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TRANSFER MAPS FOR FIBRATIONS AND DUALITY

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1. Introduction

In this paper we will describe a transfer construction for (Hurewicz) fibrations which is a generalization of that for fiber bundles studied in [4, 5]. We suppose given a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

where $p: E \to B$ is a fibration having fiber F a finite complex and base B a connected finite dimensional complex. With this data we show that there is an S-map, which we call a *transfer* map,

$$\tau(f): B^+ \to E^+$$

having the property that

$$\tilde{H}^*(B^+) \xrightarrow{p^*} \tilde{H}^*(E^+) \xrightarrow{\tau(f)^*} \tilde{H}^*(B^+)$$

is multiplication by the Lefschetz number Λ of $f': F \to F$, the restriction of f to the fiber. (Although f' is not unique we allow this abuse of language since Λ is independent of the choice of f'.)

The existence of $\tau(f)$ severely restricts the projection map of the fibration. For example

$$p_*: \{X; E^+\}_q \otimes Z[\Lambda^{-1}] \to \{X; B^+\}_q \otimes Z[\Lambda^{-1}]$$

is a split epimorphism for any (pointed) finite dimensional complex X.

We will show that the boundary map $\omega : \Omega B \to F$ arising from the Puppe sequence of the fibration $p : E \to B$ is also restricted by the transfer. Precisely, we have

(1.1) THEOREM: *Assume that F is connected. Then*

$$\Lambda \omega_* : \{X; \Omega B\}_q \to \{X; F\}_q$$

is trivial for any finite dimensional complex X.

An independent method of extending the notion of transfer from fiber bundles to fibrations is given in [7]. The method which we describe here is intrinsic and has the advantage that many basic properties of the transfer are easily derived. A. Dold [9] has also independently defined the transfer, placing somewhat different restrictions on the projection p and fiber preserving map f.

The outline of the paper is as follows. In section 2 we give a homotopy characterization of the Lefschetz number of a map. Although an elementary fact it is the key point in defining the transfer. In section 3 we deal with some homotopy properties of ex-spaces and in section 4 with the duality theory of ex-spaces. This generalization of Spanier-Whitehead duality is purely formal except for the question of the existence of dual ex-spaces (theorem 4.2). In sections 5 thru 7 we define the transfer and establish its basic properties. In section 8 we prove theorem (1.1) mentioned above and describe some consequences of the theorem. In section 9 we consider smooth fiber bundles and in this case we give a more geometric description of the transfer.

2. The Lefschetz number

Suppose that F is a finite complex with base point and $f : F \to F$ is a base point preserving map. By the reduced Lefschetz number of f we mean

$$\tilde{\Lambda}_f = \sum (-1)^i \operatorname{tr} [f_* : \tilde{H}_i(F) \to \tilde{H}_i(F)].$$

Let $\mu : S^s \to F \wedge \hat{F}$ be a duality map in the sense of Spanier [15]. Then $F \wedge \hat{F}$ is $2s$-self dual via the map

$$S^{2s} \xrightarrow{\mu \wedge \mu} (F \wedge \hat{F}) \wedge (F \wedge \hat{F}) \xrightarrow{\alpha} (F \wedge \hat{F}) \wedge (F \wedge \hat{F})$$

where $\alpha(x \wedge y \wedge x' \wedge y') = x' \wedge y \wedge x \wedge y'$. Denote this composite by ν and let $\hat{\mu} : F \wedge \hat{F} \to S^s$ be dual to μ relative to ν. The following lemma provides a homotopy description of the reduced Lefschetz number of f. It is the analogue for base point preserving maps of the Lefschetz fixed point theorem given by Dold [8, theorem (4.1)].

(2.1) LEMMA: *The composite* $S^s \xrightarrow{\mu} F \wedge \hat{F} \xrightarrow{f \wedge 1} F \wedge \hat{F} \xrightarrow{\hat{\mu}} S^s$ *has degree* $\tilde{\Lambda}_f$.

PROOF: We have the following homotopy commutative diagram

$$\begin{array}{ccc} & \xrightarrow{\nu} & (F \wedge \hat{F}) \wedge (F \wedge \hat{F}) \\ S^{2s} & & \downarrow \hat{\mu} \wedge 1 \\ & \xrightarrow{1 \wedge \mu} & S^s \wedge (F \wedge \hat{F}) \end{array} \tag{2.2}$$

Let Q denote the rational numbers and choose a generator $\gamma \in \tilde{H}_s(S^s; Q)$. Let $\{u_p\}$ be a basis for $\tilde{H}_*(F; Q)$ and $\{v_p\}$ a basis for $\tilde{H}_*(\hat{F}; Q)$. Let $d(u_p)$ and $d(v_p)$ denote respectively the dimension of u_p and v_p. Write

$$\mu_*(\gamma) = \sum_{i,j} a_{ij} u_i \wedge v_j$$

and

$$\hat{\mu}_*(u_i \wedge v_j) = b_{ij}\gamma$$

Let $A = |a_{ij}|$, $B = |b_{ij}|$, and $D = |(-1)^{d(u_i)}\delta_{ij}|$ where δ_{ij} is the Kronecker symbol. By (2.2) we have

$$(1 \wedge \mu)_*(\gamma \wedge \gamma) = (\hat{\mu} \wedge 1)_* \nu_*(\gamma \wedge \gamma).$$

By expressing each side in terms of the basis elements $\gamma \wedge u_i \wedge \nu_j$ and equating coefficients, we obtain the relation $A = DAB^TA$. Since A is non-singular $AB^T = D$.

Now suppose that $f_*(u_i) = \Sigma_k c_{ik} u_k$. We have

$$\begin{aligned} \hat{\mu}_*(f \wedge 1)_* \mu_*(\gamma) &= \hat{\mu}_*(f \wedge 1)_* \left(\sum_{i,j} a_{ij} u_i \wedge v_j\right) \\ &= \hat{\mu}_* \left(\sum_{i,j,k} a_{ij} c_{ik} u_k \wedge v_j\right) \\ &= \sum_{i,j,k} a_{ij} b_{kj} c_{ik} \gamma \\ &= \sum_{i,k} (-1)^{d(u_i)} \delta_{ik} c_{ik} \gamma \\ &= \sum_{i} (-1)^{d(u_i)} c_{ii} \gamma = \tilde{\Lambda}_f \gamma \end{aligned}$$

This completes the proof.

3. Ex-spaces

Consider a trivial fibration $p : F \to *$ and a map $f : F \to F$, where F is a finite complex. In this case the transfer map we seek is to be of the form $\tau(f): S^s \to S^s \wedge F^+$, for large s, and is to have the property that

$$S^s \xrightarrow{\tau(f)} S^s \wedge F^+ \xrightarrow{1 \wedge p^+} S^s \text{ have degree } \Lambda_f.$$

To construct $\tau(f)$ let $\mu : S^s \to F^+ \wedge \hat{F}$ be a duality map and take $\tau(f)$ to be composite

$$S^s \xrightarrow{\mu} F^+ \wedge \hat{F} \xrightarrow{(1, f^+) \wedge 1} F^+ \wedge F^+ \wedge \hat{F} \xrightarrow{1 \wedge \hat{\mu}}$$

$$F^+ \wedge S^s \longrightarrow S^s \wedge F^+.$$

Then it is immediate from the preceding lemma that $p\tau(f)$ has degree Λ_f.

In order to define $\tau(f)$ in general we intend to carry out the above construction "fiberwise". This leads naturally to the consideration of ex-spaces and duality for ex-spaces. In this section we discuss some aspects of the homotopy theory of ex-spaces and in the following section we deal with duality proper.

We shall work entirely in the category of compactly generated spaces [17]. Recall that an ex-space [13] $E = (E, B, p, \Delta)$ consists of maps $p : E \to B$ and $\Delta : B \to E$ such that $p\Delta = 1$. We assume throughout that B is a CW-complex and E has the homotopy type of a CW-complex. An *ex-map* $f : E \to E'$ is one which is both fiber and cross-section preserving, i.e. $p'f = p$ and $f\Delta = \Delta'$. The set of ex-homotopy classes of ex-maps from E to E' is denoted by $[E; E']$.

An ex-space E is an *ex-fibration* if there is a lifting function

$$\Gamma : E \times_B B^I \to E^I$$

with the property that $\Gamma(\Delta(b), \sigma) = \Delta\sigma$, when σ is a path in B beginning at b. We will also need the notion of a *well based* ex-space as in [13]. E is well based if there is a vertical retraction map $E \times I \to E \times \{0\} \cup \Delta(B) \times I$.

If $p : E \to B$ is a map we have an associated ex-space $\bar{E} = (\bar{E}, B, \bar{p}, \bar{\Delta})$ where $\bar{E}$ is the disjoint union of E and B and $\bar{p}$ and $\bar{\Delta}$ are the obvious maps. Observe that $\bar{E}$ is well based, and if $p : E \to B$ is a fibration, $\bar{E}$ is an ex-fibration.

If X is a pointed space we will also use X to denote the ex-space

$(X \times B, B, p, \Delta)$ where p is projection on the second factor and Δ is the cross section determined by the base point.

The fiberwise reduced product of ex-spaces E and E' is denoted by $E \wedge_B E'$. Let $r : E \times_B E' \to E \wedge_B E'$ denote the identification map. Because of the exponential law in the category of compactly generated spaces, $r \times 1 : (E \times_B E') \times Y \to (E \wedge_B E') \times Y$ is an identification for any space Y. From this it is easy to see that $r \times_B 1 : (E \times_B E') \times_B Y \to (E \wedge_B E') \times_B Y$ is an identification for any space Y over B. With this last observation it is easy to prove the following.

(3.1) LEMMA: *If E and E' are well based so is $E \wedge_B E'$. If E and E' are ex-fibrations so is $E \wedge_B E'$.*

(3.2) THEOREM: (Comparison theorem): *Let E and E' be ex-fibrations and suppose $g : E \to E'$ is such that its restriction to the fiber over b, $g_b : F_b \to F'_b$ is an n-equivalence, $b \in B$. Let X be a well based ex-space. Then $g_\# : [X; E] \to [X; E']$ is injective if X is n-coconnected and surjective if X is $(n+1)$-coconnected.*

The proof is the same as the proof given for bundles in [1; theorem 3.3]. For other versions of the comparison theorem, see Eggar [11; Theorem 3.9] and James [14; Theorem 3.2].

(3.3) COROLLARY: *Suppose that E and E' are well based ex-fibrations and $g : E \to E'$ is such that $g_b : F_b \to F'_b$ is a homotopy equivalence, $b \in B$. Then g is an ex-homotopy equivalence.*

Given $E = (E, B, p, \Delta)$ let $\Omega_B(E)$ denote the space of loops $\sigma : I \to E$ such that $\sigma(I) \subset F_b$ for some $b \in B$, and $\sigma(0) = \sigma(1) = \Delta(b)$. We have

$$\Omega(p) : \Omega_B(E) \to B \quad \text{and} \quad \Omega(\Delta) : B \to \Omega_B(E)$$

by $\Omega(p)(\sigma) = p(\sigma(0))$ and $\Omega(\Delta)(b) = \Delta(b) = \Delta(b)^*$ – the constant loop at $\Delta(b)$. If E is an ex-fibration so is $\Omega_B(E)$ as is easily checked.

There is the suspension map

$$\sigma : [E, E'] \to [S^1 \wedge_B E; S^1 \wedge_B E'] \tag{3.4}$$

by $f \to 1 \wedge f$. By a standard argument involving the comparison theorem and the loop space $\Omega_B(S^1 \wedge E')$, we obtain the following suspension theorem (c.f. [1; Theorem 3.14] or [14; Theorem 4.3]).

(3.5) THEOREM: *Suppose that E' is an ex-fibration such that each fiber $F_{b'}$ is $(n-1)$-connected. Let E be well based. Then σ is injective if E is $(2n-1)$-coconnected and surjective if E is $2n$-coconnected.*

Let

$$\{E; E'\}_q = \overrightarrow{\mathrm{LIM}}_k[S^{k+q} \wedge E; S^k \wedge E'] \tag{3.6}$$

with the natural abelian group structure. The cone over E is $C(E) = I \wedge_B E$ with 0 the base point of I.

Suppose that A is a subcomplex of B. Let $E_A = p^{-1}(A) \cup \Delta(B)$ regarded as an ex-space of B. Then, as in [13], we have an exact sequence

$$\cdots \to \{E \cup C(E_A); E'\}_q \to \{E; E'\}_q \to \{E_A; E'\}_q \to \cdots$$

Let E/E_A be the quotient of E obtained by identifying each fiber of E_A to its base point and let $c: E \cup C(E_A) \to E/E_A$ denote the natural map. Note that if E is well based so are E_A, E/E_A and $E \cup C(E_A)$.

(3.7) LEMMA: *If E and E' are ex-fibrations and E is well based then $c^{\#}: [E/E_A; E'] \to [E \cup C(E_A); E']$ is bijective.*

A proof is given in section 10. Now if E and E' meet the requirements of the lemma we may replace $\{E \cup C(E_A); E'\}$ in the above sequence by $\{E/E_A; E'\}$ via $c^{\#}$ and so obtain an exact sequence

$$\cdots \to \{E/E_A; E'\}_q \to \{E; E'\}_q \to \{E_A; E'\}_q \to \cdots \tag{3.8}$$

4. Duality

In this section we will outline Spanier-Whitehead duality theory in the category of ex-spaces. Some aspects of this theory have been dealt with by K. Tsuchida [18]. We restrict ourselves to ex-spaces which are well based ex-fibrations having base B a finite dimensional complex and each fiber homotopy equivalent to a finite complex. Briefly, we will refer to such ex-spaces as ex-fibrations.

An ex-map $\mu: S^s \times B \to E \wedge_B \hat{E}$ is a *duality map* if for each $b \in B$ the restricted map $\mu_b: S^s \to F_b \wedge \hat{F}_b$ is a duality map in the usual sense.

Given such a duality map and ex-fibrations X and Y we have

(4.1) $$D_\mu : \{X \wedge E; Y\}_q \to \{X; Y \wedge \hat{E}\}_{q+s}$$

defined by sending $f: S^{k+q} \wedge X \wedge E \to S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \longrightarrow S^{k+q} \wedge X \wedge S^s \xrightarrow{1 \wedge \mu} S^{k+q} \wedge X \wedge E \wedge \hat{E} \xrightarrow{f \wedge 1} S^k \wedge Y \wedge \hat{E}$$

and

(4.2) $$D^\mu : \{\hat{E} \wedge X; Y\}_q \to \{X; E \wedge Y\}_{q+s}$$

by sending $f: S^{k+q} \wedge \hat{E} \wedge X \to S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \xrightarrow{1 \wedge \mu \wedge 1} S^{k+q} \wedge E \wedge \hat{E} \wedge X \longrightarrow E \wedge S^{k+q} \wedge \hat{E} \wedge X \xrightarrow{1 \wedge f} E \wedge S^k \wedge Y \longrightarrow S^k \wedge E \wedge Y.$$

(4.3) LEMMA: *D_μ and D^μ are isomorphisms.*

This follows from the corresponding fact for pointed spaces if all the ex-spaces involved are products. The proof in general is by induction over the skeleta of B using the exact sequence (3.8). The argument is standard and will be omitted.

If $\nu : S^s \times B \to X \wedge \hat{X}$ is a second duality map we have, as in the case of pointed spaces, an isomorphism

(4.4) $$D(\mu, \nu): \{E, X\}_q \to \{\hat{X}; \hat{E}\}_q$$

defined so as to make the following diagram commutative

(4.5) $$\begin{array}{ccc} \{E; X\}_q & \xrightarrow{D(\mu,\nu)} & \{\hat{X}; \hat{E}\}_q \\ & \searrow^{D_\mu} \quad \swarrow^{D^\nu} & \\ & \{S^s \times B, X \wedge \hat{E}\}_{q+s} & \end{array}$$

In particular, $f: E \to X$ is dual to $g: \hat{X} \to \hat{E}$ relative to μ and ν if and only if the diagram

$$(4.6)\qquad \begin{array}{ccc} S^s \times B & \xrightarrow{\mu} & E \wedge \hat{E} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle f \wedge 1} \\ X \wedge \hat{X} & \xrightarrow{1 \wedge g} & X \wedge \hat{E} \end{array}$$

is stably homotopy commutative.

(4.7) THEOREM: *If E is an ex-fibration there is an integer s, an ex-fibration $\hat{E}$, and a duality map $\mu : S^s \times B \to E \wedge \hat{E}$.*

A proof is given in section 11.

5. Transfer

Let $\mathcal{F}$ denote the category of fibrations $p : E \to B$ such that B is a finite dimensional complex and each fiber is homotopy equivalent to a finite complex. We consider commutative triangles

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

where $p : E \to B$ is in $\mathcal{F}$. We will construct for such a triangle and for A a subcomplex of B a *transfer* map, which is an S-map

$$(5.1)\qquad \tau(f) : B/A \to E/E_A.$$

Here $E_A = p^{-1}(A)$.

Consider the ex-space $\bar{E}$, the disjoint union of E and B. Since $\bar{E}$ is an ex-fibration in the sense of section 4, there is an ex-fibration $\hat{E}$ and a duality map

$$\mu : S^s \times B \to \bar{E} \wedge \hat{E}.$$

Analogous to the situation for pointed spaces (see section 2), $\bar{E} \wedge \hat{E}$ is canonically $2s$-self dual. Let

$$\hat{\mu} : \bar{E} \wedge \hat{E} \to S^s \times B$$

be dual to μ. We have

$$S^s \times B \xrightarrow{\mu} \bar{E} \wedge \hat{E} \xrightarrow{(1,\bar{f})\wedge 1} \bar{E} \wedge \bar{E} \wedge \hat{E} \xrightarrow{1\wedge\hat{\mu}} \bar{E} \wedge S^s \longrightarrow S^s \wedge \bar{E}$$

which takes $S^s \times A \cup s_0 \times B$ into $S^s \times E_A \cup s_0 \times B$.

Identifying these subspaces to a point, the above map yields

$$\tau(f): S^s \wedge B/A \to S^s \wedge E/E_A.$$

We will show now that the S-homotopy class of $\tau(f)$ is well defined. Firstly, if μ is replaced by a suspension, this has the effect of replacing $\tau(f)$ by its suspension. Suppose now that

(5.2)
$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & E_i \\ & {\scriptstyle p_i}\searrow \quad \swarrow{\scriptstyle p_i} & \\ & B & \end{array}$$

$i = 1, 2$, are given and $h: E_1 \to E_2$ is a fiber homotopy equivalence such that $hf_1 = f_2 h$. Let

$$\mu_i : S^s \times B \to \bar{E}_i \wedge \hat{E}_i, \qquad i = 1, 2,$$

be duality maps and let $k: \hat{E}_2 \to \hat{E}_1$ be dual to $\bar{h}$. Then k is an ex-homotopy equivalence and we have commutativity relations

$$\begin{array}{ccc} & & \bar{E}_1 \wedge \hat{E}_1 \\ & {\scriptstyle \mu_1}\nearrow & \\ S^s \times B & & \Big\downarrow{\scriptstyle \bar{h}\wedge k^{-1}} \\ & {\scriptstyle \mu_2}\searrow & \\ & & \bar{E}_2 \wedge \hat{E}_2 \end{array}, \qquad \begin{array}{ccc} \bar{E}_1 \wedge \hat{E}_1 & & \\ & \searrow{\scriptstyle \hat{\mu}_1} & \\ \Big\downarrow{\scriptstyle \bar{h}\wedge k^{-1}} & & S^s \times B \\ & \nearrow{\scriptstyle \hat{\mu}_2} & \\ \bar{E}_2 \wedge \hat{E}_2 & & \end{array}$$

where the second triangle is obtained by dualizing the first. The following diagram is then commutative.

$$\begin{array}{ccccccc} & & \bar{E}_1 \wedge \hat{E}_1 & \xrightarrow{(1,\bar{f}_1)\wedge 1} & \bar{E}_1 \wedge \bar{E}_1 \wedge \hat{E}_1 & \xrightarrow{1\wedge\hat{\mu}_1} & \bar{E}_1 \wedge S^s \\ & {\scriptstyle \mu_1}\nearrow & & & & & \\ S^s \times B & & \Big\downarrow{\scriptstyle \bar{h}\wedge k^{-1}} & & \Big\downarrow{\scriptstyle \bar{h}\wedge\bar{h}\wedge k^{-1}} & & \Big\downarrow{\scriptstyle \bar{h}\wedge 1} \\ & {\scriptstyle \mu_2}\searrow & & & & & \\ & & \bar{E}_2 \wedge \hat{E}_2 & \xrightarrow{(1,\bar{f}_2)\wedge 1} & \bar{E}_2 \wedge \bar{E}_2 \wedge \hat{E}_1 & \xrightarrow{1\wedge\hat{\mu}_2} & \bar{E}_2 \wedge S^s \end{array}$$

Therefore $\bar{h}\tau(f_1) = \tau(f_2)$. Taking h to be the identity we see that $\tau(f)$ does not depend on the choice of duality map and moreover, $\tau(f)$ depends only on the fiber homotopy class of f.

We also established the following functorial property

(5.3) *With the data* (5.2) *if $h: E_1 \to E_2$ is a fiber homotopy equivalence such that hf_1 is fiber homotopic to f_2h then $h\tau(f_1) = \tau(f_2)$.*

Now suppose we are given

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

and a map $g: X \to B$. There is the pullback diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{g}} & E \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array} \quad \text{and the induced triangle} \quad \begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{f}} & \tilde{E} \\ & {\scriptstyle \tilde{p}}\searrow \quad \swarrow{\scriptstyle \tilde{p}} & \\ & X & \end{array}$$

(5.4) *We have $\tilde{g}\tau(\tilde{f}) = \tau(f)g$.*

This is easily checked.

We may form the sum and product of the triangles in (5.2) obtaining

$$\begin{array}{ccc} E_1 + E_2 & \xrightarrow{f_1+f_2} & E_1 + E_2 \\ & {\scriptstyle p_1+p_2}\searrow \quad \swarrow{\scriptstyle p_1+p_2} & \\ & B_1 + B_2 & \end{array} \quad , \quad \begin{array}{ccc} E_1 \times E_2 & \xrightarrow{f_1\times f_2} & E_1 \times E_2 \\ & {\scriptstyle p_1\times p_2}\searrow \quad \swarrow{\scriptstyle p_1\times p_2} & \\ & B_1 \times B_2 & \end{array}$$

where $+$ denotes disjoint union.

$$(5.5) \qquad \tau(f_1 + f_2) = \tau(f_1) \vee \tau(f_2) : (B_1/A_1) \vee (B_2/A_2) \to (E_1/E_{A_1}) \vee (E_2/E_{A_2})$$

$$(5.6) \qquad \tau(f_1 \times f_2) = \tau(f_1) \wedge \tau(f_2) : (B_1/A_1) \wedge (B_2/A_2) \to (E_1/E_{A_1}) \wedge (E_2/E_{A_2}).$$

These properties follow from standard properties of duality maps as generalized to ex-spaces.

(5.7) *For the triangle*

$$\begin{array}{ccc} B & \xrightarrow{1} & B \\ & {\scriptstyle 1}\searrow \quad \swarrow{\scriptstyle 1} & \\ & B & \end{array}$$

$\tau(1): B/A \to B/A$ *is the identity map.*

6. Products

We consider now the multiplicative properties of the cohomology homomorphism induced by the transfer. We have a commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{d} & E \times E & \xrightarrow{p \times 1} & B \times E \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle 1 \times p} \\ B & & \xrightarrow{d} & & B \times B \end{array} \tag{6.1}$$

where d is the diagonal map. From (5.3), (5.4), (5.6) and (5.7) we obtain, for subcomplexes A and C of B, a cummutative diagram

$$\begin{array}{ccccc} E/E_{A \cup C} & \xrightarrow{d} & E/E_A \wedge E/E_C & \xrightarrow{p \wedge 1} & B/A \wedge E/E_C \\ \uparrow{\scriptstyle \tau(f)} & & & & \uparrow{\scriptstyle 1 \wedge \tau(f)} \\ B/A \cup C & & \xrightarrow{d} & & B/A \wedge B/C \end{array} \tag{6.2}$$

Let $\boldsymbol{M}$ be a ring spectrum and $\boldsymbol{N}$ an $\boldsymbol{M}$-module as in [19]. From the commutativity of the above diagram we obtain the formulas

(6.3) $\quad \tau(f)^*(p^*(x) \cup y) = x \cup \tau(f)^*(y),$

$$x \in \boldsymbol{M}^s(B/A),\ y \in \boldsymbol{N}^t(E/E_C).$$

(6.4) $\quad p_*(\tau(f)_*(x) \cap y) = x \cap \tau(f)^*(y),$

$$x \in \boldsymbol{N}_s(B/A \cup C),\ y \in \boldsymbol{M}^t(E/E_C).$$

Now consider the triangle

$$\begin{array}{ccc} F & \xrightarrow{f} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & \text{pt.} & \end{array}$$

In the diagram

$$\begin{array}{ccccccc} S^s & \xrightarrow{\mu} & (F^+) \wedge \hat{F} & \xrightarrow{(1,f^+)\wedge 1} & (F^+) \wedge (F^+) \wedge \hat{F} & \xrightarrow{1 \wedge \hat{\mu}} & (F^+) \wedge S^s \\ & & \downarrow{\scriptstyle f^+ \wedge 1} & & & & \downarrow{\scriptstyle p \wedge 1} \\ & & (F^+) \wedge \hat{F} & & \xrightarrow{\hat{\mu}} & & S^s \end{array}$$

the composite $(1 \wedge \hat{\mu})((1, f^+) \wedge 1)\mu$ represents $\tau(f)$. Hence, by lemma (2.1) and the commutativity of the diagram we have (identifying pt.$^+$ with S^0).

(6.5) *$p\tau(f) : S^0 \to S^0$ has degree $\tilde{\Lambda}(f^+) = \Lambda(f)$ – the Lefschetz number of f.*

We can now establish the fundamental property of the transfer. Consider

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

with $p : E \to B$ in $\mathscr{F}$. Let $f_b : F_b \to F_b$ denote the restriction of f to the fiber over $b \in B$ and let Λ denote the Lefschetz number of f_b. Let $H(\ \ ;\Gamma)$ denote singular theory with coefficients in the abelian group Γ.

(6.6) THEOREM: *If B is connected the composite*

$$\tilde{H}^*(B/A\,;\Gamma) \xrightarrow{p^*} \tilde{H}^*(E/E_A\,;\Gamma) \xrightarrow{\tau(f)^*} \tilde{H}^*(B/A\,;\Gamma)$$

is multiplication by Λ.

PROOF: Consider the inclusion

$$\begin{array}{ccc} F_b & \xrightarrow{j_b} & E \\ \Big\downarrow{\scriptstyle p_b} & & \Big\downarrow{\scriptstyle p} \\ \{b\} & \xrightarrow{i_b} & B \end{array}$$

By (6.5), for $1 \in \tilde{H}^\circ(B^+; Z)$

$$i_b^* \tau(f)^* p^*(1) = \tau(f_b)^* p_b^*(1) = \Lambda.$$

Since $i_b^* : \tilde{H}^\circ(B^+; Z) \to \tilde{H}^\circ(\{b\}^+; Z)$ is an isomorphism,

$$\tau(f)^*(1) = \tau(f)^* p^*(1) = \Lambda$$

Applying (6.3), we have for $x \in H^*(B/A; \Gamma)$,

$$\begin{aligned} \tau(f)^* p^*(x) &= \tau(f)^*(p^*(x) \cup 1) \\ &= x \cup \tau(f)^*(1) = \Lambda x. \end{aligned}$$

7. The retraction property

In this section we compare the transfer for a fibration with that of a retract up to homotopy.

Suppose that $p : E \to B$ and $q : D \to B$ are fibrations in $\mathcal{F}$ and

$$D \xrightarrow{\lambda} E \xrightarrow{\rho} D$$

are fiber preserving maps such that $\rho\lambda \simeq 1$ over the identity. Then if $f : D \to D$ is a fiber preserving map we have triangles

$$\begin{array}{ccc} D & \xrightarrow{f} & D \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array} , \qquad \begin{array}{ccc} E & \xrightarrow{\lambda f \rho} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

(7.1) THEOREM: $\lambda\tau(f) = \tau(\lambda f\rho): B/A \to E/E_A$.

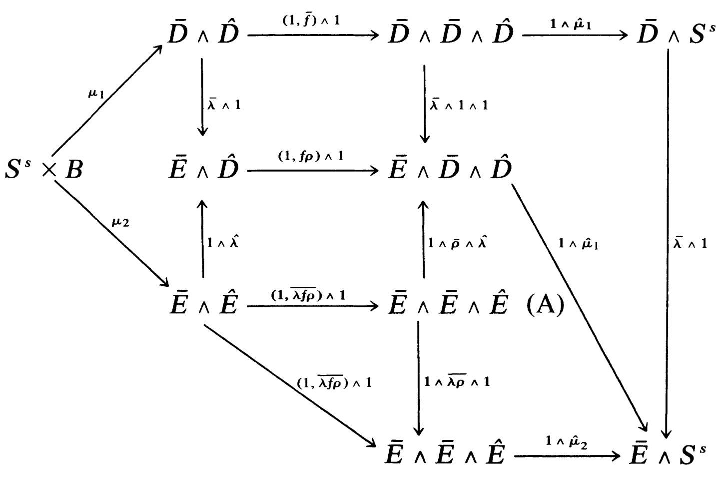
PROOF: Let

$$\mu_1: S^s \times B \to \bar{D} \wedge \hat{D}$$
$$\mu_2: S^s \times B \to \bar{E} \wedge \hat{E}$$

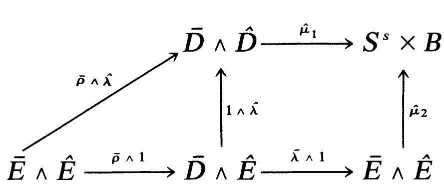
be duality maps. Let $\hat{\lambda}: \hat{E} \to \hat{D}$ be dual to $\bar{\lambda}: \bar{D} \to \bar{E}$ relative to μ_1 and μ_2, so that

(7.2)
$$\begin{array}{ccc} S^s \times B & \xrightarrow{\mu_1} & \bar{D} \wedge \hat{D} \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \bar{\lambda} \wedge 1} \\ \bar{E} \wedge \hat{E} & \xrightarrow{1 \wedge \hat{\lambda}} & \bar{E} \wedge \hat{D} \end{array}$$

is commutative. Consider the diagram

(7.3)


The commutativity of the triangle (A) follows from the commutativity of the diagram



where the square is the dual of (7.2). The remaining commutativity relations in (7.3) are easily checked. The theorem follows by comparing the two outside paths in (7.3) from $S^s \times B$ to $\bar{E} \wedge S^s$.

8. Proof of theorem (1.1)

We begin with an observation concerning the transfer map when the base space is a suspension. Suppose that X is a finite dimensional complex with base point x_0 and we are given

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{f}} & \tilde{E} \\ & {\scriptstyle\tilde{p}}\searrow \quad \swarrow{\scriptstyle\tilde{p}} & \\ & S(X) & \end{array}$$

with $\tilde{p}: \tilde{E} \to S(X)$ in $\mathscr{F}$. Let F denote the fiber over x_0 and choose a base point $e_0 \in F$. Let

$$\Delta : S(X) \to \tilde{E}/F \tag{8.1}$$

be defined by $\Delta(e^{2\pi i t} \wedge x) = \tilde{\Gamma}(e_0, \sigma(t, x))$ (1), where $\tilde{\Gamma}$ is a lifting function and $\sigma(t, x): I \to S(X)$ is the path

$$\sigma(t, x)(\lambda) = e^{2\pi i t\lambda} \wedge x.$$

Let Λ denote the Lefschetz number of $f': F \to F$, the restriction of f to F.

(8.2) Lemma: *Assume that F is connected. Then $\Lambda\Delta$ is stably homotopic to $\tau(\tilde{f}): S(X) \to \tilde{E}/F$.*

Proof: Let $C(X) = I \wedge X$ denote the reduced cone of X (with 0 the base point of I) and consider

$$\begin{array}{ccc} C(X) \times F & \xrightarrow{\psi} & \tilde{E} \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \tilde{p}} \\ C(X) & \xrightarrow{\rho} & S(X) \end{array}$$

where ρ is the natural identification and

$$\psi(t \wedge x, y) = \tilde{\Gamma}(y, \sigma(t, x))(1).$$

Then ψ is a homotopy equivalence on each fiber and the restriction of ψ to the fiber over x_0 is the identity. It follows that

$$\begin{array}{ccc} C(X) \times F & \xrightarrow{1 \times f'} & C(X) \times F \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \tilde{E} & \xrightarrow{\tilde{f}} & \tilde{E} \end{array}$$

is fiber homotopy commutative. Therefore, by (5.3), (5.4) and (5.6) $\tau(\tilde{f})\rho = \psi\tau(1 \times f') = \psi(1 \wedge \tau(f')) : C(X)/X \to \tilde{E}/F$, where $\tau(f') : S^\circ \to F^+$ is associated with the trivial triangle

$$\begin{array}{ccc} F & \xrightarrow{f'} & F \\ & \searrow \quad \swarrow & \\ & * & \end{array}$$

Let $i_0 : S^\circ \to F^+$ by $i_0(+1) = e_0$ and $i_0(-1) = +$. Since F is connected it is clear from the behaviour of the homomorphism in singular homology induced by $\tau(f')$ that $\tau(f') = \Lambda i_0$. Note that

(8.3)
$$\begin{array}{ccc} C(X)/X \wedge F^+ & \xrightarrow{\psi} & \tilde{E}/F \\ \uparrow{\scriptstyle 1 \wedge i_0 = j_0} & & \uparrow{\scriptstyle \Delta} \\ C(X)/X & \xrightarrow{\rho} & S(X) \end{array}$$

is commutative. Therefore

$$\tau(\tilde{f})\rho = \psi(1 \wedge \tau(f')) = \psi(1 \wedge \Lambda i_0) = \Lambda\Delta\rho.$$

Since $\rho : C(X)/X \to S(X)$ is a homeomorphism, $\tau(\tilde{f}) = \Lambda\Delta$ and the proof is complete.

Now let

(8.4)
$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

be given where the fiber F of $p : E \to B$ is a finite complex and B is a complex (not necessarily finite dimensional). Choose base points $b_0 \in B$ and $e_0 \in F = p^{-1}(b_0)$ and let $\omega : \Omega B \to F$ denote the boundary map arising from the fibration $p : E \to B$.

If X is a finite dimensional complex with base point and $g : X \to \Omega B$ is a base point preserving map let

$$(8.5) \qquad \begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{f}} & \tilde{E} \\ & \searrow^{\tilde{p}} \quad {}^{\tilde{p}}\swarrow & \\ & S(X) & \end{array}$$

be induced by $S(X) \xrightarrow{S(g)} S(\Omega B) \xrightarrow{\epsilon} B$ where ϵ is the adjoint of the identity map. We will show now that

$$(8.6) \qquad \begin{array}{ccc} \tilde{E}/F & \xrightarrow{k} & S(F) \\ \uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle S(\omega)} \\ S(X) & \xrightarrow{S(g)} & S(\Omega B) \end{array}$$

is commutative, where Δ is as in (8.1) and k is from the Puppe sequence of the cofibration $F \to \tilde{E}$.

We have

$$(8.7) \qquad \begin{array}{ccccc} \tilde{E}/F & \xrightarrow{k} & S(F) & & \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle S(\psi)} & \nwarrow^{S(\omega)} & \\ C(X) \times F/X \times F & \xrightarrow{k} & S(X \times F) & & S(\Omega B) \\ \uparrow{\scriptstyle j_0} & & \uparrow{\scriptstyle S(j_0)} & \swarrow_{S(g)} & \\ C(X)/X & \xrightarrow{k=\rho} & S(X) & & \end{array}$$

where k in each case is from the appropriate Puppe sequence and j_0 is the inclusion $y \to (y, e_0)$. The commutativity of the right hand triangle is by direct calculation. The commutativity of (8.6) now follows from the commutativity of (8.3) and (8.7).

We are now in a position to prove the theorem of the introduction.

(1.1) THEOREM: *Assume that F is connected. Then*

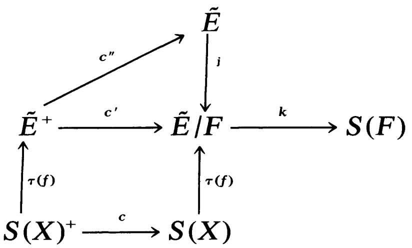
$$\Lambda\omega_*:\{X;\Omega B\}_q \to \{X;F\}_q$$

is trivial for any finite dimensional complex X.

PROOF: ΩB has the homotopy type of a CW-complex Y. If $\phi : Y \to \Omega B$ is a homotopy equivalence it is sufficient, to prove the theorem, to show that $\Lambda\omega_*(\{g\}) = 0$ if X is a finite dimensional subcomplex of Y and $g : X \to \Omega B$ is the inclusion followed by ϕ. By the commutativity of (8.6) and Lemma (8.2)

$$\Lambda\omega_*(\{g\}) = \Lambda\{k\Delta\} = \{k\tau(f)\}.$$

We have a commutative diagram



where c, c', c'', and j are quotient maps. Since $\{kj\} = 0$ we have $\{k\tau(f)c\} = c^*\{k\tau(f)\} = 0$. Since c^* is monomorphic, $\{k\tau(f)\} = 0$ and the proof is complete.

REMARKS: (1) The map ω frequently appears in other forms, hence theorem (1.1) applies for (a) coset maps $\rho : G \to G/H$, or more generally for (b) maps $\omega : M \to X$ which factor through the evaluation map $\mathcal{H}(X) \to X$ where $\mathcal{H}(X)$ is the space of homotopy equivalences of X, or for (c) fibre inclusions of principal bundles. Theorem (1.1) states that $\Lambda\omega_* = 0$ and $\Lambda\omega^* = 0$ for all homology and cohomology theories on the category of finite dimensional complexes. This is an extension of two results of [7], wherein theorem (1.1) was proved only for singular cohomology and for homotopy groups in the stable range. See also [5].

(8.8) COROLLARY: *Let* $\alpha \in \pi_i(S^{2n})$. *Then* $[\alpha, \iota_{2n}] = 0$ *implies that* $2\{\alpha\} = 0$, *where* $\{\alpha\}$ *denotes the stable homotopy element represented by*

α and $[\alpha, \iota_{2n}]$ is the Whitehead product of α with the generator of $\pi_{2n}(S^{2n})$.

PROOF: The fact that $[\alpha, \iota_{2n}] = 0$ implies there is a map $F: S^i \times S^{2n} \to S^{2n}$ such that F restricted to $* \times S^{2n}$ is the identity and F restricted to $S^i \times *$ represents α. Taking adjoints, we see that α factors through $\omega: M \to S^{2n}$ where M is the space of degree one maps on S^{2n}, and ω is the evaluation map given by evaluation at the base point. Thus (1.1) may be applied to α in view of the remark. In this case $\Lambda = \chi(S^{2n}) = 2$.

Let G be a compact connected Lie group, H a closed subgroup of G, and $\rho: G \to G/H$ the projection.

(8.9) COROLLARY: *As an S-map $\chi(G/H)\rho: G \to G/H$ is trivial, where $\chi(G/H)$ is the Euler characteristic of G/H.*

In particular, if N is the normalizer of a maximal torus in G then $\rho: G \to G/N$ is stably trivial since $\chi(G/N) = 1$ [6, 12]. On the other hand, it is interesting to note that $\rho_*: \pi_k(G) \to \pi_k(G/N)$ is an isomorphism for $k > 2$.

9. Smooth fiber bundles

In the case of a smooth fiber bundle $p: E \to B$ a more geometric description can be given for the transfer associated with a fiber preserving map. We assume that B and F are closed manifolds.

Let $\tilde{p}: E \to B \times R^s$ be a fiber preserving embedding. Its normal bundle β is inverse to the bundle α of tangents along the fiber and we have an isomorphism $\alpha \oplus \beta \simeq R^s$ associated with the embedding. Let

$$c: B^+ \wedge S^s \to E$$

denote the Pontryagin-Thom map of this trivialization.

The diagonal inclusion into the fiber square, $d: E \to E^2$, has normal bundle α so that we have

$$c': (E^2)^{\pi_1^*(\beta)} \longrightarrow E^{\alpha \oplus \beta} = E^+ \wedge S^s$$

where $\pi_1: E^2 \to E$ is projection onto the first factor.

If $f: E \to E$ is a fiber preserving map let

$$\widetilde{(1,f)}: E^{\beta} \to (E^2)^{\pi_1^*(\beta)}$$

send v_e to $(e, f(e), v_e)$.

(9.1) PROPOSITION: *$\tau(f): B^+ \to E^+$ is represented by*

$$B^+ \wedge S^s \xrightarrow{c} E^{\beta} \xrightarrow{\widetilde{(1,f)}} (E^2)^{\pi_1^*(\beta)} \xrightarrow{c'} E.$$

First observe that the S-map determined by $c'\widetilde{(1,f)}c$ is independent of the choice of embedding $\tilde{p}$ and of the tabular neighborhood maps used in constructing c and c'.

Now, for $\tilde{p}: E \to B \times R^s$, let β denote the normal bundle, let $S(\beta)$ denote the total space of the unit sphere bundle, and let $\hat{E}$ denote the quotient of $D(\beta)$ obtained by identifying each fiber of $S(\beta)$ to a point. We regard $\hat{E}$ as an ex-space of B by $\hat{p}: \hat{E} \to B$ and $\hat{\Delta}: B \to \hat{E}$ where $\hat{p}([v_e]) = p(e)$ and $\hat{\Delta}(p(e)) = [v'_e]$, where $v'_e \in S(\beta)$. Then $\hat{p}$ is the projection of a fiber bundle whose fiber over b is the Thom space $F_b^{\nu_b}$, where ν_b is the normal bundle of the embedding $F_b \to \{b\} \times R^s$.

Choose a fiber preserving tubular neighborhood $D(\beta) \subset B \times R^s$ and let $\theta: B \times S^s \to \hat{E}$ denote the associated Pontryagin-Thom map. Let

$$\mu: B \times S^s \to \bar{E} \wedge_B \hat{E} \tag{9.2}$$

be the composite $B \times S^s \xrightarrow{\theta} \hat{E} \xrightarrow{d} \bar{E} \wedge_B \hat{E}$, where $d(v_e) = e \wedge v_e$. By Atiyah's duality theorem [1] μ is a duality map.

The diagonal embedding $E \to E \times_B D(\beta)$ has normal bundle $\alpha \oplus \beta = E \times R^s$. Choosing a fiber preserving tubular neighborhood $E \times D^s \subset E \times_B D(\beta)$, we obtain

$$\theta': \bar{E} \wedge_B \hat{E} \to \bar{E} \wedge_B S^s. \tag{9.3}$$

Let $\theta'': \bar{E} \wedge_B \hat{E} \to B \times S^s$ denote θ' followed by the projection $\bar{E} \wedge_B S^s \to B \times S^s$.

(9.4) LEMMA: *θ'' is dual to μ.*

PROOF: We must show that

$$\begin{array}{ccc} B \times S^s \wedge S^s & \xrightarrow{\nu} & (\bar{E} \wedge_B \hat{E}) \wedge_B (\bar{E} \wedge_B \hat{E}) \\ & \searrow^{\mu \wedge 1} & \downarrow{\scriptstyle 1 \wedge \theta''} \\ & & (\bar{E} \wedge_B \hat{E}) \wedge S^s \end{array}$$

is homotopy commutative. Since isotopic embeddings determine homotopic duality maps, the duality map determined by

$$E \xrightarrow{(\bar{p}, 0)} B \times R^s \times R^s$$

is homotopic to that determined by

$$E \xrightarrow{(\bar{p}, \bar{p})} B \times R^s \times R^s.$$

The former duality map is $\mu \wedge 1$ whereas, using the factorisation

$$E \xrightarrow{d} E^2 \xrightarrow{\bar{p} \times \bar{p}} B \times R^s \times R^s,$$

the latter is easily seen to be homotopic to $(1 \wedge \theta'')\nu$.

Proposition (9.1) is now a consequence of the following commutative diagram.

$$\begin{array}{ccccccc} B \times S^s & \xrightarrow{\mu} & \bar{E} \wedge_B \hat{E} & \xrightarrow{(1,f) \wedge 1} & \bar{E} \wedge_B \bar{E} \wedge_B \hat{E} & \xrightarrow{1 \wedge \hat{\mu}} & \bar{E} \wedge S^s \\ \downarrow & \searrow^{\theta} & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d'} & \nearrow^{\theta'} & \downarrow \\ & & \hat{E}^\beta & \xrightarrow{g} & \bar{E} \wedge \hat{E} & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B^+ \wedge S^s & \xrightarrow{c} & E^\beta & \xrightarrow{\widetilde{(1,f)}} & (E^2)^{\pi_1^*(\beta)} & \xrightarrow{c'} & E^+ \wedge S^s. \end{array}$$

Here $d'(e' \wedge v_e) = e \wedge e' \wedge v_e$, $g(v_e) = f(e) \wedge v_e$, and $h(e' \wedge v_e) = (e, e', v_e)$. The unlabeled arrows denote the natural identification map. The commutativity of the upper right hand triangle follows from the fact that $\hat{\mu} = \theta''$.

REMARK It follows from Proposition (9.1) and the retraction property (7.1) that the two methods of constructing the transfer which are outlined in [3], do in fact lead to the same map.

10. Proof of (3.7)

Suppose that $E = (E, B, p, \Delta)$ and $E' = (E', B', p', \Delta')$ are ex-spaces. If $h: E \to E'$ and $f: B \to B'$ are such that $p'h = fp$ and $h\Delta = \Delta' f$ we will say that h is a map *over* f. If E' is an ex-fibration then we obtain from the special nature of the lifting function for E' the following covering homotopy property.

(10.1) *Given* $F: B \times I \to B'$ *and a map* $h: E \to E'$ *over* F_0 *there is* $H: E \times I \to E'$ *such that* $H_0 = h$ *and* H_t *is a map over* F_t, $0 \le t \le 1$.

Suppose that E is an ex-space of B and $A \subset B$ is a subcomplex. As before, let $c: E \cup C(E_A) \to E/E_A$ denote the natural map. Let $\lambda: B \times I \to B \times \{1\} \cup A \times I$ be a retraction map and let $F: B \times I \to B$ denote λ followed by projection onto B.

(10.2) Lemma: *Suppose that* E *is an ex-fibration. There is* $q: E/E_A \to E \cup C(E_A)$ *over* F_0 *and homotopies* $H: E/E_A \times I \to E/E_A$ *and* $K: E \cup C(E_A) \times I \to E \cup C(E_A)$ *such that*

(a) $H_0 = cq$, $H_1 = 1$, *and* H_t is over F_t, $0 \le t \le 1$.

(b) $K_0 = qc$, $K_1 = 1$, *and* K_t is over F_t, $0 \le t \le 1$.

Proof: Consider

$$\begin{array}{ccc} E \times \{1\} & \xrightarrow{i_1} & E \times 1 \\ \downarrow{\scriptstyle p \times 1} & & \downarrow{\scriptstyle p \times 1} \\ B \times I & \xrightarrow{\lambda} & B \times I. \end{array}$$

Applying (10.1) for the ex-fibration $E \times I$ there is $M: E \times I \to E \times I$ such that $M_1 = i_1$ and M_t is over λ_t, $0 \le t \le 1$. Then we actually have

$$M: E \times I \to E \times \{1\} \cup E_A \times I.$$

Let q' denote

$$E \xrightarrow{M_0} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Since $M_0(E_A) \subset E_A \times \{0\}$ we have $q'(E_A) \subset \Delta(B)$. Let $q: E/E_A \to E \cup C(E_A)$ denote the collapse of q'. Then q is a map over F_0.

To construct H, the map

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A) \xrightarrow{c} E/E_A$$

has a quotient $H: E/E_A \times I \to E/E_A$ which is the desired map.

To construct K let $N: E \times I \to E \cup C(E_A)$ denote the composite

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Define $K: E \cup C(E_A) \times I \to E \cup C(E_A)$ by $K(e, t) = N(e, t)$, if $e \in E$, and $K([e, s], t) = s * N(e, t)$, if $e \in E_A$, where $s * [e', t] = [e', st]$ for $[e', t] \in C(E_A)$. This completes the proof.

We will now prove (3.7) which asserts that

$$c^{\#}: [E/E_A; E'] \to [E \cup C(E_A); E']$$

is bijective provided E and E' are ex-fibrations.

To show that $c^{\#}$ is onto let $\theta: E \cup C(E_A) \to E'$ be an ex-map and consider

$$\begin{array}{ccc} E/E_A \times \{0\} & \xrightarrow{\theta q} & E' \\ \downarrow{\scriptstyle p \times 1} & & \downarrow{\scriptstyle p'} \\ B \times I & \xrightarrow{F} & B. \end{array}$$

There is $N: E/E_A \times I \to E'$ over F such that $N_0 = \theta q$. Let $\psi = N_1: E/E_A \to E'$. Then ψ is an ex-map and we will show that $c^{\#}([\psi]) = [\theta]$. We have homotopies

$$E \cup C(E_A) \times I \xrightarrow{c \times 1} E/E_A \times I \xrightarrow{N} E'$$

over F, and

$$E \cup C(E_A) \times I \xrightarrow{K} E \cup C(E_A) \xrightarrow{\theta} E'$$

also over F. Then

$$(\theta K^{-1}) \circ (N(c \times 1)): E \cup C(E_A) \times I \to E'$$

is a homotopy from θ to ψc over $F^{-1} \circ F$. By a standard argument involving the covering homotopy property (10.1) we see that θ is ex-homotopic to ψc.

To show that $c^{\#}$ is one-one let $\psi_0, \psi_1 : E/E_A \to E'$ be such that $\psi_0 c$ is ex-homotopic to $\psi_1 c$ by $p : E \cup C(E_A) \times I \to E'$ say. We have

$$\begin{array}{ccc} E/E_A \times (I \times \{0\} \cup \dot{I} \times I) & \xrightarrow{\;Q\;} & E' \\ \Big\downarrow{\scriptstyle p \times 1 \times 1} & & \Big\downarrow{\scriptstyle p'} \\ B \times I \times I & \xrightarrow{\;\tilde{F}\;} & B \end{array}$$

where $\tilde{F}(b, t, \lambda) = F(b, \lambda)$ and $Q(e, t, 0) = p(q(e), t)$, $Q(e, 0, \lambda) = \psi_0 H(e, \lambda)$, and $Q(e, 1, \lambda) = \psi_1 H(e, \lambda)$. There is then $R : E/E_A \times I \times I \to E'$ over $\tilde{F}$ which extends Q. Then $S : E/E_A \times I \to E'$ by $S(e, t) = R(e, t, 1)$ is an ex-homotopy from ψ_0 to ψ_1.

11. Proof of (4.7)

Let $E = (E, B, p, \Delta)$ be an ex-fibration in the sense of section 4. Let $\Sigma(E)$ denote the unreduced fiberwise suspension of E and $\Sigma(p) : \Sigma(E) \to B$ the projection. There is the "south pole" cross section $\delta : B \to \Sigma(E)$ given by $\delta(b) = [e, 0]$ where $p(e) = b$ and we let $\Sigma_0(E) = (\Sigma(E), B, \Sigma(p), \delta)$. It is easy to to check that $\Sigma_0(E)$ is an ex-fibration as in section 4. The quotient map $\pi : \Sigma_0(E) \to S^1 \wedge E$ is an ex-map and is a homotopy equivalence on each fiber. Hence by the comparison theorem (3.2), π is an ex-homotopy equivalence. Note that if $p : E \to B$ is a fibration with fiber a finite complex (not necessarily equipped with a cross section) we may still form $\Sigma_0(E)$ which is an ex-fibration.

Now let E be an ex-fibration. To construct an ex-fibration $\hat{E}$ and a duality map

$$\mu : S^s \times B \to E \wedge \hat{E}$$

we proceed by induction over the skeleta of B. Let B have dimension n and let A denote the $(n-1)$-skeleton of B. Assume there is an ex-fibration $D(E|A)$, with fiber $\hat{F}$ say, and a duality map

(11.1) $$\omega : S^s \times A \to E|A \wedge D(E|A).$$

Let B be obtained from A by adjoining cells via $\lambda_j : S^{n-1} \to A$, $j \in J$, and let $\bar{\lambda}_j : D^n \to B$ denote the characteristic map. Let $\bar{\psi}_j : F \times D^m \to \bar{\lambda}_j^*(E)$ be an ex-fiber homotopy equivalence and $\psi_j : F \times S^{n-1} \to \lambda_j^*(E)$ its restriction.

We have a duality map

(11.2) $$\eta_j : S^s \times S^{n-1} \to \lambda_j^*(E) \wedge \lambda_j^*(D(E|A))$$

induced by ω. Choose a duality map

(11.3) $$\nu : S^s \times S^{n-1} \to (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1})$$

and let $\phi_j : \lambda_j^*(D(E|A)) \to \hat{F} \times S^{n-1}$ be dual to ψ_j relative to η_j and ν. Then ϕ_j is an ex-homotopy equivalence and we have a homotopy commutative triangle.

(11.4) $$\begin{array}{ccc} & \overset{\eta_j}{\nearrow} & \lambda_j^*(E) \wedge \lambda_j^*(D(E|A)) \\ S^s \times S^{n-1} & & \uparrow \psi_j \wedge \phi_j^{-1} \\ & \underset{\nu}{\searrow} & (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \end{array}$$

Let $\theta_j : \hat{F} \times S^{n-1} \to D(E|A)$ denote the composite

$$\hat{F} \times S^{n-1} \xrightarrow{\phi_j^{-1}} \lambda_j^*(D(E|A)) \xrightarrow{\bar{\lambda}_j} D(E|A),$$

and let X' be obtained by adjoining, for each $j \in J$, $\hat{F} \times D^n$ to $D(E|A)$ via θ_j. We have an ex-space (X', B, p', Δ') where p' and Δ' are the obvious maps. Moreover, by results of Dold and Thom [10 (2.2) and (2.10)], $p' : X' \to B$ is a quasifibration. We replace p' by a fibration in the usual way obtaining a commutative square

(11.5) $$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ p' \downarrow \uparrow \Delta' & & p \downarrow \uparrow \Delta \\ B & \xrightarrow{1} & B, \end{array}$$

where $X = \{(x, \sigma) \in X' \times B^I | p'(x) = \sigma(0)\}$, $p(x, \sigma) = \sigma(1)$, $\Delta(b) = (\Delta'(b), b^*)$ where b^* denotes the constant path at b, and $\alpha(x) = (x, p'(x)^*)$.

Now $p : X \to B$ is a fibration with each fiber $p^{-1}(b)$ of the weak homotopy type of $\hat{F}$. Since X (being homotopy equivalent to X') has the homotopy type of a CW-complex, it follows from [16 Proposition

0] that $p^{-1}(b)$ has the homotopy type of a CW-complex, hence is homotopy equivalent to $\hat{F}$.

Let $\hat{E} = \Sigma_0(X)$. Then $\hat{E}$ is an ex-fibration as in section 4. We will show now that there is a duality map

$$\mu : S^{s+1} \times B \to E \wedge \hat{E}.$$

From (11.4) we obtain the following homotopy commutative diagram.

$$\begin{array}{ccccc}
S^1 \wedge \lambda_j^*(E) \wedge \lambda_j^*(D(E|A)) \to E \wedge S^1 \wedge D(E|A) & \xrightarrow{1\wedge\pi^{-1}} & E \wedge \Sigma_0(D(E|A)) \\
\uparrow 1\wedge\eta_j \qquad\qquad \uparrow \tilde{\lambda}_j\psi_j \wedge 1 \wedge \theta_j & & \downarrow \tilde{\lambda}_j\psi_j \wedge \Sigma(\theta_j) \\
S^{s+1} \times S^{n-1} & & \\
\downarrow 1\wedge v & & \\
S^1 \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1}) \to (F \times S^{n-1}) \wedge ((S^1 \wedge \hat{F}) \times S^{n-1}) & \xrightarrow{1\wedge\pi^{-1}} & (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1})
\end{array} \tag{11.6}$$

Let ω' denote the duality map

$$S^{s+1} \times A \xrightarrow{1\wedge\omega} S^1 \wedge E|A \wedge D(E|A) \to E|A \wedge S^1 \wedge D(E|A)$$
$$\xrightarrow{1\wedge\pi^{-1}} E|A \wedge \Sigma_0(D(E|A)) \xrightarrow{1\wedge\Sigma(\alpha)} E|A \wedge \hat{E}|A,$$

and let ν' denote the duality map

$$S^{s+1} \times S^{n-1} \xrightarrow{1\wedge\nu} S^1 \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1})$$
$$\longrightarrow (F \times S^{n-1}) \wedge ((S^1 \wedge \hat{F}) \times S^{n-1}) \xrightarrow{1\wedge\pi^{-1}} (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1})$$

We have from (11.6) a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & A \times S^{s+1} & & \\
 & \nearrow \lambda_j \times 1 & & \searrow \omega' & \\
S^{s+1} \times S^{n-1} & & & & E \wedge \hat{E} \\
 & \searrow \nu' & & \nearrow \tilde{\lambda}_j\psi_j \wedge \Sigma(\alpha\theta_j) & \\
 & & (F \times S^{n-1}) \wedge (\Sigma_0(\hat{F}) \times S^{n-1}) & &
\end{array}$$

It follows now that $\omega'(\lambda_j \times 1)$ has an extension over $S^{s+1} \times D^n$, for each $j \in J$, and therefore ω' has an extension $\mu : S^{s+1} \times B \to E \wedge \hat{E}$, which is clearly a duality map.

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