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# TRANSFER MAPS FOR FIBRATIONS AND DUALITY

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## 1. Introduction

In this paper we will describe a transfer construction for (Hurewicz) fibrations which is a generalization of that for fiber bundles studied in [4, 5]. We suppose given a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

where $p: E \to B$ is a fibration having fiber $F$ a finite complex and base $B$ a connected finite dimensional complex. With this data we show that there is an $S$-map, which we call a *transfer* map,

$$\tau(f): B^+ \to E^+$$

having the property that

$$\tilde{H}^*(B^+) \xrightarrow{p^*} \tilde{H}^*(E^+) \xrightarrow{\tau(f)^*} \tilde{H}^*(B^+)$$

is multiplication by the Lefschetz number $\Lambda$ of $f': F \to F$, the restriction of $f$ to the fiber. (Although $f'$ is not unique we allow this abuse of language since $\Lambda$ is independent of the choice of $f'$.)

The existence of $\tau(f)$ severely restricts the projection map of the fibration. For example

$$p_*: \{X; E^+\}_q \otimes Z[\Lambda^{-1}] \to \{X; B^+\}_q \otimes Z[\Lambda^{-1}]$$

is a split epimorphism for any (pointed) finite dimensional complex $X$.

We will show that the boundary map $\omega : \Omega B \to F$ arising from the Puppe sequence of the fibration $p : E \to B$ is also restricted by the transfer. Precisely, we have

(1.1) THEOREM: *Assume that $F$ is connected. Then*

$$\Lambda \omega_* : \{X; \Omega B\}_q \to \{X; F\}_q$$

*is trivial for any finite dimensional complex $X$.*

An independent method of extending the notion of transfer from fiber bundles to fibrations is given in [7]. The method which we describe here is intrinsic and has the advantage that many basic properties of the transfer are easily derived. A. Dold [9] has also independently defined the transfer, placing somewhat different restrictions on the projection $p$ and fiber preserving map $f$.

The outline of the paper is as follows. In section 2 we give a homotopy characterization of the Lefschetz number of a map. Although an elementary fact it is the key point in defining the transfer. In section 3 we deal with some homotopy properties of ex-spaces and in section 4 with the duality theory of ex-spaces. This generalization of Spanier-Whitehead duality is purely formal except for the question of the existence of dual ex-spaces (theorem 4.2). In sections 5 thru 7 we define the transfer and establish its basic properties. In section 8 we prove theorem (1.1) mentioned above and describe some consequences of the theorem. In section 9 we consider smooth fiber bundles and in this case we give a more geometric description of the transfer.

## 2. The Lefschetz number

Suppose that $F$ is a finite complex with base point and $f : F \to F$ is a base point preserving map. By the reduced Lefschetz number of $f$ we mean

$$\tilde{\Lambda}_f = \sum (-1)^i \operatorname{tr} [f_* : \tilde{H}_i(F) \to \tilde{H}_i(F)].$$

Let $\mu : S^s \to F \wedge \hat{F}$ be a duality map in the sense of Spanier [15]. Then $F \wedge \hat{F}$ is $2s$-self dual via the map

$$S^{2s} \xrightarrow{\mu \wedge \mu} (F \wedge \hat{F}) \wedge (F \wedge \hat{F}) \xrightarrow{\alpha} (F \wedge \hat{F}) \wedge (F \wedge \hat{F})$$

where $\alpha(x \wedge y \wedge x' \wedge y') = x' \wedge y \wedge x \wedge y'$. Denote this composite by $\nu$ and let $\hat{\mu}: F \wedge \hat{F} \to S^s$ be dual to $\mu$ relative to $\nu$. The following lemma provides a homotopy description of the reduced Lefschetz number of $f$. It is the analogue for base point preserving maps of the Lefschetz fixed point theorem given by Dold [8, theorem (4.1)].

(2.1) LEMMA: *The composite* $S^s \xrightarrow{\mu} F \wedge \hat{F} \xrightarrow{f \wedge 1} F \wedge \hat{F} \xrightarrow{\hat{\mu}} S^s$ *has degree* $\tilde{\Lambda}_f$.

PROOF: We have the following homotopy commutative diagram

(2.2)
$$\begin{tikzcd} S^{2s} \arrow[r, "\nu"] \arrow[dr, "1 \wedge \mu"'] & (F \wedge \hat{F}) \wedge (F \wedge \hat{F}) \arrow[d, "\hat{\mu} \wedge 1"] \\ & S^s \wedge (F \wedge \hat{F}) \end{tikzcd}$$

Let $Q$ denote the rational numbers and choose a generator $\gamma \in \tilde{H}_s(S^s; Q)$. Let $\{u_p\}$ be a basis for $\tilde{H}_*(F; Q)$ and $\{v_p\}$ a basis for $\tilde{H}_*(\hat{F}; Q)$. Let $d(u_p)$ and $d(v_p)$ denote respectively the dimension of $u_p$ and $v_p$. Write

$$\mu_*(\gamma) = \sum_{i,j} a_{ij} u_i \wedge v_j$$

and

$$\hat{\mu}_*(u_i \wedge v_j) = b_{ij}\gamma$$

Let $A = |a_{ij}|$, $B = |b_{ij}|$, and $D = |(-1)^{d(u_i)}\delta_{ij}|$ where $\delta_{ij}$ is the Kronecker symbol. By (2.2) we have

$$(1 \wedge \mu)_*(\gamma \wedge \gamma) = (\hat{\mu} \wedge 1)_* \nu_*(\gamma \wedge \gamma).$$

By expressing each side in terms of the basis elements $\gamma \wedge u_i \wedge v_j$ and equating coefficients, we obtain the relation $A = DAB^TA$. Since $A$ is non-singular $AB^T = D$.

Now suppose that $f_*(u_i) = \Sigma_k c_{ik} u_k$. We have

$$\begin{aligned} \hat{\mu}_*(f \wedge 1)_* \mu_*(\gamma) &= \hat{\mu}_*(f \wedge 1)_* \left(\sum_{i,j} a_{ij} u_i \wedge v_j\right) \\ &= \hat{\mu}_* \left(\sum_{i,j,k} a_{ij} c_{ik} u_k \wedge v_j\right) \\ &= \sum_{i,j,k} a_{ij} b_{kj} c_{ik} \gamma \\ &= \sum_{i,k} (-1)^{d(u_i)} \delta_{ik} c_{ik} \gamma \\ &= \sum_i (-1)^{d(u_i)} c_{ii} \gamma = \tilde{\Lambda}_f \gamma \end{aligned}$$

This completes the proof.

## 3. Ex-spaces

Consider a trivial fibration $p: F \to *$ and a map $f: F \to F$, where $F$ is a finite complex. In this case the transfer map we seek is to be of the form $\tau(f): S^s \to S^s \wedge F^+$, for large $s$, and is to have the property that

$$S^s \xrightarrow{\tau(f)} S^s \wedge F^+ \xrightarrow{1 \wedge p^+} S^s \text{ have degree } \Lambda_f.$$

To construct $\tau(f)$ let $\mu: S^s \to F^+ \wedge \hat{F}$ be a duality map and take $\tau(f)$ to be composite

$$S^s \xrightarrow{\mu} F^+ \wedge \hat{F} \xrightarrow{(1,f^+)\wedge 1} F^+ \wedge F^+ \wedge \hat{F} \xrightarrow{1 \wedge \hat{\mu}} F^+ \wedge S^s \longrightarrow S^s \wedge F^+.$$

Then it is immediate from the preceding lemma that $p\tau(f)$ has degree $\Lambda_f$.

In order to define $\tau(f)$ in general we intend to carry out the above construction "fiberwise". This leads naturally to the consideration of ex-spaces and duality for ex-spaces. In this section we discuss some aspects of the homotopy theory of ex-spaces and in the following section we deal with duality proper.

We shall work entirely in the category of compactly generated spaces [17]. Recall that an ex-space [13] $E = (E, B, p, \Delta)$ consists of maps $p: E \to B$ and $\Delta: B \to E$ such that $p\Delta = 1$. We assume throughout that $B$ is a $CW$-complex and $E$ has the homotopy type of a $CW$-complex. An *ex-map* $f: E \to E'$ is one which is both fiber and cross-section preserving, i.e. $p'f = p$ and $f\Delta = \Delta'$. The set of ex-homotopy classes of ex-maps from $E$ to $E'$ is denoted by $[E; E']$.

An ex-space $E$ is an *ex-fibration* if there is a lifting function

$$\Gamma: E \times_B B^I \to E^I$$

with the property that $\Gamma(\Delta(b), \sigma) = \Delta\sigma$, when $\sigma$ is a path in $B$ beginning at $b$. We will also need the notion of a *well based* ex-space as in [13]. $E$ is well based if there is a vertical retraction map $E \times I \to E \times \{0\} \cup \Delta(B) \times I$.

If $p: E \to B$ is a map we have an associated ex-space $\bar{E} = (\bar{E}, B, \bar{p}, \bar{\Delta})$ where $\bar{E}$ is the disjoint union of $E$ and $B$ and $\bar{p}$ and $\bar{\Delta}$ are the obvious maps. Observe that $\bar{E}$ is well based, and if $p: E \to B$ is a fibration, $\bar{E}$ is an ex-fibration.

If $X$ is a pointed space we will also use $X$ to denote the ex-space

$(X \times B, B, p, \Delta)$ where $p$ is projection on the second factor and $\Delta$ is the cross section determined by the base point.

The fiberwise reduced product of ex-spaces $E$ and $E'$ is denoted by $E \wedge_B E'$. Let $r: E \times_B E' \to E \wedge_B E'$ denote the identification map. Because of the exponential law in the category of compactly generated spaces, $r \times 1: (E \times_B E') \times Y \to (E \wedge_B E') \times Y$ is an identification for any space $Y$. From this it is easy to see that $r \times_B 1: (E \times_B E') \times_B Y \to (E \wedge_B E') \times_B Y$ is an identification for any space $Y$ over $B$. With this last observation it is easy to prove the following.

(3.1) LEMMA: *If $E$ and $E'$ are well based so is $E \wedge_B E'$. If $E$ and $E'$ are ex-fibrations so is $E \wedge_B E'$.*

(3.2) THEOREM: (Comparison theorem): *Let $E$ and $E'$ be ex-fibrations and suppose $g: E \to E'$ is such that its restriction to the fiber over $b$, $g_b: F_b \to F'_b$ is an $n$-equivalence, $b \in B$. Let $X$ be a well based ex-space. Then $g_{\#}: [X; E] \to [X; E']$ is injective if $X$ is $n$-coconnected and surjective if $X$ is $(n+1)$-coconnected.*

The proof is the same as the proof given for bundles in [1; theorem 3.3]. For other versions of the comparison theorem, see Eggar [11; Theorem 3.9] and James [14; Theorem 3.2].

(3.3) COROLLARY: *Suppose that $E$ and $E'$ are well based ex-fibrations and $g: E \to E'$ is such that $g_b: F_b \to F'_b$ is a homotopy equivalence, $b \in B$. Then $g$ is an ex-homotopy equivalence.*

Given $E = (E, B, p, \Delta)$ let $\Omega_B(E)$ denote the space of loops $\sigma: I \to E$ such that $\sigma(I) \subset F_b$ for some $b \in B$, and $\sigma(0) = \sigma(1) = \Delta(b)$. We have

$$\Omega(p): \Omega_B(E) \to B \quad \text{and} \quad \Omega(\Delta): B \to \Omega_B(E)$$

by $\Omega(p)(\sigma) = p(\sigma(0))$ and $\Omega(\Delta)(b) = \Delta(b) = \Delta(b)^*$ – the constant loop at $\Delta(b)$. If $E$ is an ex-fibration so is $\Omega_B(E)$ as is easily checked.

There is the suspension map

$$\sigma: [E, E'] \to [S^1 \wedge_B E; S^1 \wedge_B E'] \tag{3.4}$$

by $f \to 1 \wedge f$. By a standard argument involving the comparison theorem and the loop space $\Omega_B(S^1 \wedge E')$, we obtain the following suspension theorem (c.f. [1; Theorem 3.14] or [14; Theorem 4.3]).

(3.5) THEOREM: *Suppose that $E'$ is an ex-fibration such that each fiber $F_{b'}$ is $(n-1)$-connected. Let $E$ be well based. Then $\sigma$ is injective if $E$ is $(2n-1)$-coconnected and surjective if $E$ is $2n$-coconnected.*

Let

$$(3.6) \qquad \{E; E'\}_q = \overrightarrow{\mathrm{LIM}}_k[S^{k+q} \wedge E; S^k \wedge E']$$

with the natural abelian group structure. The cone over $E$ is $C(E) = I \wedge_B E$ with 0 the base point of $I$.

Suppose that $A$ is a subcomplex of $B$. Let $E_A = p^{-1}(A) \cup \Delta(B)$ regarded as an ex-space of $B$. Then, as in [13], we have an exact sequence

$$\cdots \to \{E \cup C(E_A); E'\}_q \to \{E; E'\}_q \to \{E_A; E'\}_q \to \cdots$$

Let $E/E_A$ be the quotient of $E$ obtained by identifying each fiber of $E_A$ to its base point and let $c: E \cup C(E_A) \to E/E_A$ denote the natural map. Note that if $E$ is well based so are $E_A$, $E/E_A$ and $E \cup C(E_A)$.

(3.7) LEMMA: *If $E$ and $E'$ are ex-fibrations and $E$ is well based then $c^{\#}: [E/E_A; E'] \to [E \cup C(E_A); E']$ is bijective.*

A proof is given in section 10. Now if $E$ and $E'$ meet the requirements of the lemma we may replace $\{E \cup C(E_A); E'\}$ in the above sequence by $\{E/E_A; E'\}$ via $c^{\#}$ and so obtain an exact sequence

$$(3.8) \qquad \cdots \to \{E/E_A; E'\}_q \to \{E; E'\}_q \to \{E_A; E'\}_q \to \cdots$$

## 4. Duality

In this section we will outline Spanier-Whitehead duality theory in the category of ex-spaces. Some aspects of this theory have been dealt with by K. Tsuchida [18]. We restrict ourselves to ex-spaces which are well based ex-fibrations having base $B$ a finite dimensional complex and each fiber homotopy equivalent to a finite complex. Briefly, we will refer to such ex-spaces as ex-fibrations.

An ex-map $\mu: S^s \times B \to E \wedge_B \hat{E}$ is a *duality map* if for each $b \in B$ the restricted map $\mu_b: S^s \to F_b \wedge \hat{F}_b$ is a duality map in the usual sense.

Given such a duality map and ex-fibrations $X$ and $Y$ we have

(4.1) $$D_\mu : \{X \wedge E; Y\}_q \to \{X; Y \wedge \hat{E}\}_{q+s}$$

defined by sending $f: S^{k+q} \wedge X \wedge E \to S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \longrightarrow S^{k+q} \wedge X \wedge S^s \xrightarrow{1 \wedge \mu} S^{k+q} \wedge X \wedge E \wedge \hat{E} \xrightarrow{f \wedge 1} S^k \wedge Y \wedge \hat{E}$$

and

(4.2) $$D^\mu : \{\hat{E} \wedge X; Y\}_q \to \{X; E \wedge Y\}_{q+s}$$

by sending $f: S^{k+q} \wedge \hat{E} \wedge X \to S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \xrightarrow{1 \wedge \mu \wedge 1} S^{k+q} \wedge E \wedge \hat{E} \wedge X \longrightarrow E \wedge S^{k+q} \wedge \hat{E} \wedge X \xrightarrow{1 \wedge f} E \wedge S^k \wedge Y \longrightarrow S^k \wedge E \wedge Y.$$

(4.3) LEMMA: *$D_\mu$ and $D^\mu$ are isomorphisms.*

This follows from the corresponding fact for pointed spaces if all the ex-spaces involved are products. The proof in general is by induction over the skeleta of $B$ using the exact sequence (3.8). The argument is standard and will be omitted.

If $\nu : S^s \times B \to X \wedge \hat{X}$ is a second duality map we have, as in the case of pointed spaces, an isomorphism

(4.4) $$D(\mu, \nu): \{E, X\}_q \to \{\hat{X}; \hat{E}\}_q$$

defined so as to make the following diagram commutative

(4.5)
$$\begin{tikzcd}
\{E; X\}_q \arrow[rr, "D(\mu,\nu)"] \arrow[dr, "D_\mu"'] & & \{\hat{X}; \hat{E}\}_q \arrow[dl, "D^\nu"] \\
& \{S^s \times B, X \wedge \hat{E}\}_{q+s} &
\end{tikzcd}$$

In particular, $f: E \to X$ is dual to $g : \hat{X} \to \hat{E}$ relative to $\mu$ and $\nu$ if and only if the diagram

$$
\text{(4.6)} \qquad
\begin{array}{ccc}
S^s \times B & \xrightarrow{\mu} & E \wedge \hat{E} \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle f \wedge 1} \\
X \wedge \hat{X} & \xrightarrow{1 \wedge g} & X \wedge \hat{E}
\end{array}
$$

is stably homotopy commutative.

(4.7) THEOREM: *If $E$ is an ex-fibration there is an integer $s$, an ex-fibration $\hat{E}$, and a duality map $\mu : S^s \times B \to E \wedge \hat{E}$.*

A proof is given in section 11.

## 5. Transfer

Let $\mathcal{F}$ denote the category of fibrations $p : E \to B$ such that $B$ is a finite dimensional complex and each fiber is homotopy equivalent to a finite complex. We consider commutative triangles

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E \\
& {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\
& B &
\end{array}
$$

where $p : E \to B$ is in $\mathcal{F}$. We will construct for such a triangle and for $A$ a subcomplex of $B$ a *transfer* map, which is an $S$-map

$$\text{(5.1)} \qquad \tau(f) : B/A \to E/E_A.$$

Here $E_A = p^{-1}(A)$.

Consider the ex-space $\bar{E}$, the disjoint union of $E$ and $B$. Since $\bar{E}$ is an ex-fibration in the sense of section 4, there is an ex-fibration $\hat{E}$ and a duality map

$$\mu : S^s \times B \to \bar{E} \wedge \hat{E}.$$

Analogous to the situation for pointed spaces (see section 2), $\bar{E} \wedge \hat{E}$ is canonically $2s$-self dual. Let

$$\hat{\mu} : \bar{E} \wedge \hat{E} \to S^s \times B$$

be dual to $\mu$. We have

$$S^s \times B \xrightarrow{\mu} \bar{E} \wedge \hat{E} \xrightarrow{(1,\bar{f})\wedge 1} \bar{E} \wedge \bar{E} \wedge \hat{E} \xrightarrow{1\wedge\hat{\mu}} \bar{E} \wedge S^s \longrightarrow S^s \wedge \bar{E}$$

which takes $S^s \times A \cup s_0 \times B$ into $S^s \times E_A \cup s_0 \times B$.

Identifying these subspaces to a point, the above map yields

$$\tau(f): S^s \wedge B/A \to S^s \wedge E/E_A.$$

We will show now that the $S$-homotopy class of $\tau(f)$ is well defined. Firstly, if $\mu$ is replaced by a suspension, this has the effect of replacing $\tau(f)$ by its suspension. Suppose now that

(5.2)
$$\begin{tikzcd} E_i \arrow[rr, "f_i"] \arrow[dr, "p_i"'] & & E_i \arrow[dl, "p_i"] \\ & B & \end{tikzcd}$$

$i = 1, 2$, are given and $h: E_1 \to E_2$ is a fiber homotopy equivalence such that $hf_1 = f_2h$. Let

$$\mu_i : S^s \times B \to \bar{E}_i \wedge \hat{E}_i, \qquad i = 1, 2,$$

be duality maps and let $k: \hat{E}_2 \to \hat{E}_1$ be dual to $\bar{h}$. Then $k$ is an ex-homotopy equivalence and we have commutativity relations

$$\begin{tikzcd} & \bar{E}_1 \wedge \hat{E}_1 \arrow[dd, "\bar{h}\wedge k^{-1}"] \\ S^s \times B \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\ & \bar{E}_2 \wedge \hat{E}_2 \end{tikzcd}, \qquad \begin{tikzcd} \bar{E}_1 \wedge \hat{E}_1 \arrow[dd, "\bar{h}\wedge k^{-1}"'] \arrow[dr, "\hat{\mu}_1"] & \\ & S^s \times B \\ \bar{E}_2 \wedge \hat{E}_2 \arrow[ur, "\hat{\mu}_2"'] & \end{tikzcd}$$

where the second triangle is obtained by dualizing the first. The following diagram is then commutative.

$$\begin{tikzcd} & \bar{E}_1 \wedge \hat{E}_1 \arrow[r, "(1,\bar{f}_1)\wedge 1"] \arrow[dd, "\bar{h}\wedge k^{-1}"] & \bar{E}_1 \wedge \bar{E}_1 \wedge \hat{E}_1 \arrow[r, "1\wedge\hat{\mu}_1"] \arrow[dd, "\bar{h}\wedge\bar{h}\wedge k^{-1}"] & \bar{E}_1 \wedge S^s \arrow[dd, "\bar{h}\wedge 1"] \\ S^s \times B \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & & & \\ & \bar{E}_2 \wedge \hat{E}_2 \arrow[r, "(1,\bar{f}_2)\wedge 1"] & \bar{E}_2 \wedge \bar{E}_2 \wedge \hat{E}_1 \arrow[r, "1\wedge\hat{\mu}_2"] & \bar{E}_2 \wedge S^s \end{tikzcd}$$

Therefore $\bar{h}\tau(f_1) = \tau(f_2)$. Taking $h$ to be the identity we see that $\tau(f)$ does not depend on the choice of duality map and moreover, $\tau(f)$ depends only on the fiber homotopy class of $f$.

We also established the following functorial property

(5.3) *With the data* (5.2) *if* $h : E_1 \to E_2$ *is a fiber homotopy equivalence such that* $hf_1$ *is fiber homotopic to* $f_2h$ *then* $h\tau(f_1) = \tau(f_2)$.

Now suppose we are given

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

and a map $g : X \to B$. There is the pullback diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{p}"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B \end{tikzcd} \quad \text{and the induced triangle} \quad \begin{tikzcd} \tilde{E} \arrow[rr, "\tilde{f}"] \arrow[dr, "\tilde{p}"'] & & \tilde{E} \arrow[dl, "\tilde{p}"] \\ & X & \end{tikzcd}$$

(5.4) *We have* $\tilde{g}\tau(\tilde{f}) = \tau(f)g$.

This is easily checked.

We may form the sum and product of the triangles in (5.2) obtaining

$$\begin{tikzcd} E_1 + E_2 \arrow[rr, "f_1+f_2"] \arrow[dr, "p_1+p_2"'] & & E_1 + E_2 \arrow[dl, "p_1+p_2"] \\ & B_1 + B_2 & \end{tikzcd}, \quad \begin{tikzcd} E_1 \times E_2 \arrow[rr, "f_1\times f_2"] \arrow[dr, "p_1\times p_2"'] & & E_1 \times E_2 \arrow[dl, "p_1\times p_2"] \\ & B_1 \times B_2 & \end{tikzcd}$$

where + denotes disjoint union.

(5.5) $\tau(f_1 + f_2) = \tau(f_1) \vee \tau(f_2) : (B_1/A_1) \vee (B_2/A_2) \to (E_1/E_{A_1}) \vee (E_2/E_{A_2})$

(5.6) $\tau(f_1 \times f_2) = \tau(f_1) \wedge \tau(f_2) : (B_1/A_1) \wedge (B_2/A_2) \to (E_1/E_{A_1}) \wedge (E_2/E_{A_2})$.

These properties follow from standard properties of duality maps as generalized to ex-spaces.

(5.7) *For the triangle*

$$\begin{tikzcd} B \arrow[rr, "1"] \arrow[dr, "1"'] & & B \arrow[dl, "1"] \\ & B & \end{tikzcd}$$

$\tau(1): B/A \to B/A$ *is the identity map.*

## 6. Products

We consider now the multiplicative properties of the cohomology homomorphism induced by the transfer. We have a commutative diagram

(6.1)
$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "p"] & E \times E \arrow[r, "p \times 1"] & B \times E \arrow[d, "1 \times p"] \\ B \arrow[rr, "d"] & & B \times B \end{tikzcd}$$

where $d$ is the diagonal map. From (5.3), (5.4), (5.6) and (5.7) we obtain, for subcomplexes $A$ and $C$ of $B$, a cummutative diagram

(6.2)
$$\begin{tikzcd} E/E_{A\cup C} \arrow[r, "d"] & E/E_A \wedge E/E_C \arrow[r, "p \wedge 1"] & B/A \wedge E/E_C \\ B/A \cup C \arrow[u, "\tau(f)"] \arrow[rr, "d"] & & B/A \wedge B/C \arrow[u, "1 \wedge \tau(f)"'] \end{tikzcd}$$

Let $\boldsymbol{M}$ be a ring spectrum and $\boldsymbol{N}$ an $\boldsymbol{M}$-module as in [19]. From the commutativity of the above diagram we obtain the formulas

(6.3) $\quad \tau(f)^*(p^*(x) \cup y) = x \cup \tau(f)^*(y),$

$$x \in \boldsymbol{M}^s(B/A),\ y \in \boldsymbol{N}^t(E/E_C).$$

(6.4) $\quad p_*(\tau(f)_*(x) \cap y) = x \cap \tau(f)^*(y),$

$$x \in \boldsymbol{N}_s(B/A \cup C),\ y \in \boldsymbol{M}^t(E/E_C).$$

Now consider the triangle

$$\begin{tikzcd} F \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "p"] \\ & \text{pt.} & \end{tikzcd}$$

In the diagram

$$\begin{tikzcd} S^s \arrow[r, "\mu"] & (F^+) \wedge \hat{F} \arrow[r, "(1,f^+)\wedge 1"] \arrow[d, "f^+ \wedge 1"] & (F^+) \wedge (F^+) \wedge \hat{F} \arrow[r, "1 \wedge \hat{\mu}"] & (F^+) \wedge S^s \arrow[d, "p \wedge 1"] \\ & (F^+) \wedge \hat{F} \arrow[rr, "\hat{\mu}"] & & S^s \end{tikzcd}$$

the composite $(1 \wedge \hat{\mu})((1, f^+) \wedge 1)\mu$ represents $\tau(f)$. Hence, by lemma (2.1) and the commutativity of the diagram we have (identifying pt.$^+$ with $S^0$).

(6.5) *$p\tau(f): S^0 \to S^0$ has degree $\tilde{\Lambda}(f^+) = \Lambda(f)$ – the Lefschetz number of $f$.*

We can now establish the fundamental property of the transfer. Consider

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

with $p: E \to B$ in $\mathscr{F}$. Let $f_b: F_b \to F_b$ denote the restriction of $f$ to the fiber over $b \in B$ and let $\Lambda$ denote the Lefschetz number of $f_b$. Let $H(\ \ ;\Gamma)$ denote singular theory with coefficients in the abelian group $\Gamma$.

(6.6) THEOREM: *If $B$ is connected the composite*

$$\tilde{H}^*(B/A;\Gamma) \xrightarrow{p^*} \tilde{H}^*(E/E_A;\Gamma) \xrightarrow{\tau(f)^*} \tilde{H}^*(B/A;\Gamma)$$

*is multiplication by $\Lambda$.*

PROOF: Consider the inclusion

$$\begin{array}{ccc} F_b & \xrightarrow{j_b} & E \\ \Big\downarrow{\scriptstyle p_b} & & \Big\downarrow{\scriptstyle p} \\ \{b\} & \xrightarrow{i_b} & B \end{array}$$

By (6.5), for $1 \in \tilde{H}^\circ(B^+; Z)$

$$i_b^* \tau(f)^* p^*(1) = \tau(f_b)^* p_b^*(1) = \Lambda.$$

Since $i_b^*: \tilde{H}^\circ(B^+; Z) \to \tilde{H}^\circ(\{b\}^+; Z)$ is an isomorphism,

$$\tau(f)^*(1) = \tau(f)^* p^*(1) = \Lambda$$

Applying (6.3), we have for $x \in H^*(B/A; \Gamma)$,

$$\begin{aligned} \tau(f)^* p^*(x) &= \tau(f)^*(p^*(x) \cup 1) \\ &= x \cup \tau(f)^*(1) = \Lambda x. \end{aligned}$$

## 7. The retraction property

In this section we compare the transfer for a fibration with that of a retract up to homotopy.

Suppose that $p: E \to B$ and $q: D \to B$ are fibrations in $\mathscr{F}$ and

$$D \xrightarrow{\lambda} E \xrightarrow{\rho} D$$

are fiber preserving maps such that $\rho\lambda \simeq 1$ over the identity. Then if $f: D \to D$ is a fiber preserving map we have triangles

$$\begin{array}{ccc} D & \xrightarrow{f} & D \\ & {\scriptstyle q}\searrow \;\; \swarrow{\scriptstyle q} & \\ & B & \end{array}, \qquad \begin{array}{ccc} E & \xrightarrow{\lambda f \rho} & E \\ & {\scriptstyle p}\searrow \;\; \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

(7.1) THEOREM: $\lambda\tau(f) = \tau(\lambda f\rho) : B/A \to E/E_A$.

PROOF: Let

$$\mu_1 : S^s \times B \to \bar{D} \wedge \hat{D}$$
$$\mu_2 : S^s \times B \to \bar{E} \wedge \hat{E}$$

be duality maps. Let $\hat{\lambda} : \hat{E} \to \hat{D}$ be dual to $\bar{\lambda} : \bar{D} \to \bar{E}$ relative to $\mu_1$ and $\mu_2$, so that

(7.2)
$$\begin{tikzcd}
S^s \times B \arrow[r, "\mu_1"] \arrow[d, "\mu_2"] & \bar{D} \wedge \hat{D} \arrow[d, "\bar{\lambda} \wedge 1"] \\
\bar{E} \wedge \hat{E} \arrow[r, "1 \wedge \hat{\lambda}"] & \bar{E} \wedge \hat{D}
\end{tikzcd}$$

is commutative. Consider the diagram

(7.3)
$$\begin{tikzcd}
 & \bar{D} \wedge \hat{D} \arrow[r, "(1, \bar{f}) \wedge 1"] \arrow[d, "\bar{\lambda} \wedge 1"] & \bar{D} \wedge \bar{D} \wedge \hat{D} \arrow[r, "1 \wedge \hat{\mu}_1"] \arrow[d, "\bar{\lambda} \wedge 1 \wedge 1"] & \bar{D} \wedge S^s \arrow[ddd, "\bar{\lambda} \wedge 1"] \\
S^s \times B \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \bar{E} \wedge \hat{D} \arrow[r, "(1, f\rho) \wedge 1"] & \bar{E} \wedge \bar{D} \wedge \hat{D} \arrow[ddr, "1 \wedge \hat{\mu}_1"] & \\
 & \bar{E} \wedge \hat{E} \arrow[r, "(1, \overline{\lambda f \rho}) \wedge 1"] \arrow[u, "1 \wedge \hat{\lambda}"] \arrow[dr, "(1, \overline{\lambda f \rho}) \wedge 1"'] & \bar{E} \wedge \bar{E} \wedge \hat{E} \ \text{(A)} \arrow[u, "1 \wedge \bar{\rho} \wedge \hat{\lambda}"] \arrow[d, "1 \wedge \overline{\lambda \rho} \wedge 1"] & \\
 & & \bar{E} \wedge \bar{E} \wedge \hat{E} \arrow[r, "1 \wedge \hat{\mu}_2"] & \bar{E} \wedge S^s
\end{tikzcd}$$

The commutativity of the triangle (A) follows from the commutativity of the diagram

$$\begin{tikzcd}
 & \bar{D} \wedge \hat{D} \arrow[r, "\hat{\mu}_1"] & S^s \times B \\
\bar{E} \wedge \hat{E} \arrow[ur, "\bar{\rho} \wedge \hat{\lambda}"] \arrow[r, "\bar{\rho} \wedge 1"] & \bar{D} \wedge \hat{E} \arrow[r, "\bar{\lambda} \wedge 1"] \arrow[u, "1 \wedge \hat{\lambda}"] & \bar{E} \wedge \hat{E} \arrow[u, "\hat{\mu}_2"]
\end{tikzcd}$$

where the square is the dual of (7.2). The remaining commutativity relations in (7.3) are easily checked. The theorem follows by comparing the two outside paths in (7.3) from $S^s \times B$ to $\bar{E} \wedge S^s$.

## 8. Proof of theorem (1.1)

We begin with an observation concerning the transfer map when the base space is a suspension. Suppose that $X$ is a finite dimensional complex with base point $x_0$ and we are given

$$\begin{tikzcd} \tilde{E} \arrow[rr, "\tilde{f}"] \arrow[dr, "\tilde{p}"'] & & \tilde{E} \arrow[dl, "\tilde{p}"] \\ & S(X) & \end{tikzcd}$$

with $\tilde{p} : \tilde{E} \to S(X)$ in $\mathscr{F}$. Let $F$ denote the fiber over $x_0$ and choose a base point $e_0 \in F$. Let

(8.1) $$\Delta : S(X) \to \tilde{E}/F$$

be defined by $\Delta(e^{2\pi it} \wedge x) = \tilde{\Gamma}(e_0, \sigma(t, x))$ (1), where $\tilde{\Gamma}$ is a lifting function and $\sigma(t, x) : I \to S(X)$ is the path

$$\sigma(t, x)(\lambda) = e^{2\pi it\lambda} \wedge x.$$

Let $\Lambda$ denote the Lefschetz number of $f' : F \to F$, the restriction of $f$ to $F$.

(8.2) Lemma: *Assume that $F$ is connected. Then $\Lambda\Delta$ is stably homotopic to $\tau(\tilde{f}) : S(X) \to \tilde{E}/F$.*

Proof: Let $C(X) = I \wedge X$ denote the reduced cone of $X$ (with 0 the base point of $I$) and consider

$$\begin{tikzcd} C(X) \times F \arrow[r, "\psi"] \arrow[d, "\pi_1"'] & \tilde{E} \arrow[d, "\tilde{p}"] \\ C(X) \arrow[r, "\rho"] & S(X) \end{tikzcd}$$

where $\rho$ is the natural identification and

$$\psi(t \wedge x, y) = \tilde{\Gamma}(y, \sigma(t, x))(1).$$

Then $\psi$ is a homotopy equivalence on each fiber and the restriction of $\psi$ to the fiber over $x_0$ is the identity. It follows that

$$\begin{tikzcd} C(X) \times F \arrow[r, "1\times f'"] \arrow[d, "\psi"] & C(X) \times F \arrow[d, "\psi"] \\ \tilde{E} \arrow[r, "\tilde{f}"] & \tilde{E} \end{tikzcd}$$

is fiber homotopy commutative. Therefore, by (5.3), (5.4) and (5.6) $\tau(\tilde{f})\rho = \psi\tau(1 \times f') = \psi(1 \wedge \tau(f')) : C(X)/X \to \tilde{E}/F$, where $\tau(f') : S^\circ \to F^+$ is associated with the trivial triangle

$$\begin{tikzcd} F \arrow[rr, "f'"] \arrow[dr] & & F \arrow[dl] \\ & * & \end{tikzcd}$$

Let $i_0 : S^\circ \to F^+$ by $i_0(+1) = e_0$ and $i_0(-1) = +$. Since $F$ is connected it is clear from the behaviour of the homomorphism in singular homology induced by $\tau(f')$ that $\tau(f') = \Lambda i_0$. Note that

(8.3)
$$\begin{tikzcd} C(X)/X \wedge F^+ \arrow[r, "\psi"] & \tilde{E}/F \\ C(X)/X \arrow[r, "\rho"] \arrow[u, "1 \wedge i_0 = j_0"] & S(X) \arrow[u, "\Delta"] \end{tikzcd}$$

is commutative. Therefore

$$\tau(\tilde{f})\rho = \psi(1 \wedge \tau(f')) = \psi(1 \wedge \Lambda i_0) = \Lambda\Delta\rho.$$

Since $\rho : C(X)/X \to S(X)$ is a homeomorphism, $\tau(\tilde{f}) = \Lambda\Delta$ and the proof is complete.

Now let

(8.4)
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

be given where the fiber $F$ of $p: E \to B$ is a finite complex and $B$ is a complex (not necessarily finite dimensional). Choose base points $b_0 \in B$ and $e_0 \in F = p^{-1}(b_0)$ and let $\omega: \Omega B \to F$ denote the boundary map arising from the fibration $p: E \to B$.

If $X$ is a finite dimensional complex with base point and $g: X \to \Omega B$ is a base point preserving map let

(8.5)
$$\begin{tikzcd} \tilde{E} \arrow[rr, "\tilde{f}"] \arrow[dr, "\tilde{p}"'] & & \tilde{E} \arrow[dl, "\tilde{p}"] \\ & S(X) & \end{tikzcd}$$

be induced by $S(X) \xrightarrow{S(g)} S(\Omega B) \xrightarrow{\epsilon} B$ where $\epsilon$ is the adjoint of the identity map. We will show now that

(8.6)
$$\begin{tikzcd} \tilde{E}/F \arrow[r, "k"] & S(F) \\ S(X) \arrow[u, "\Delta"] \arrow[r, "S(g)"] & S(\Omega B) \arrow[u, "S(\omega)"'] \end{tikzcd}$$

is commutative, where $\Delta$ is as in (8.1) and $k$ is from the Puppe sequence of the cofibration $F \to \tilde{E}$.

We have

(8.7)
$$\begin{tikzcd} \tilde{E}/F \arrow[r, "k"] & S(F) & \\ C(X) \times F/X \times F \arrow[u, "\psi"] \arrow[r, "k"] & S(X \times F) \arrow[u, "S(\psi)"'] & S(\Omega B) \arrow[ul, "S(\omega)"'] \arrow[dl, "S(g)"] \\ C(X)/X \arrow[u, "j_0"] \arrow[r, "k=\rho"] & S(X) \arrow[u, "S(j_0)"'] & \end{tikzcd}$$

where $k$ in each case is from the appropriate Puppe sequence and $j_0$ is the inclusion $y \to (y, e_0)$. The commutativity of the right hand triangle is by direct calculation. The commutativity of (8.6) now follows from the commutativity of (8.3) and (8.7).

We are now in a position to prove the theorem of the introduction.

(1.1) THEOREM: *Assume that $F$ is connected. Then*

$$\Lambda\omega_*:\{X;\Omega B\}_q \to \{X;F\}_q$$

*is trivial for any finite dimensional complex $X$.*

PROOF: $\Omega B$ has the homotopy type of a $CW$-complex $Y$. If $\phi: Y \to \Omega B$ is a homotopy equivalence it is sufficient, to prove the theorem, to show that $\Lambda\omega_*(\{g\}) = 0$ if $X$ is a finite dimensional subcomplex of $Y$ and $g: X \to \Omega B$ is the inclusion followed by $\phi$. By the commutativity of (8.6) and Lemma (8.2)

$$\Lambda\omega_*(\{g\}) = \Lambda\{k\Delta\} = \{k\tau(f)\}.$$

We have a commutative diagram

$$\begin{tikzcd}
 & \tilde{E} \arrow[d, "j"] & \\
\tilde{E}^+ \arrow[r, "c'"] \arrow[ur, "c''"] & \tilde{E}/F \arrow[r, "k"] & S(F) \\
S(X)^+ \arrow[u, "\tau(f)"] \arrow[r, "c"] & S(X) \arrow[u, "\tau(f)"] &
\end{tikzcd}$$

where $c$, $c'$, $c''$, and $j$ are quotient maps. Since $\{kj\} = 0$ we have $\{k\tau(f)c\} = c^*\{k\tau(f)\} = 0$. Since $c^*$ is monomorphic, $\{k\tau(f)\} = 0$ and the proof is complete.

REMARKS: (1) The map $\omega$ frequently appears in other forms, hence theorem (1.1) applies for (a) coset maps $\rho: G \to G/H$, or more generally for (b) maps $\omega: M \to X$ which factor through the evaluation map $\mathcal{H}(X) \to X$ where $\mathcal{H}(X)$ is the space of homotopy equivalences of $X$, or for (c) fibre inclusions of principal bundles. Theorem (1.1) states that $\Lambda\omega_* = 0$ and $\Lambda\omega^* = 0$ for all homology and cohomology theories on the category of finite dimensional complexes. This is an extension of two results of [7], wherein theorem (1.1) was proved only for singular cohomology and for homotopy groups in the stable range. See also [5].

(8.8) COROLLARY: *Let $\alpha \in \pi_i(S^{2n})$. Then $[\alpha, \iota_{2n}] = 0$ implies that $2\{\alpha\} = 0$, where $\{\alpha\}$ denotes the stable homotopy element represented by*

*$\alpha$ and $[\alpha, \iota_{2n}]$ is the Whitehead product of $\alpha$ with the generator of $\pi_{2n}(S^{2n})$.*

PROOF: The fact that $[\alpha, \iota_{2n}] = 0$ implies there is a map $F: S^i \times S^{2n} \to S^{2n}$ such that $F$ restricted to $* \times S^{2n}$ is the identity and $F$ restricted to $S^i \times *$ represents $\alpha$. Taking adjoints, we see that $\alpha$ factors through $\omega: M \to S^{2n}$ where $M$ is the space of degree one maps on $S^{2n}$, and $\omega$ is the evaluation map given by evaluation at the base point. Thus (1.1) may be applied to $\alpha$ in view of the remark. In this case $\Lambda = \chi(S^{2n}) = 2$.

Let $G$ be a compact connected Lie group, $H$ a closed subgroup of $G$, and $\rho: G \to G/H$ the projection.

(8.9) COROLLARY: *As an $S$-map $\chi(G/H)\rho: G \to G/H$ is trivial, where $\chi(G/H)$ is the Euler characteristic of $G/H$.*

In particular, if $N$ is the normalizer of a maximal torus in $G$ then $\rho: G \to G/N$ is stably trivial since $\chi(G/N) = 1$ [6, 12]. On the other hand, it is interesting to note that $\rho_*: \pi_k(G) \to \pi_k(G/N)$ is an isomorphism for $k > 2$.

## 9. Smooth fiber bundles

In the case of a smooth fiber bundle $p: E \to B$ a more geometric description can be given for the transfer associated with a fiber preserving map. We assume that $B$ and $F$ are closed manifolds.

Let $\tilde{p}: E \to B \times R^s$ be a fiber preserving embedding. Its normal bundle $\beta$ is inverse to the bundle $\alpha$ of tangents along the fiber and we have an isomorphism $\alpha \oplus \beta \simeq R^s$ associated with the embedding. Let

$$c: B^+ \wedge S^s \to E$$

denote the Pontryagin-Thom map of this trivialization.

The diagonal inclusion into the fiber square, $d: E \to E^2$, has normal bundle $\alpha$ so that we have

$$c': (E^2)^{\pi_1^*(\beta)} \longrightarrow E^{\alpha \oplus \beta} = E^+ \wedge S^s$$

where $\pi_1: E^2 \to E$ is projection onto the first factor.

If $f: E \to E$ is a fiber preserving map let

$$\widetilde{(1,f)}: E^{\beta} \to (E^2)^{\pi_1^*(\beta)}$$

send $v_e$ to $(e, f(e), v_e)$.

(9.1) PROPOSITION: *$\tau(f): B^+ \to E^+$ is represented by*

$$B^+ \wedge S^s \xrightarrow{\quad c \quad} E^{\beta} \xrightarrow{\quad \widetilde{(1,f)} \quad} (E^2)^{\pi_1^*(\beta)} \xrightarrow{\quad c' \quad} E.$$

First observe that the $S$-map determined by $c'\widetilde{(1,f)}c$ is independent of the choice of embedding $\tilde{p}$ and of the tabular neighborhood maps used in constructing $c$ and $c'$.

Now, for $\tilde{p}: E \to B \times R^s$, let $\beta$ denote the normal bundle, let $S(\beta)$ denote the total space of the unit sphere bundle, and let $\hat{E}$ denote the quotient of $D(\beta)$ obtained by identifying each fiber of $S(\beta)$ to a point. We regard $\hat{E}$ as an ex-space of $B$ by $\hat{p}: \hat{E} \to B$ and $\hat{\Delta}: B \to \hat{E}$ where $\hat{p}([v_e]) = p(e)$ and $\hat{\Delta}(p(e)) = [v'_e]$, where $v'_e \in S(\beta)$. Then $\hat{p}$ is the projection of a fiber bundle whose fiber over $b$ is the Thom space $F_b^{\nu_b}$, where $\nu_b$ is the normal bundle of the embedding $F_b \to \{b\} \times R^s$.

Choose a fiber preserving tubular neighborhood $D(\beta) \subset B \times R^s$ and let $\theta: B \times S^s \to \hat{E}$ denote the associated Pontryagin-Thom map. Let

$$\mu: B \times S^s \to \bar{E} \wedge_B \hat{E} \tag{9.2}$$

be the composite $B \times S^s \xrightarrow{\theta} \hat{E} \xrightarrow{d} \bar{E} \wedge_B \hat{E}$, where $d(v_e) = e \wedge v_e$. By Atiyah's duality theorem [1] $\mu$ is a duality map.

The diagonal embedding $E \to E \times_B D(\beta)$ has normal bundle $\alpha \oplus \beta = E \times R^s$. Choosing a fiber preserving tubular neighborhood $E \times D^s \subset E \times_B D(\beta)$, we obtain

$$\theta': \bar{E} \wedge_B \hat{E} \to \bar{E} \wedge_B S^s. \tag{9.3}$$

Let $\theta'': \bar{E} \wedge_B \hat{E} \to B \times S^s$ denote $\theta'$ followed by the projection $\bar{E} \wedge_B S^s \to B \times S^s$.

(9.4) LEMMA: *$\theta''$ is dual to $\mu$.*

PROOF: We must show that

$$\begin{tikzcd}
B \times S^s \wedge S^s \arrow[r, "\nu"] \arrow[dr, "\mu \wedge 1"'] & (\bar{E} \wedge_B \hat{E}) \wedge_B (\bar{E} \wedge_B \hat{E}) \arrow[d, "1 \wedge \theta''"] \\
 & (\bar{E} \wedge_B \hat{E}) \wedge S^s
\end{tikzcd}$$

is homotopy commutative. Since isotopic embeddings determine homotopic duality maps, the duality map determined by

$$E \xrightarrow{(\bar{p},\,0)} B \times R^s \times R^s$$

is homotopic to that determined by

$$E \xrightarrow{(\bar{p},\,\bar{p})} B \times R^s \times R^s.$$

The former duality map is $\mu \wedge 1$ whereas, using the factorisation

$$E \xrightarrow{d} E^2 \xrightarrow{\bar{p} \times \bar{p}} B \times R^s \times R^s,$$

the latter is easily seen to be homotopic to $(1 \wedge \theta'')\nu$.

Proposition (9.1) is now a consequence of the following commutative diagram.

$$\begin{tikzcd}
B \times S^s \arrow[r, "\mu"] \arrow[dr, "\theta"] \arrow[dd] & \bar{E} \wedge_B \hat{E} \arrow[r, "(1,f) \wedge 1"] & \bar{E} \wedge_B \bar{E} \wedge_B \hat{E} \arrow[r, "1 \wedge \hat{\mu}"] & \bar{E} \wedge S^s \arrow[dd] \\
 & \hat{E}^{\beta} \arrow[u, "d"] \arrow[r, "g"] \arrow[d] & \bar{E} \wedge \hat{E} \arrow[u, "d'"] \arrow[ur, "\theta'"] \arrow[d] & \\
B^+ \wedge S^s \arrow[r, "c"] & E^{\beta} \arrow[r, "\widetilde{(1,f)}"] & (E^2)^{\pi_1^*(\beta)} \arrow[r, "c'"] & E^+ \wedge S^s.
\end{tikzcd}$$

Here $d'(e' \wedge v_e) = e \wedge e' \wedge v_e$, $g(v_e) = f(e) \wedge v_e$, and $h(e' \wedge v_e) = (e, e', v_e)$. The unlabeled arrows denote the natural identification map. The commutativity of the upper right hand triangle follows from the fact that $\hat{\mu} = \theta''$.

REMARK It follows from Proposition (9.1) and the retraction property (7.1) that the two methods of constructing the transfer which are outlined in [3], do in fact lead to the same map.

## 10. Proof of (3.7)

Suppose that $E = (E, B, p, \Delta)$ and $E' = (E', B', p', \Delta')$ are ex-spaces. If $h: E \to E'$ and $f: B \to B'$ are such that $p'h = fp$ and $h\Delta = \Delta' f$ we will say that $h$ is a map *over* $f$. If $E'$ is an ex-fibration then we obtain from the special nature of the lifting function for $E'$ the following covering homotopy property.

(10.1) *Given $F: B \times I \to B'$ and a map $h: E \to E'$ over $F_0$ there is $H: E \times I \to E'$ such that $H_0 = h$ and $H_t$ is a map over $F_t$*, $0 \le t \le 1$.

Suppose that $E$ is an ex-space of $B$ and $A \subset B$ is a subcomplex. As before, let $c: E \cup C(E_A) \to E/E_A$ denote the natural map. Let $\lambda: B \times I \to B \times \{1\} \cup A \times I$ be a retraction map and let $F: B \times I \to B$ denote $\lambda$ followed by projection onto $B$.

(10.2) Lemma: *Suppose that $E$ is an ex-fibration. There is $q: E/E_A \to E \cup C(E_A)$ over $F_0$ and homotopies $H: E/E_A \times I \to E/E_A$ and $K: E \cup C(E_A) \times I \to E \cup C(E_A)$ such that*

(a) *$H_0 = cq$, $H_1 = 1$, and $H_t$* is over *$F_t$*, $0 \le t \le 1$.

(b) *$K_0 = qc$, $K_1 = 1$, and $K_t$* is over *$F_t$*, $0 \le t \le 1$.

Proof: Consider

$$\begin{tikzcd} E \times \{1\} \arrow[r, "i_1"] \arrow[d, "p \times 1"] & E \times 1 \arrow[d, "p \times 1"] \\ B \times I \arrow[r, "\lambda"] & B \times I. \end{tikzcd}$$

Applying (10.1) for the ex-fibration $E \times I$ there is $M: E \times I \to E \times I$ such that $M_1 = i_1$ and $M_t$ is over $\lambda_t$, $0 \le t \le 1$. Then we actually have

$$M: E \times I \to E \times \{1\} \cup E_A \times I.$$

Let $q'$ denote

$$E \xrightarrow{M_0} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Since $M_0(E_A) \subset E_A \times \{0\}$ we have $q'(E_A) \subset \Delta(B)$. Let $q: E/E_A \to E \cup C(E_A)$ denote the collapse of $q'$. Then $q$ is a map over $F_0$.

To construct $H$, the map

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A) \xrightarrow{c} E/E_A$$

has a quotient $H : E/E_A \times I \to E/E_A$ which is the desired map.

To construct $K$ let $N : E \times I \to E \cup C(E_A)$ denote the composite

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Define $K : E \cup C(E_A) \times I \to E \cup C(E_A)$ by $K(e, t) = N(e, t)$, if $e \in E$, and $K([e, s], t) = s * N(e, t)$, if $e \in E_A$, where $s * [e', t] = [e', st]$ for $[e', t] \in C(E_A)$. This completes the proof.

We will now prove (3.7) which asserts that

$$c^{\#} : [E/E_A ; E'] \to [E \cup C(E_A); E']$$

is bijective provided $E$ and $E'$ are ex-fibrations.

To show that $c^{\#}$ is onto let $\theta : E \cup C(E_A) \to E'$ be an ex-map and consider

$$\begin{tikzcd} E/E_A \times \{0\} \arrow[r, "\theta q"] \arrow[d, "p \times 1"] & E' \arrow[d, "p'"] \\ B \times I \arrow[r, "F"] & B. \end{tikzcd}$$

There is $N : E/E_A \times I \to E'$ over $F$ such that $N_0 = \theta q$. Let $\psi = N_1 : E/E_A \to E'$. Then $\psi$ is an ex-map and we will show that $c^{\#}([\psi]) = [\theta]$. We have homotopies

$$E \cup C(E_A) \times I \xrightarrow{c \times 1} E/E_A \times I \xrightarrow{N} E'$$

over $F$, and

$$E \cup C(E_A) \times I \xrightarrow{K} E \cup C(E_A) \xrightarrow{\theta} E'$$

also over $F$. Then

$$(\theta K^{-1}) \circ (N(c \times 1)) : E \cup C(E_A) \times I \to E'$$

is a homotopy from $\theta$ to $\psi c$ over $F^{-1} \circ F$. By a standard argument involving the covering homotopy property (10.1) we see that $\theta$ is ex-homotopic to $\psi c$.

To show that $c^{\#}$ is one-one let $\psi_0, \psi_1 : E/E_A \to E'$ be such that $\psi_0 c$ is ex-homotopic to $\psi_1 c$ by $p : E \cup C(E_A) \times I \to E'$ say. We have

$$\begin{array}{ccc} E/E_A \times (I \times \{0\} \cup \dot{I} \times I) & \xrightarrow{\quad Q \quad} & E' \\ \Big\downarrow{\scriptstyle p \times 1 \times 1} & & \Big\downarrow{\scriptstyle p'} \\ B \times I \times I & \xrightarrow{\quad \tilde{F} \quad} & B \end{array}$$

where $\tilde{F}(b, t, \lambda) = F(b, \lambda)$ and $Q(e, t, 0) = p(q(e), t)$, $Q(e, 0, \lambda) = \psi_0 H(e, \lambda)$, and $Q(e, 1, \lambda) = \psi_1 H(e, \lambda)$. There is then $R : E/E_A \times I \times I \to E'$ over $\tilde{F}$ which extends $Q$. Then $S : E/E_A \times I \to E'$ by $S(e, t) = R(e, t, 1)$ is an ex-homotopy from $\psi_0$ to $\psi_1$.

## 11. Proof of (4.7)

Let $E = (E, B, p, \Delta)$ be an ex-fibration in the sense of section 4. Let $\Sigma(E)$ denote the unreduced fiberwise suspension of $E$ and $\Sigma(p) : \Sigma(E) \to B$ the projection. There is the "south pole" cross section $\delta : B \to \Sigma(E)$ given by $\delta(b) = [e, 0]$ where $p(e) = b$ and we let $\Sigma_0(E) = (\Sigma(E), B, \Sigma(p), \delta)$. It is easy to to check that $\Sigma_0(E)$ is an ex-fibration as in section 4. The quotient map $\pi : \Sigma_0(E) \to S^1 \wedge E$ is an ex-map and is a homotopy equivalence on each fiber. Hence by the comparison theorem (3.2), $\pi$ is an ex-homotopy equivalence. Note that if $p : E \to B$ is a fibration with fiber a finite complex (not necessarily equipped with a cross section) we may still form $\Sigma_0(E)$ which is an ex-fibration.

Now let $E$ be an ex-fibration. To construct an ex-fibration $\hat{E}$ and a duality map

$$\mu : S^s \times B \to E \wedge \hat{E}$$

we proceed by induction over the skeleta of $B$. Let $B$ have dimension $n$ and let $A$ denote the $(n-1)$-skeleton of $B$. Assume there is an ex-fibration $D(E|A)$, with fiber $\hat{F}$ say, and a duality map

$$\omega : S^s \times A \to E|A \wedge D(E|A). \tag{11.1}$$

Let $B$ be obtained from $A$ by adjoining cells via $\lambda_j : S^{n-1} \to A$, $j \in J$, and let $\bar{\lambda}_j : D^n \to B$ denote the characteristic map. Let $\bar{\psi}_j : F \times D^m \to \bar{\lambda}_j^*(E)$ be an ex-fiber homotopy equivalence and $\psi_j : F \times S^{n-1} \to \lambda_j^*(E)$ its restriction.

We have a duality map

(11.2) $$\eta_j : S^s \times S^{n-1} \to \lambda_j^*(E) \wedge \lambda_j^*(D(E|A))$$

induced by $\omega$. Choose a duality map

(11.3) $$\nu : S^s \times S^{n-1} \to (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1})$$

and let $\phi_j : \lambda_j^*(D(E|A)) \to \hat{F} \times S^{n-1}$ be dual to $\psi_j$ relative to $\eta_j$ and $\nu$. Then $\phi_j$ is an ex-homotopy equivalence and we have a homotopy commutative triangle.

(11.4)
$$\begin{tikzcd} & \lambda_j^*(E) \wedge \lambda_j^*(D(E|A)) \\ S^s \times S^{n-1} \arrow[ur, "\eta_j"] \arrow[dr, "\nu"'] & \\ & (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \arrow[uu, "\psi_j \wedge \phi_j^{-1}"'] \end{tikzcd}$$

Let $\theta_j : \hat{F} \times S^{n-1} \to D(E|A)$ denote the composite

$$\hat{F} \times S^{n-1} \xrightarrow{\phi_j^{-1}} \lambda_j^*(D(E|A)) \xrightarrow{\bar{\lambda}_j} D(E|A),$$

and let $X'$ be obtained by adjoining, for each $j \in J$, $\hat{F} \times D^n$ to $D(E|A)$ via $\theta_j$. We have an ex-space $(X', B, p', \Delta')$ where $p'$ and $\Delta'$ are the obvious maps. Moreover, by results of Dold and Thom [10 (2.2) and (2.10)], $p' : X' \to B$ is a quasifibration. We replace $p'$ by a fibration in the usual way obtaining a commutative square

(11.5)
$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d, shift right, "p'"'] & X \arrow[d, shift right, "p"'] \\ B \arrow[r, "1"] \arrow[u, shift right, "\Delta'"'] & B, \arrow[u, shift right, "\Delta"'] \end{tikzcd}$$

where $X = \{(x, \sigma) \in X' \times B^I | p'(x) = \sigma(0)\}$, $p(x, \sigma) = \sigma(1)$, $\Delta(b) = (\Delta'(b), b^*)$ where $b^*$ denotes the constant path at $b$, and $\alpha(x) = (x, p'(x)^*)$.

Now $p : X \to B$ is a fibration with each fiber $p^{-1}(b)$ of the weak homotopy type of $\hat{F}$. Since $X$ (being homotopy equivalent to $X'$) has the homotopy type of a $CW$-complex, it follows from [16 Proposition

0] that $p^{-1}(b)$ has the homotopy type of a $CW$-complex, hence is homotopy equivalent to $\hat{F}$.

Let $\hat{E} = \Sigma_0(X)$. Then $\hat{E}$ is an ex-fibration as in section 4. We will show now that there is a duality map

$$\mu : S^{s+1} \times B \to E \wedge \hat{E}.$$

From (11.4) we obtain the following homotopy commutative diagram.

$$\begin{tikzcd}
S^1 \wedge \lambda_j^*(E) \wedge \lambda_j^*(D(E|A)) \arrow[r] & E \wedge S^1 \wedge D(E|A) \arrow[r, "1 \wedge \pi^{-1}"] & E \wedge \Sigma_0(D(E|A)) \arrow[dd, "\tilde{\lambda}_j\psi_j \wedge \Sigma(\theta_j)"] \\
S^{s+1} \times S^{n-1} \arrow[u, "1 \wedge \eta_j"] \arrow[d, "1 \wedge \nu"] & & \\
S^1 \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \arrow[r] & (F \times S^{n-1}) \wedge ((S^1 \wedge \hat{F}) \times S^{n-1}) \arrow[uu, "\tilde{\lambda}_j\psi_j \wedge 1 \wedge \theta_j"] \arrow[r, "1 \wedge \pi^{-1}"] & (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1})
\end{tikzcd} \tag{11.6}$$

Let $\omega'$ denote the duality map

$$S^{s+1} \times A \xrightarrow{1 \wedge \omega} S^1 \wedge E|A \wedge D(E|A) \to E|A \wedge S^1 \wedge D(E|A)$$
$$\xrightarrow{1 \wedge \pi^{-1}} E|A \wedge \Sigma_0(D(E|A)) \xrightarrow{1 \wedge \Sigma(\alpha)} E|A \wedge \hat{E}|A,$$

and let $\nu'$ denote the duality map

$$S^{s+1} \times S^{n-1} \xrightarrow{1 \wedge \nu} S^1 \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1})$$
$$\longrightarrow (F \times S^{n-1}) \wedge ((S^1 \wedge \hat{F}) \times S^{n-1}) \xrightarrow{1 \wedge \pi^{-1}} (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1})$$

We have from (11.6) a homotopy commutative diagram

$$\begin{tikzcd}
& A \times S^{s+1} \arrow[dr, "\omega'"] & \\
S^{s+1} \times S^{n-1} \arrow[ur, "\lambda_j \times 1"] \arrow[dr, "\nu'"'] & & E \wedge \hat{E} \\
& (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1}) \arrow[ur, "\tilde{\lambda}_j\psi_j \wedge \Sigma(\alpha\theta_j)"'] &
\end{tikzcd}$$

It follows now that $\omega'(\lambda_j \times 1)$ has an extension over $S^{s+1} \times D^n$, for each $j \in J$, and therefore $\omega'$ has an extension $\mu : S^{s+1} \times B \to E \wedge \hat{E}$, which is clearly a duality map.

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