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SIMPLE BIRATIONAL EXTENSIONS OF TWO DIMENSIONAL AFFINE RATIONAL DOMAINS

Peter Russell

Let k be a field. For any ring R let $R^{(n)}$ denote the polynomial ring in n variables over R. In this paper we investigate affine k-domains A with the property that $A[a/b] \simeq k^{(2)}$ for some $a, b \in A$. By our main result (see 1.3), if A is a unique factorization domain (UFD), then $A \simeq k^{(2)}$ under various fairly mild additional assumptions. A corollary (see 1.4) is the following little piece of information on the "cancellation problem" for $k^{(2)}$ (see [3]): Let k be perfect, A a k-algebra and t transcendental over A. Assume that $A[t] \simeq k^{(3)}$. If a variable in $k^{(3)}$ is linear as a polynomial in t, then $A \simeq k^{(2)}$.

This work was inspired by [9], where the following is shown: Let k be of characteristic 0, $A \simeq k^{(2)}$, $a, b \in A$ with $b \neq 0$ and $H = bw - a \in A[w] \simeq k^{(3)}$ such that $A[w]/HA[w] = A[a/b]$ is isomorphic to $k^{(2)}$. Then there exist $F, G \in A[w]$ such that $k[F, G, H] = A[w]$. We extend this result to fields of arbitrary characteristic (see 2.3).

The proof of 1.3 runs like this: An irreducible factor x of b in $k^{(2)}$ contracts to a maximal ideal in A and, since $k^{(2)}$ is generated by one element over A, defines a line in $k^{(2)}$ (i.e. $k^{(2)}/xk^{(2)} \simeq k^{(1)}$). The crucial problem lies in showing that there exists $y \in k^{(2)}$ such that $k[x, y] = k^{(2)}$. If char $k = 0$, this is assured by [1]. Under suitable restrictions on A (not involving char k), however, we can also reach this conclusion exploiting further the fact that x contracts birationally to a maximal ideal in A. One then shows $A = k[x, by]$ without much difficulty.

I would like to express my thanks here to W. Heinzer. Numerous conversations with him were instrumental in getting this research off the ground.

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We begin with an elementary result.

1.1. PROPOSITION: *Let A be a UFD and a, a', b, $b' \in A$ with $GCD(a, b) = 1 = GCD(a', b')$. Suppose $A[a/b] = B = A[a'/b']$. Then*

(i) *b and b' have the same irreducible factors,*

(ii) $a' = a_0 b' + ca$

$a = a'_0 b + c'a'$

with a_0, a'_0, c, $c' \in A$ and $cc' \equiv 1 \bmod GCD(b, b')$,

(iii) *if $b' = qb$ with $q \in A$, then a and a' are units* mod *q and q is a unit in B.*

PROOF: We have

$$\frac{a'}{b'} = a_0 + a_1 \frac{a}{b} + \cdots + a_n \frac{a^n}{b^n} \tag{1}$$

with $a_i \in A$. Since $GCD(a', b') = 1$, $b' | b^n$. Similarly, $b | b'^m$ for some m. This proves (i). From (1) we obtain

$$a' = a_0 b' + ca \tag{2}$$

where

$$c = a_1 \frac{b'}{b} + a_2 a \frac{b'}{b^2} + \cdots + a_n a^{n-1} \frac{b'}{b^n}. \tag{3}$$

Since $ca \in A$ and $GCD(a, b) = 1$, $c \in A$. Similarly,

$$a = a'_0 b + c'a' \tag{2'}$$

with a'_0, $c' \in A$. Now $(1 - cc')a' = a_0 b' + ca'_0 b$ and hence $GCD(b, b') | 1 - cc'$. This proves (ii).

Suppose $b' = qb$ with $q \in A$. By (ii), c is a unit mod q, and we obtain from (3)

$$c = a_1 q + ad$$

with $d \in A$. Hence a is a unit mod q, and so a' is a unit mod q by (2). Write $\alpha a' = 1 + \beta q$ with α, $\beta \in A$. Then $b\alpha(a'/b') = 1/q + \beta$ and hence $1/q \in B$. This proves (iii).

1.2. COROLLARY: *Let p be an irreducible factor of $GCD(b, b')$ and suppose the p-orders of b and b' are different. Then a and a' are units* mod *p in A_h, where h is the product of the prime factors of $GCD(b, b')$ different from p.*

PROOF: Replace A by A_h and apply (ii).

1.3. THEOREM: *Let A be a UFD finitely generated over k and a, $b \in A$ with $GCD(a, b) = 1$. Let $B = A[a/b]$ and suppose $B \simeq k^{(2)}$. Assume also that one of the following conditions holds:*

(i) *A contains a field generator, i.e. there exists $f \in A$ such that $qtA = k(f, q)$ for some $q \in qtA$ (qtA = field of quotients of A),*

(ii) char $k = 0$,

(iii) *k is perfect and A is regular.*

Then $A \simeq k^{(2)}$. More precisely, there exist x, $y \in B$ such that $B = k[x, y]$, $b \in k[x]$ and $A = k[x, by]$.

PROOF: We assume $b \notin k$ and claim (see [9], proof of Lemma 3): (*) Let $b_1, \ldots, b_r$ be the irreducible factors of b in B. Then $(b_i, b_j)B = B$ for $i \neq j$, $b_iB \cap A = M_i$ is a maximal ideal, $M_i \neq M_j$ for $i \neq j$ and $B/b_iB \simeq (A/M_i)^{(1)}$. If $c \in B$ is irreducible and $cB \cap A$ a maximal ideal, then $cB = b_iB$ for some i.

In fact, if b_i is an irreducible factor of b, then $M_i = b_iB \cap A \supset (a, b)A$, and since $GCD(a, b) = 1$, M_i is maximal. Hence $M_iB \cap A = M_i$ and $B/M_iB = A/M_i[z]$, where z is the image of a/b mod M_iB. Now $M_iB \subset b_iB$, so z is transcendental over A/M_i and M_iB is prime. Hence $M_iB = b_iB$, $B/b_iB \simeq (A/M_i)^{(1)}$ and $M_i \neq M_j$, and so $(b_i, b_j)B = B$, for $i \neq j$. The last assertion follows from $A_b = B_b$.

Under each of the conditions (i), (ii), (iii) we will show by different methods:

(**) There exist x, $y \in B$ such that $B = k[x, y]$ and $b \in k[x]$.

Suppose this has been done. Then x is integral over A and hence $x \in A$. Since $A_b = B_b$, $b^m y \in A$ for some m and there exists $b' \in k[x]$ of smallest degree such that $v = b'y \in A$. Then b and b' have the same irreducible factors (note that these are the same whether taken in $k[x]$, A or B) and $k[x, v]_{b'} = A_{b'} = k[x, y]_{b'}$. Suppose there is an irreducible $c \in A$ such that $cA \cap k[x, v]$ is maximal. Then c is an irreducible factor of b and $v = b'y \in cA$. Hence $b''y \in A$, where $b'' = b/c \in k[x]$ is of smaller degree than b', and this is impossible. So no height one prime in A contracts to a maximal ideal in $k[x, v]$ and the birational morphism Spec $A \to$ Spec $k[x, v]$ has finite fibres. By Zariski's Main Theorem (see [6, Cor. 2, p. 42]), it is an open immersion. Since $k[x, v]$ is a UFD, A is a localization of $k[x, v]$, and since the units of A are constant, $k[x, v] = A$. Now $A[v/b'] = k[x, y] = A[a/b]$, and $b = b'$ follows from 1.2.

It remains to establish (**).

Case 1: $f \in A$ is a field generator. We keep the notation of (*). There exist monic polynomials P_i with coefficients in k such that $b_i | P_i(f)$ in B (the minimal polynomials of $f \bmod M_i$, for instance). Now f is a field generator in $B \simeq k^{(2)}$ as well as in A, and by [8, 3.7 and 4.5] we can find $x, y \in B$ such that $B = k[x, y]$ and (α) the degree form of f is a monomial in x and y, (β) f is not tangent to the line at infinity of $k[x, y]$. (Equivalently, $f = x^m y^n + g$ where $\deg g < m + n$, $\deg_x g \leq m$, $\deg_y g \leq n$.) The operations of forming a polynomial (with coefficients in k) and of taking a factor preserve these properties and hence each b_i satisfies (α) and (β). On the other hand, since B/b_iB is a polynomial ring over a field, the degree form of b_i is a monomial in x alone or y alone and hence b_i is a polynomial in either x or y. (This argument slightly generalizes [8, 4.8].) Since $(b_i, b_j)B = B$ for $i \neq j$, x and y cannot appear both, and we may assume that each b_i, and therefore b, is a polynomial in x.

By Lemma 1.6 below, we can assume that k is algebraically closed in verifying (**) under conditions (ii) and (iii). (Unique factorization will not be used again, and A remains regular over an algebraic closure of k if we assume (iii).)

Case 2: k algebraically closed, $\operatorname{char} k = 0$. Let $x = b_1$ be an irreducible factor of b. Then $B/xB \simeq k^{(1)}$, and by the main result of [1], there exists $y \in B$ such that $B = k[x, y]$. If b_i is any other irreducible factor of b, then $b_i = \gamma_i x + \delta_i$ with $\gamma_i, \delta_i \in k$ since $(b_i, b_1)B = B$ (see [9, Lemma 1]). Hence $b \in k[x]$.

Case 3: k algebraically closed, A regular. Let $x = b_1$. As in case 2, $B/xB \simeq k^{(1)}$, $b_i = \gamma_i x + \delta_i$ with $\gamma_i, \delta_i \in k$, and $b \in k[x]$. Hence $x \in A$. Let X and Y be complete non-singular surfaces containing respectively $\operatorname{Spec} B$ and $\operatorname{Spec} A$ as dense open subsets, with $X = \mathbb{P}^2_k$. The birational morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ induces a birational map $\varphi : X \to Y$ and (see [12, part II] or [10, Ch. IV, §3] for basic facts from the theory of birational correspondences of surfaces used below) there exists a nonsingular surface Z and birational morphisms $\varphi_1 : Z \to X$, $\varphi_2 : Z \to Y$ such that $\varphi \circ \varphi_1 = \varphi_2$ and φ_1, φ_2 are composites of locally quadratic transformations. (The centres of these we call the fundamental points of φ_1 and φ_2 respectively.) Replacing Z, if necessary, by a surface Z^* dominated by Z we may assume that

(a_1) no irreducible exceptional curve E of the first kind on Z (this means $E \simeq \mathbb{P}^1_k$ and $(E, E) = -1$, where $(-, -)$ denotes the intersection pairing) shrinks to a point on both X and Y.

For any curve C on X or Y let C' denote its proper transform on

Z. For $\lambda \in k$, let C_λ be the curve on X whose ideal in B is $(x-\lambda)B$. Put $d = \deg C_\lambda$ and $L = X - \operatorname{Spec} B$. The curves C_λ together with dL form a linear pencil $\Lambda = \Lambda(x)$ (see [8, 1.2]). Let $p \in \operatorname{Spec} A \subset Y$ be the closed point with ideal $M_1 = xB \cap A$ in A. By (*), $C'_0 \subset \varphi_2^{-1}(p)$. Let E be an irreducible component of $\varphi_2^{-1}(p)$ such that $(E, E) = -1$. Then $E \neq L'$. In fact,

(a$_2$) $\varphi_2(L') \subset Y - \operatorname{Spec} A$.

Otherwise $\varphi_2(D') \subset \operatorname{Spec} A$ for almost all lines $D \subset X = \mathbb{P}^2$ and $\operatorname{Spec} A$ carries a complete curve, which is impossible. Also, E does not contract to a point on X by (a_1) and hence $E = C'$, where $C \subset X$ is an irreducible curve such that $C \cap \operatorname{Spec} B \neq \emptyset$. By (*), $C = C_0$ and hence

(a$_3$) $(C'_0, C'_0) = -1$ and C'_0 is the only irreducible component of $\varphi_2^{-1}(p)$ with this property.

For general $\lambda \in k$, $\varphi_2(C'_\lambda)$ is a curve on Y whose ideal in A is $I_\lambda = (x-\lambda)B \cap A$. Since $x \in M_1 \subset A$, we have $x - \lambda \in I_\lambda$ and $I_\lambda \not\subset M_1$. Hence $p \notin \varphi_2(C'_\lambda)$ and

(a$_4$) no base point of Λ (see [8, 2.5]) is on C'_0.

Let $q_1, \ldots, q_s$ be the base points of Λ, let $\nu_i = \mu(\Lambda, q_i)$ (see [8, 2.5], this is the multiplicity at q_i of the proper transform of a general member of Λ) and $\mu_i = \mu(C_0, q_i)$ (see [8, 2.3], this is the multiplicity at q_i of the proper transform of C_0). Note $\nu_i \geq 1$ and $\mu_i \geq 0$. By (a$_4$), all q_i with $\mu_i > 0$ are fundamental points of φ_1. By (a$_3$), and since $(C_0, C_0) = d^2$, $\Sigma\, \mu_i^2 \leq d^2 + 1$. Intersecting C_0 with a general member C_λ of Λ we find $\Sigma\, \mu_i \nu_i = d^2$. Also, since Λ is a pencil, $\Sigma\, \nu_i^2 = d^2$ (see [8, 2.10]). By Schwarz's inequality, $\Sigma\, \mu_i^2 \geq d^2$. If $\Sigma\, \mu_i^2 = d^2 + 1$, then $\Sigma\, \nu_i(\mu_i - \nu_i) = 0$ and $\Sigma\, (\mu_i - \nu_i)^2 = 1$, which cannot be satisfied in integers μ_i, ν_i with $\nu_i \geq 1$. Hence $\Sigma\, \mu_i^2 = d^2$ and $\mu_i = \nu_i$ for all i. Since $C'_0 \simeq \mathbb{P}^1_k$, any multiple point of C_0 is a fundamental point of φ_1 and hence one of the q_i. For if not, $(C'_0, C'_0) \leq d^2 - \Sigma\, \mu_i^2 - 4 \leq -4$ in contradiction to (a$_3$). Hence $\Sigma\, \mu_i(\mu_i - 1) = (d-1)(d-2) = \Sigma\, \nu_i(\nu_i - 1)$ and the generic member Λ_η of Λ is a curve of genus zero over $k(x)$ (see [8, 2.8 and 2.11]). Since k is algebraically closed, Λ_η is a rational curve, i.e. qtB, the function field of Λ_η, is purely transcendental over $k(x)$. (There is a conic in $\mathbb{P}^2_{k(x)}$ birationally equivalent to Λ_η (see [2, Ch. II, §6]). By Tsen's Theorem, Λ_η has a place rational over $k(x)$ (see [11, Ch. II, 3.2 and 3.3]) and hence is rational (see [2, Ch. II, §3]).) Equivalently, x is a field generator in B, and by [8, 4.8] there exists $y \in B$ such that $B = k[x, y]$.

1.4. Corollary: *Let k be perfect, A a k-algebra, t transcendental over A and assume $A[t] = k[x, y, z] \simeq k^{(3)}$. Assume also that $z = bt - a$ with $a, b \in A$. Then $A \simeq k^{(2)}$.*

Proof: If $b = 0$, the result is proven in [3, Theorem 4.1]. If $b \neq 0$, the homomorphism $\sigma : A \to A[t] \to A[t]/zA[t] = B$ is injective. Clearly A is a regular UFD finitely generated over k and $GCD(a, b) = 1$. Identifying A with $\sigma(A)$ we have $B \simeq A[a/b]$. By 1.3 (iii), $A \simeq k^{(2)}$.

1.5. Lemma: *Let k'/k be a separable algebraic extension. Let $x \in B \simeq k^{(2)}$ such that $B \otimes_k k' = k'[x, y']$ for some $y' \in B \otimes_k k'$. Then there exists $y \in B$ such that $B = k[x, y]$.*

Proof: Let t be transcendental over k and put $R = B \otimes_k k(t)/(x - t)B \otimes_k k(t)$. Then $R \otimes_{k(t)} k'(t) \simeq k'(t)^{(1)}$, and since $k'(t)/k(t)$ is separable, $R \simeq k(t)^{(1)}$, as is well known (see [7, 1.1]). Hence x is a field generator (see [8, 1.3]) and we can apply [8, 4.8].

1.6. Lemma: *Let k'/k be a separable algebraic extension. Let $b \in B \simeq k^{(2)}$ such that there exist $x', y' \in B \otimes_k k'$ with $B \otimes_k k' = k'[x', y']$ and $b \in k'[x']$. Then there exist $x, y \in B$ such that $B = k[x, y]$ and $b \in k[x]$.*

Proof: We can find $\alpha, \beta \in k'$, $\alpha \neq 0$, such that $\alpha x' + \beta \in B$. The proof is the same as the proof of Corollary 1 of [9]. Put $x = \alpha x' + \beta$. Then $k'[x, y'] = k'[x', y']$ and $b \in k'[x] \cap B = k[x]$. The existence of y such that $B = k[x, y]$ follows from 1.5.

2

In this section we extend the result of [9] on "linear planes" to fields of arbitrary characteristic. It is possible to do this, once 1.3 is established, by referring to details in the proof of [9]. It may be worthwhile, nevertheless, to write down an argument more directly adapted to our line of reasoning. Also, 2.2 below (which, more or less, can be found hidden in [9]), giving a construction for somewhat unfamiliar (since in general not "tame") automorphisms of the polynomial ring $R^{(2)}$, where R is any commutative ring, deserves to be mentioned explicitly.

We record the following well known facts (see also the remark after Lemma 5 in [9]):

2.1. Let S be a commutative ring and $\alpha \in S[T] \simeq S^{(1)}$, $\alpha = \Sigma\, \alpha_i T^i$ with $\alpha_i \in S$. Then

(i) α is nilpotent $\Leftrightarrow \alpha_i$ is nilpotent for all i,

(ii) α is a unit $\Leftrightarrow \alpha_0$ is a unit and α_i is nilpotent for $i \geq 1$,

(iii) $S[\alpha] = S[T] \Leftrightarrow \alpha_1$ is a unit and α_i is nilpotent for $i \geq 2$.

2.2. PROPOSITION: *Let R be a commutative ring, $b \in R$, $\alpha = \Sigma\, \alpha_i v^i \in R[v] \simeq R^{(1)}$ with $\alpha_i \in R$, and $H = bw + \alpha \in R[v, w] \simeq R^{(2)}$. Assume that α_1 is a unit* mod bR *and that α_i is nilpotent* mod bR *for $i \geq 2$. Then there exists $\varphi(T) \in R[T]$ such that $\varphi(\alpha) \equiv v$* mod $bR[v]$. *For any such φ,*

$$\varphi(H) = v + bG$$

with $G \in R[v, w]$. Moreover,

$$R[G, H] = R[v, w].$$

PROOF: The existence of φ follows from 2.1(iii) applied to $S = R/bR$. Let $\varphi(T) = \Sigma_{i \geq 0}\, \beta_i T^i$ with $\beta_i \in R$. Then $\alpha_1\beta_1 - 1$ and the β_i for $i \geq 2$ are nilpotent mod b. Write

$$\varphi(\alpha) = v + b\beta$$

with $\beta \in R[v]$. We have

$$\varphi(H) = \varphi(\alpha + bw) = \sum_{i \geq 0} \varphi^{(i)}(\alpha) b^i w^i,$$

where $\varphi^{(i)}(T) = \Sigma_{j \geq i} \binom{j}{i} \beta_j T^{j-i}$. Hence $\varphi(H) = v + b\beta + bwP$ with

$$P = \sum_{i \geq 1} \varphi^{(i)}(\alpha) b^{i-1} w^{i-1}.$$

So $\varphi(H)$ is of the desired form with

$$G = \beta + wP.$$

Moreover, $\varphi^{(1)}(\alpha)$ is a unit mod b and $\varphi^{(i)}(\alpha)$ is nilpotent mod b for $i \geq 2$. Hence P is a unit mod b in $R[v, bw]$ (apply 2.1(ii) to $S = R[v]/bR[v]$) and there exist Q, $Q_1 \in R[v, bw]$ such that $PQ = 1 + bQ_1$. Then $w = (G - \beta)Q - wbQ_1$. Clearly $v = \varphi(H) - bG \in R[G, H]$ and hence $bw = H - \alpha \in R[G, H]$. It follows that β, Q, $Q_1 \in R[G, H]$. Hence $w \in R[G, H]$.

2.3. THEOREM: *Let k be a field, a, $b \in A \simeq k^{(2)}$ with $b \neq 0$, and $H = bw - a \in A[w] \simeq k^{(3)}$. Suppose $A[w]/HA[w] \simeq k^{(2)}$. Then there exist u, $v \in A$ such that $A = k[u, v]$, $b \in k[u]$ and $a - v$ is nilpotent* mod bA. *Moreover, there exists* $G \in A[w] = k[u, v, w]$ *such that* $k[u, v, w] = k[u, G, H]$.

PROOF: Let $B = A[a/b]$. Then $B \simeq A[w]/HA[w] \simeq k^{(2)}$. Clearly $GCD(a, b) = 1$. Also, A contains a field generator, and by 1.3(i), we can find $x, y \in B$ such that $B = k[x, y]$, $b \in k[x]$ and $A = k[x, by]$. Put $u = x$ and $v' = by$. Then $B = A[v'/b]$, and by 1.1(ii), $a = a_0 b + cv'$ where a_0, $c \in A$ and c is a unit mod b. By 2.1(ii), $c = \Sigma\, c_i v'^i$ with $c_i \in k[u]$, c_0 a unit mod b and c_i nilpotent mod b for $i \geq 1$. Finally, $c_0 = \Sigma\, \gamma_i u^i$ with $\gamma_i \in k$, $\gamma_0 \neq 0$ and $c_0 - \gamma_0$ nilpotent mod b. Put $v = \gamma_0 v'$. Then $a - v$ is nilpotent mod b. The existence of G follows from 2.2 applied to $R = k[u]$.

3

The conditions under which we proved 1.3 may not be the best possible. Unique factorization for A, however, is essential. For an easy example, consider

(3.1) $$A = k[x, xy, xy^2] \subset B = k[x, y] = A\left[\frac{xy}{x}\right].$$

Also, it does not help to assume that A is regular, as the next example shows (which the author learned from W. Heinzer).

(3.2) $$A = k\left[u, v, \frac{v(v-1)}{u}\right] \subset B = k[x, y] = A\left[\frac{v}{u}\right],$$

where $u = x$ and $v = xy$.
Here Spec $B \to$ Spec A is an open immersion, and this is typical in a way. For we have

3.3. THEOREM: *Let k be perfect, A a finitely generated regular k-domain and a, $b \in A$ with $b \neq 0$. Assume $B = A[a/b] \simeq k^{(2)}$. Then there exist x, $y \in B$ and $b' \in k[x]$ such that $B = k[x, y]$, $A \subset k[x, b'y] = B'$ and* Spec $B' \to$ Spec A *is an open immersion.*

PROOF: Not all irreducible factors of b in B now necessarily contract to maximal ideals in A, but the claim (*) we made in the proof of 1.3 holds for those irreducible factors $b_1, \ldots, b_r$ that do. If there are none, we are done by Zariski's Main Theorem. Otherwise we can, exactly as under condition (iii) of 1.3, find x, $y \in B$ with $B = k[x, y]$ and $b_i \in k[x]$ for all i. We claim $A \subset k[x, b_i y]$, and to finish the proof of the theorem we choose $b' \in k[x]$ of maximal degree such that $A \subset k[x, b'y]$. It will be enough to establish the claim in case k is algebraically closed. For then, if $b_{i1}, \ldots, b_{is}$ are the (distinct) irreducible factors of b_i in $k'[x, y] = B'$, where k' is an algebraic closure of k, we have $b_{i1} = b_i \cdots b_{is}$, each b_{ij} contracts to a maximal ideal in $A' = A \otimes_k k'$ and $A' \subset k'[x, b_{ij}y]$ for all j. Hence $A' \subset k'[x, b_i y]$ and $A \subset k'[x, b_i y] \cap k[x, y] = k[x, b_i y]$.

Let then k be algebraically closed. We may set $x = b_1$. Let

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\varphi_1}\swarrow & & \searrow{\scriptstyle\varphi_2} \\ \operatorname{Spec} B \subset X & \xrightarrow[\varphi]{} & Y \supset \operatorname{Spec} A \end{array}$$

be as in case 3 of the proof of 1.3, with the embedding $\operatorname{Spec} B \subset X = \mathbb{P}^2$ so chosen that C_0, the curve whose ideal in B is xB, has degree $d = 1$. Let $\sigma_1: Z_1 \to X$ be the locally quadratic transformation with centre $p_1 = C_0 \cap L$ ($L = X - \operatorname{Spec} B$), $E_1 = \sigma_1^{-1}(p_1)$ and $\sigma_2: Z_2 \to Z_1$ the locally quadratic transformation with centre $p_2 = C_0^{(1)} \cap E_1$, where $C_0^{(1)}$ is the proper transform of C_0 on Z_1. The proper transform on Z_2 of a curve C on X or Z_1 will be denoted by $C^{(2)}$, and the proper transform on Z of a curve C on X, Z_1 or Z_2 by C'. Put $\sigma = \sigma_1 \circ \sigma_2$. Since $(C'_0, C'_0) = -1$ (see (a_3) in the proof of case 3) and since there are no fundamental points of φ_1 on $\operatorname{Spec} B$, p_1 and p_2 are precisely the fundamental points of φ_1 on C_0. It follows that there exists a morphism $\psi: Z \to Z_2$ such that $\varphi_1 = \sigma \circ \psi$. We have, putting $E_2 = \sigma_2^{-1}(p_2)$,

(b_1) $\sigma^{-1}(C_0 \cup L) = C_0^{(2)} \cup E_2 \cup E_1^{(2)} \cup L^{(2)}$

with $C_0^{(2)}$ and E_2, E_2 and $E_1^{(2)}$, $E_1^{(2)}$ and $L^{(2)}$ meeting normally in one point and all other intersections empty.

There are no fundamental points of ψ on $C_0^{(2)}$ and hence $\rho = \varphi \circ \sigma$ is a morphism in a neighbourhood of $C_0^{(2)}$. In particular, since C_0 contracts to a point in $\operatorname{Spec} A$,

(b_2) $\rho(E_2) \cap \operatorname{Spec} A \neq \emptyset$.

We claim that there are no fundamental points of ψ on $E_2 - E_1^{(2)}$. In fact, $F = Y - \operatorname{Spec} A$ is connected (see [4, Ch. II, 6.2]) and hence

(b_3) $\varphi_2^{-1}(F)$ is connected.

Since φ is a morphism on $\operatorname{Spec} B$, $\varphi_2^{-1}(F) \subset \varphi_1^{-1}(L) = \psi^{-1}(E_2 \cup E_1^{(2)} \cup L^{(2)})$. Now $L' \subset \varphi_2^{-1}(F)$ by (a_2), but $E'_2 \not\subset \varphi_2^{-1}(F)$ by (b_2). It follows from (b_1) and (b_3) that if $q \in E_2 - E_1^{(2)}$ is a fundamental point of ψ, then $\psi^{-1}(q) \cap \varphi^{-1}(F) = \emptyset$. Hence $\varphi_2(\psi^{-1}(q)) \subset \text{Spec } A$ and therefore is a point, and this contradicts (a_1).

By what we have shown, ρ is a morphism on $Z_2 - (E_1^{(2)} \cup L^{(2)})$. Let $\tau: Z_2 \to X_1$ be the contraction of $C_0^{(2)}$. (Note $(C_0^{(2)}, C_0^{(2)}) = -1$ and $C_0^{(2)} \simeq \mathbb{P}_k^1$. X_1 is isomorphic to the ruled surface F_2.) Let $\rho = \rho' \circ \tau$. Then ρ' is a morphism on $U = X_1 - \tau(E_1^{(2)} \cup L^{(2)})$. Also $\tau \circ \sigma^{-1}(\text{Spec } B) \subset U$. It is easily verified that U is affine and that $\Gamma(U) = k[x, xy]$ ($\Gamma(U)$ = ring of functions defined on U). Hence $A \subset k[x, xy]$.

3.4. Remark: The proof given above actually shows:

Let k be perfect, A a finitely generated regular k-domain, $A \subset B \simeq k^{(2)}$ with $qtA = qtB$. Let $x \in B$ such that $xB \cap A$ is maximal and $B/xB \simeq k^{(1)}$. Let X, Y, Z be as above. If the proper transform on Z of the curve on X defined by x is exceptional of the first kind (briefly, "x shrinks first"; one can see that this is independent of the choice of X, Y, and Z as long as Z satisfies (a_1)) then there exists $y \in B$ such that $B = k[x, y]$ and $A \subset k[x, xy]$. The example

$$A = k[xy, xy^2] \subset k[x, y] \subset B$$

shows that it is not enough to assume that $xB \cap A$ is maximal. (Here y has to shrink before x can shrink.)

3.5. Remark: Suppose $A[t] = k[x, y, z]$ as in 1.4. One may ask what information our methods give if, say, $t = bz - a$ with a, $b \in k[x, y]$, $b \neq 0$. Not much, unfortunately. We have $k[x, y, a/b] = A$, but this does not guarantee $A \simeq k^{(2)}$ under the best of conditions for A, as we will see. If f is an irreducible factor of b in A, then $A/fA \simeq k^{(1)}$ (assume k is algebraically closed), so $k[x, y, z]/fk[x, y, z] \simeq k^{(2)}$. If f is linear in x, y or z, we can use 2.3 and refer to [3, 4.1], but there is no reason why it should.

Now put $a = x$ and let $b \in k[x, y] = B$ be irreducible such that $(x, b)B = (x, y)B = M$ and $B/bB \not\simeq k^{(1)}$ (for instance $b = xy^2 + y + x^2$). Let $A = B[a/b]$. It is easily checked that A is regular with constant units. Also, $bA \cap B \supset (x, b)B = M$, so $bA \cap B = M$ and $A/bA \simeq k^{(1)}$. Hence bA is prime. Since $A_b = B_b$ is a UFD, A is a UFD (see [5]). On the other hand, $A \not\simeq k^{(2)}$, for otherwise $B/bB \simeq k^{(1)}$ by 1.3.

3.6. Remark: 1.5 remains true for purely inseparable extensions k'/k if $B/xB \simeq k^{(1)}$ is included in the assumptions. (The proof is more complicated.) Otherwise the conclusion is false in general. In fact, let $\text{char } k = p > 0$ and $x = v^p + u + \alpha u^p \in B = k[u, v]$ with $\alpha \in k - k^p$. Then $B/xB \not\simeq k^{(1)}$ (see [7]) and $k[x, y] \neq B$ for all $y \in B$, but $k'[u, v] =$

$k'[x, v+\beta u]$, where $\beta^p = \alpha$ and $k' = k(\beta)$. This also shows, with $b = x$, that 1.6 may fail if k'/k is not separable. More interesting in the present context is the fact that it does not help to assume $B/bB \simeq K^{(1)}$, where K is a field. For let $b = x^p - \alpha$. If b is a polynomial in some $w \in k[u, v]$, then clearly w is a power of $\gamma x + \delta$ with $\gamma, \delta \in k$, and again $B/wB \neq k^{(1)}$. However, letting $\bar{}$ denote residues mod b, we have $\bar{x} = (\bar{v} + \bar{x}\bar{u})^p + \bar{u}$ and hence $B/bB = k[\bar{x}, \bar{v} + \bar{x}\bar{u}] = k'[\bar{v} + \bar{x}\bar{u}] \simeq k'^{(1)}$.

3.7. Remark: If char $k = p > 0$, there exists $a \in A = k[u, v] \simeq k^{(2)}$ such that $A/aA \simeq k^{(1)}$, but $A \neq k[a, y]$ for all $y \in A$, e.g. $a = u^{p^2} + v + v^{rp}$, with $r > 1$, $GCD(r, p) = 1$ (see the introduction of [1]). One deduces easily that the assumption $b \neq 0$ in 2.3 cannot be dropped.

We conclude by raising two questions suggested by the proof of 1.3.

3.8. Question: If A is a finitely generated k-domain such that $qtA \simeq qtk^{(2)}$, when does A contain a field generator? One should assume that the units of A are constant and may want to impose additional conditions, such as, in some combination,

(i) $A \subset k[x, y] \simeq k^{(2)}$ with $qtA = k(x, y)$,
(i') $A[a/b] = k[x, y]$ for some $a, b \in A$,
(ii) A is regular,
(iii) A is a UFD.

3.9. Question: Let $f \in B \simeq k^{(2)}$ be irreducible. When does there exist a regular k-domain (regular UFD) $A \subset B$ with $qtA = qtB$ such that fB contracts to a maximal ideal in A? (Clearly, if k is algebraically closed, Spec B/fB is a nonsingular rational curve, but what else?) One could require in addition that fB is the only height 1 prime in B contracting to a maximal ideal, or, somewhat weaker, that f has the property of x in 3.4 of "shrinking first".

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