$

Let $D(g)$ be a fundamental domain for N_g. Then

$$\begin{aligned} \int_D \operatorname{Tr}(Q_h(z,z) - Q_h^H(z,z))d\sigma(z) &= q_h(i,i)\sigma(D^{\pi/2}) \operatorname{Tr}(I) \\ &\quad + \sum_{g \in \{G\}_0} \operatorname{Tr}(\chi(g)) \int_{D(g)} q_h(z, gz) d\sigma(z) \\ &\quad + \sum_{j=1}^{p} \sum_{g \in G_{\alpha(j)} \backslash \{I\}} \operatorname{Tr}(\chi(g)) \int_{D(g) \backslash D_j} q_h(z, gz) d\sigma(z). \end{aligned}$$

The calculation of the integrals offer no difficulties and we obtain

$$I_1 = \sigma(D^{\pi/2})(4\pi)^{-1}\dim(V)\int_{-\infty}^{+\infty} r\tanh \pi r h(1/2+ir)dr$$
$$+\sum_{\gamma\in H}\sum_{j=1}^{\infty} b(\gamma)\operatorname{Tr}(\chi(\gamma^j))\log N(\gamma)g(j\log N(\gamma))/(N(\gamma)^{j/2}-N(\gamma)^{-j/2})$$

where

H = set of elements representing the primitive hyperbolic conjugacy classes,

$N(\gamma)$ = multiplier of γ, and

$b(\gamma) = 1$ if γ is not conjugate to a generator of a $G_{\alpha(j)}$,
$= 1/2$ otherwise.

$$g(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{-irx}h(1/2+ir)dr.$$

Thus combining our results we see that for $h(s)$ satisfying (i), . . ., (v) the following version of the Selberg trace formula holds

$$\sum_{s_\mu>1/2} h(s_\mu) = (\sigma(D^{\pi/2})/4\pi)\cdot\dim(V)\int_{-\infty}^{+\infty} r\cdot\tanh\pi r h(1/2+ir)dr$$
$$+\sum_{\gamma\in H}\sum_{j=1}^{\infty} b(\gamma)\frac{\operatorname{Tr}(\chi(\gamma^j))\cdot\log N(\gamma)}{N(\gamma)^{j/2}-N(\gamma)^{-j/2}}g(j\cdot\log N(\gamma))$$
$$+(4\pi)^{-1}\int_L(\Phi'(s)/\Phi(s)-\psi_1(s))h(s)ds. \tag{29}$$

It is not difficult to see that condition (v) can in general be removed. This follows by a fairly simple and standard approximation argument but we shall not discuss it here. If $\delta(G)<1/2$ then, as remarked at the end of §5, condition (iv) can be dropped. In this case it also follows that the left-hand side is zero as there are no eigenfunctions (cf. [9]).

As in the case of groups of the first kind, $\sigma(D^\pi/2)$ is a topological invariant. In fact, if G is generated by $A_1, B_1, \ldots, A_g, B_g, E_1, \ldots, E_r, H_1, \ldots, H_p, \pi_1, \ldots, \pi_q$ satisfying

$$[A_1, B_1]\cdots[A_g, B_g]E_1\cdots E_r\cdot H_1\cdots H_p\cdot\pi_1\cdots\pi_q = I,$$

and

$$E_j^{e(j)} = I \quad (1\le j\le r)$$

so that

(a) any elliptic element of G is conjugate to a power of some E_j,
(b) H_j is conjugate to a generator of $G_{\alpha(j)}$, and

(c) any parabolic element of G is conjugate to some power of some π_j.

Then an application of the Gauss–Bonnet theorem shows that

$$\sigma(D^{\pi/2})/(2\pi) = 2g - 2 + p + q + \sum_{j=1}^{r} (1 - e(j)^{-1}).$$

Let us now consider the case $\delta(G) < 1/2$. We define the Selberg zeta-function in the region $\operatorname{Re}(s) > \delta(G)$,

$$Z(s) = \prod_{\gamma \in H} \prod_{k=0}^{\infty} \det (I - \chi(\gamma) N(\gamma)^{-s-k})^{2b(\gamma)}.$$

Then, using a well-known identity,

$$-Z'(s)/Z(s) = \sum_{\gamma \in H} \sum_{k=0}^{\infty} 2b(\gamma) \sum_{j=1}^{\infty} \operatorname{Tr}(\chi(\gamma^j)) \log N(\gamma) \cdot N(\gamma)^{-s-k}.$$

Summing over k gives

$$-Z'(s)/Z(s) = \sum_{\gamma \in H} \sum_{j=1}^{\infty} 2b(\gamma) \frac{\operatorname{Tr}(\chi(\gamma^j)) \cdot \log N(\gamma) \cdot N(\gamma)^{-j(s-1/2)}}{N(\gamma)^{j/2} - N(\gamma)^{-j/2}}.$$

From this it follows that

$$-(8\pi)^{-1} \int_{-\infty}^{\infty} ((Z'/Z)(1/2 + ir) + (Z'/Z)(1/2 - ir)) h(1/2 + ir) dr$$
$$= \sum_{\gamma \in H} \sum_{j=1}^{\infty} b(\gamma) \frac{\operatorname{Tr}(\chi(\gamma^j)) \log N(\gamma)}{N(\gamma)^{j/2} - N(\gamma)^{-j/2}} g(j \log N(\gamma)).$$

Putting this into the Selberg trace formula gives

$$0 = \int_{-\infty}^{+\infty} (\sigma(D^{\pi/2})(4\pi)^{-1} \dim (V) r \tanh \pi r - (8\pi)^{-1}((Z'/Z)(1/2 + ir)$$
$$+ (Z'/Z)(1/2 - ir)) + (4\pi)^{-1}((\Phi'/\Phi)(1/2 + ir)$$
$$- \psi_1(1/2 + ir)) h(1/2 + ir) dr.$$

Since we can find a sequence of admissible h approximating $\delta_{r_0} + \delta_{-r_0}$ (δ_x is the Dirac δ-'function' at x) it follows that the expression in brackets, which is even, is zero. That is

$$-2\sigma(D^{\pi/2}) \dim (V)(s - 1/2) \cot \pi s + (Z'/Z)(s)$$
$$+ (Z'/Z)(1 - s) + 2(\Phi'/\Phi)(s) - 2\psi_1(s) = 0. \tag{30}$$

If $k = 0$ it follows that

$$q_0^+(s, a) = q_0^+(s, -a).$$

This, with [10; Eq. (37)] gives

$$I_0(s)J_0(1-s)+I_0(1-s)J_0(s)=0.$$

Then a simple calculation shows that

$$\psi_1(s)+\psi_1(1-s)=0.$$

As

$$\psi_1(s)-\psi_1(1-s)=\psi'(s)/\psi(s)$$

we have

$$2\psi_1(s)=\psi'(s)/\psi(s). \tag{31}$$

It is worth noting that in this case

$$\begin{aligned}\psi(s)&=\prod_{\alpha\in Z^*}\prod_{j=1}^{n}(D_0(s)D_0(1-s)/(q_0^+(s,a_j)q_0^+(1-s,a_j))\\ &=\prod_{\alpha\in Z^*}\prod_{j=1}^{n}(\cosh 2\pi^2 a_j-\cos 2\pi s)/(2\cosh^2\pi^2 a_j),\end{aligned} \tag{32}$$

where $\alpha=(a_1,\ldots,a_n)$. This formula shows quite clearly the analytic structure of $\psi(s)$.

Now note that the residues at the poles of

$$2\pi(s-1/2)\cot\pi s$$

are integral, and so we define

$$\omega(s)=\exp\left(2\pi\int_{1/2}^{s}(t-1/2)\cot\pi t dt\right)$$

exists as a meromorphic function, with a pole of order $2m+1$ at $s=-m$, $m\in\boldsymbol{Z}$, $m>0$.

(30) is valid at first only for $\mathrm{Re}(s)=1/2$, but as the left-hand side is analytic and defined in $\delta(G)<\mathrm{Re}(s)<1-\delta(G)$ it follows that it is also valid there. Integrating, and using (31) we obtain an equation which can be written in the two forms

$$\Phi(s)^2=\psi(s)Z(1-s)Z(s)^{-1}\omega(s)^{2M} \tag{33}$$

$$Z(s)\Phi(s)=Z(1-s)\Phi(1-s)\omega(s)^{2M} \tag{34}$$

valid for all s. The second gives the analytic continuation and the analytic properties of $Z(s)$. The first can be regarded as an explicit formula for $\Phi(s)$, useful in $\delta(G) < \mathrm{Re}(s) < 1-\delta(G)$.

$$M = \dim(V) \cdot \sigma(D^{\pi/2})/2\pi.$$

7. A special case

The case when $\delta(G) = 1/2$ exhibits some special features, especially in respect to the behaviour at 1/2. The object of this section is to examine this case. So, unless otherwise stated, $\delta(G) = 1/2$ but k, χ are allowed to be arbitrary. Suppose firstly that $k - 1/2 \notin \boldsymbol{Z}$.

The starting point is again [10; Th. 4]. Fix φ and let

$$\tau_{\alpha\beta}(s) = \sigma_{\alpha\beta}(s) + \kappa(\alpha)e(\boldsymbol{k})\delta_{\alpha\beta}\,\mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha)$$

and

$$I(s, \alpha) = \kappa(\alpha)(\mathfrak{R}_k^{++}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha)\mathfrak{R}_k^{--}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha) - \mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha)\mathfrak{R}_k^{-+}(\varphi_{i(\alpha)}s, \bar{s}, \alpha)).$$

With these notations [10; Eq. (51)] gives

$$(s(1-s) - \bar{s}(1-\bar{s}))\int_{D^\varphi} E^!_\alpha(z, s)^* E^!_\alpha(z, s)\,d\sigma(z) = \sum_\beta \kappa(\beta)^{-1}(\tau_{\beta\alpha}(s)^*\mathfrak{R}_k^{--}(\varphi_{i(\beta)}, s, \bar{s}, \beta)\tau_{\beta\alpha}(s)) + I(\alpha, s)\mathfrak{R}_k^{--}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha)^{-1}.$$

Now, as remarked previously, in $\mathrm{Re}(s) > 1/2$

$$(s(1-s) - \bar{s}(1-\bar{s}))^{-1}\mathfrak{R}_k^{--}(\varphi_{i(\beta)}, s, \bar{s}, \beta)$$

has negative entries. Clearly

$$\int_{D^\varphi} \mathrm{Tr}(E^!_\alpha(z, s)^* E^!_\alpha(z, s))\,d\sigma(z) > 0.$$

$I(s, \alpha)$ was mentioned in [10; §6], where it was noted that

$$I(s, \alpha) = |\partial\mathfrak{Q}_k^+(\theta, \alpha, s)/\partial\theta\,\mathfrak{Q}_k^-(\theta, \alpha, s) - \partial\mathfrak{Q}_k^-(\theta, \alpha, s)/\partial\theta \cdot \mathfrak{Q}_k^+(\theta, \alpha, s)|^2$$

and so, as $\mathfrak{Q}_k^+(\theta, \alpha, s)$ and $\mathfrak{Q}_k^-(\theta, \alpha, s)$ are linearly independent at

$s = 1/2$, $I(s, \alpha)$ is positive definite. Thus

$$|\mathrm{Tr}\,(\tau_{\beta\alpha}(s)^* \mathfrak{R}_k^{--}(\varphi_{i(\beta)}, s, \bar{s}, \beta)\tau_{\beta\alpha}(s))| \leq |\mathrm{Tr}\,(I(s, \alpha)\mathfrak{R}_k^{--}(\varphi_{i(\alpha)}, s, \bar{s}, \alpha)^{-1})|.$$

Using (6) it follows that

$$|\tau_{\beta\alpha}(s)| \leq c(\alpha, \beta)|\mathrm{Im}\,(s)|^{-1}.$$

The technique of the lemma of §6 shows now that

$$(s - 1/2)\tau_{\alpha\beta}(s),$$

with fixed α, β, is bounded in a neighbourhood of $1/2$ in $\mathrm{Re}\,(s) > 1/2$. The method of [10; §§7, 8] shows that the same is true of $(s - 1/2)E_\zeta(z, s)$.

Again by the method of [10; §7] there is a sequence $s_m \to 1/2$ along which $(s_m - 1/2)E_\zeta(z, s_m)$ converges uniformly on compact subsets (of z and ζ). Suppose that

$$(s_m - 1/2)\sigma_{\alpha\beta}(s_m) \to t_{\alpha\beta}.$$

Let s'_m be another such sequence with corresponding $t'_{\alpha\beta}$.

Taking the limit in [10; Eq. (51)] along (s_m) shows that, for a certain function $f_\beta(z)$,

$$(s(1-s) - 1/4)\int_{D^\varphi} f_\beta(z)E^!_\alpha(z, s)d\sigma(z) = e(k)t^*_{\alpha\beta}\mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, s, 1/2, \alpha)$$
$$+\sum_\gamma \kappa(\gamma)^{-1}t^*_{\gamma\beta}\mathfrak{R}_k^{--}(\varphi_{i(\gamma)}, s, 1/2, \gamma)\sigma_{\gamma\alpha}(s).$$

Taking now the limit along (s'_m)

$$0 = e(k)t^*_{\alpha\beta}\mathfrak{R}_k^{+-}(\varphi_{i(\alpha)}, 1/2, 1/2, \alpha) + \sum_\gamma \kappa(\gamma)^{-1}t^*_{\gamma\beta}\mathfrak{M}_k(\varphi_{i(\gamma)}, 1/2, \gamma)t'_{\gamma\alpha}.$$

Interchanging the roles of (s_m) and (s'_m) and using [10; Th. 6] shows that

$$t_{\alpha\beta} = t'_{\alpha\beta}.$$

Hence $E_\zeta(z, s)(s - 1/2)$ has a continuous extension to $\mathrm{Re}\,(s) = 1/2$. Hence so does $\mathfrak{Q}_k^-(\theta, \alpha, s)\sigma_{\alpha\beta}(s)(s - 1/2)$, for fixed θ, uniformly in α, β. Consequently, by [10; Eqs. (62), (64)], and the fact that $J_k(1/2)^{-1} = 0$, $\Phi(s) = \Phi_k(s, \chi)$ has also a continuous extension to $\mathrm{Re}\,(s) = 1/2$ (cf.

[5]). However

$$\Phi(s)\Phi(1-s)=\psi(s)$$

shows that $\Phi(1/2)\neq 0$. Thus $\Phi(s)\neq 0$ in a neighbourhood of 1/2 in $\mathrm{Re}(s)\geq 1/2$. Hence $\Phi(s)$ is bounded (by the functional equation) in a neighbourhood of 1/2. By Riemann's isolated singularities theorem $\Phi(s)$ can be continued to a regular function at 1/2. By [10; Th. 8] $E_\zeta(z,s)$ has no singularities at any points in $\mathrm{Re}(s)\geq 1/2$ except perhaps at $s=1/2$. As $\Phi(s)\neq 0$ in a neighbourhood of 1/2 it follows from the functional equation that

$$\Sigma_0(s)J_k(s)^{-1}$$

has a continuous extension to $\mathrm{Re}(s)=1/2$ in $\mathrm{Re}(s)\leq 1/2$. Thus $E_\zeta(z,s)(s-1/2)$ has a continuous extension to 1/2. Again, by Riemann's theorem on removable singularities it follows that $E_\zeta(z,s)(s-1/2)$ can be continued to a regular function at $s=1/2$.

Now suppose that $k-1/2\in \boldsymbol{Z}$. The argument above fails at the point where we showed that $(s-1/2)E_\zeta(z,s)$ has a unique limit at 1/2, since $R_k^{+-}(\varphi,1/2,1/2,\alpha)=0$. However, since

$$\mathfrak{M}_k(\varphi,1/2,\alpha)=\mathfrak{q}_k^-(1/2,\alpha)\mathfrak{q}_k^-(1/2,\alpha)^*$$

(from (6)) the same argument shows that $t_{\alpha\beta}=0$. Thus $(s-1/2)E_\zeta(z,s)$ has a unique limit 0 at $s=1/2$. The rest of the argument goes through as before, remembering that now $J_k(s)$ is regular and non-zero at $s=1/2$. (There is one exception to this last statement, for if $k-1/2\in\boldsymbol{Z}$ then $q_k^+(s,0)=0$ but this can easily be dealt with.) Thus

Theorem 4: *Suppose that $\delta(G)=1/2$. Then $E_\zeta(z,s)(s-1/2)$ and $\Phi_k(s,\chi)$ are regular in a neighbourhood of $s=1/2$, and hence meromorphic in the s-plane.*

Now let

$$f(z,w;s)=\sum_{g\in G}\sigma(z,gw)^{-s}.$$

This has abscissa of convergence $\delta(G)$ and essentially the same behaviour as $s\to\delta(G)$ as $E_\zeta(z,s)$, $\chi=Id$ ([11]). By the corollary to theorem 2 and theorem 4 we see that if $\delta(G)\geq 1/2$ then $E_\zeta(z,s)$ is meromorphic in a neighbourhood of $\delta(G)$, whereas if $\delta(G)<1/2$ this was proved in [10; §4]. By Landau's theorem, since the $E_\zeta(z,s)$ are

Dirichlet series with positive terms when $\chi = Id$, that $E_\zeta(z, s)$ has a singularity at $s = \delta(G)$. From the results of [11] and theorem 4 this pole is first-order if $\delta(G) \geq 1/2$. Consequently

THEOREM 5: *Suppose* $k = 0$, $\chi = Id$. *Then* $E_\zeta(z, s)$ *has a pole at* $s = \delta(G)$, *of the first order if* $\delta(G) \geq 1/2$. *Furthermore* $f(z, w; s)(s - \delta(G))$ *is bounded below by a positive quantity as* $s \to \delta(G)$.

This shows that not only is $\delta(G)$ the abscissa of convergence of $f(z, w; s)$ but also $f(z, w; s)$ diverges at $s = \delta(G)$. Using the spectral decomposition of $f(z, w; s)$ it is not possible to obtain much finer information, but we shall not deal with this topic here.

8. Concluding remarks

The results of this paper are far from complete. The most obvious outstanding problem arising from it is to extend theorem 3 to arbitrary k. This seems to be technical in so far that good lower bounds for $Q_k^-(\theta, X, s)$ are needed.

A much more fundamental problem, however is that of deciding whether or not the set $\{s_\mu\}$ appearing in theorem 2 can really be infinite. This seems to require more than the very general methods which are used here.

It is worth remarking that the results here have a great deal in common with scattering theory, and the theory of the Schrödinger equation (cf. [16]). The techniques used there to produce generalised eigenfunctions do not seem to apply in our case, although very similiar methods can be used in the theory of groups of the first kind with parabolic elements ([4], [6], [8], [15]). Could such a method be found, it would, presumably, shed quite a lot of light on the analytic nature of $\Phi(s)$, which one would like to exhibit as some sort of Fredholm determinant of a resolvent kernel.

The case of groups of the first kind can be regarded as a limiting case of the theory described in this series. This appears formally on letting the $G_{\alpha(j)}$ 'shrink' to parabolic. This is rather suggestive and leads one to hope that if the theory here could be extended to more general Lie groups than $SL(2, \boldsymbol{R})$, that one could infer something about the spectral resolution for these groups. However, even the simplest case, that of Kleinian groups in $SL(2, \boldsymbol{C})$ (also of rank 1) offers formidable, but very interesting difficulties. In this case the spectral resolution is known (Selberg). It is also worth noting that the space W, which classifies the continuous spectrum, is the space of L^2 functions from the boundary of the Riemann surface $G\backslash H$, with values in V. This phenomenon appears also in Euclidean spaces and it would be interesting to know just how general it is.

Notes added in proof

1. Since this paper was written I have received a preprint from Prof. John D. Fay, "Fourier Coefficients of the Resolvent for a Fuchsian Group", where the same problems are treated in a rather different way. In particular, some of the open questions mentioned above are no longer open.

2. In [9] there are some misprints and unclarities. In §3 the argument is rather sketchy; this is more fully dealt with in a forthcoming paper, "Spectral Theory and Fuchsian Groups", to appear in Math. Proc. Camb. Phil. Soc. Also, in §3, conditions (i), (ii), (iii) should always be replaced by (i′), (ii′), (iii′). The most serious misprints are listed below. I am indebted to J. Elstrodt for a set of very detailed comments on [9] from which these are taken.

p. 89, line 9, for: "the kernel $q(\ldots$"	*read*: "the point-pair invariant kernel $q(\ldots$".
p. 90, line 19, for: "$+\frac{1}{4\pi}\cdots$"	*read*: "$-\frac{1}{4\pi}\cdots$"
p. 91, line −4, for: $\Gamma(2k-m), \Gamma(2k-2m-1)$"	*read*: "$\Gamma(2\|k\|-m), \Gamma(2\|k\|-2m-1)$"
p. 92, line 2, for: "a Fuchsian..."	*read*: "a finitely-generated Fuchsian..."
p. 93, line 10, for: "$(0 \leq m \leq \|k\|)$"	*read*: "$(0 \leq m < \|k\|)$"
p. 100, line −5, for: "a Fuchsian..."	*read*: "a finitely-generated Fuchsian..."
p. 104, line 4, for: "a diagonal..."	*read*: "an invertible diagonal..."

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(Oblatum 28–VII–1975)

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