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INJECTIVE AND PROJECTIVE NEAR-RING MODULES

Gordon Mason

1. Introduction

Maxson [10] has given sufficient homological conditions for a unital near-ring R to be semi-simple, conditions which in the case of rings are also necessary. One of these conditions is that every R-module M be injective, another is that every M be strictly projective (see definition 2.3). Now the category of groups, viewed as the near-ring modules over the integers, is an example where there exist no non-trivial injective or strictly projective modules. In this paper we define alternate forms of "injectivity" which give necessary and sufficient conditions for all modules over a near-ring to be semi-simple. We also prove that non-trivial injectives do not exist in the category of near-ring modules over R for any R and discuss the possibilities in categories of unital near-ring modules. Finally the injectivity conditions are related to essential extensions.

With the exception of Thm. 2.6, all modules M will be unital left modules over the unital right near-ring R where R satisfies $x \cdot 0 = 0$ for all x (see e.g. [3], [6] for basic terminology). We use the term "R-subgroup of M" to mean a subgroup of $(M, +)$ closed under left R-multiplication, and "R-submodule" to mean a normal R-subgroup A satisfying $r(m + a) - rm \in A$ for all $r \in R$, $m \in M$, $a \in A$. M is *simple* if it has no proper non-zero R-submodules and *irreducible* if it has no proper non-zero R-subgroups. The R-submodules of R are called left ideals. The words injective and projective are used in the usual categorical sense.

By the phrase "in the d.g. case" we mean (see [6]) that $(R, +)$ is generated by a multiplicative semigroup S of left distributive elements and the category of (R, S)-modules consists of all unital left modules M such that $s(x + y) = sx + sy$ for all $s \in S$, x, $y \in M$. Thus the category of groups is an example of a "d.g. case" with $R = Z$ and $S = \{1\}$. Note however, that if S is taken to be $R = Z$ itself the (R, S)-modules are just the abelian groups. In this paper it may be assumed that S is a minimal distributive generating semigroup of R whereas in [2] results have been obtained assuming S is the set of all distributive elements of R.

Since references [3] and [10] are not easily available, proofs have been sketched for results taken from these sources.

2. Injectivity

We begin with the following definitions and results, some taken from [10].

DEFINITION 2.1 [10]: (a) An R-homomorphism $f : M \to N$ is *normal* if $f(M)$ is an R-submodule of N. (b) An exact sequence $M \xrightarrow{f} N \to 0$ *splits* if there exists a normal $g : N \to M$ such that $fg = 1_N$. (c) The short exact sequence (s.e.s.) $0 \to L \to M \to N \to 0$ *splits* if the sequence $M \to N \to 0$ splits. (d) The exact sequence $0 \to L \xrightarrow{f} M$ *splits* if there exists $g : M \to L$ such that $gf = 1_L$.

If L and N are R-submodules of M such that $M = L + N$ and $L \cap N = 0$ we write $M = L \oplus N$.

LEMMA 2.1 [10]: *The following are equivalent.*

(a) The s.e.s. $0 \to L \xrightarrow{h} M \xrightarrow{f} N \to 0$ *splits.*
(b) $M = h(L) \oplus g(N) \simeq L \oplus N$ *where* g *is the normal splitting map for* f.
(c) The exact sequence $0 \to L \xrightarrow{h} M$ *splits.*

PROOF: (a)⇒(b) If $g : N \to M$ is the normal splitting map then $fg = 1_N$ and $g(N)$ is an R-submodule of M. For all $m \in M$, $f(m - gf(m)) = 0$ so $m \in \operatorname{ke} f + g(N)$. It is easily checked that $\operatorname{ke} f \cap g(N) = 0$ so $M = \operatorname{ke} f \oplus g(N)$. Since the sequence is exact at M $\operatorname{ke} f = h(L)$ so $M = h(L) \oplus g(N)$. Moreover since both h and g are monomorphisms, $M \simeq L \oplus N$.
(b)⇒(c) For all $m \in M$, $m = h(x) + g(y)$, $x \in L$, $y \in N$. Define $k : M \to L$ by $k(m) = h^{-1}(g(y))$. Then $kh = 1_L$.
(c)⇒(a) Let $k : M \to L$ be such that $kh = 1_L$. As in the first part of the

proof, $M = h(L) + \text{ke}\, k$ and $h(L) \cap \text{ke}\, k = 0$. Since $L = h(L) = \text{ke}\, f$, L is an R-submodule of M so $M = h(L) \oplus \text{ke}\, k = \text{ke}\, f \oplus \text{ke}\, k$. Therefore $N = f(m) = f(\text{ke}\, k)$. If f_1 is the restriction of f to $\text{ke}\, k$, it is an isomorphism. Putting $g = (f_1)^{-1}$ we have $g(N) = \text{ke}\, k$ so g is normal and clearly $fg = 1_N$.

DEFINITION 2.2: The s.e.s. $0 \to L \to M \xrightarrow{f} N \to 0$ *almost splits* if there exists $g : N \to M$ (not necessarily normal) such that $fg = 1_N$.

DEFINITION 2.3 [10]: The module P is *strictly projective* if every s.e.s. $0 \to L \to M \to P \to 0$ splits.

PROPOSITION 2.2 [10]: *Every strictly projective module is projective in the d.g. case.*

PROOF: It is shown in [6] that in the d.g. case every module P is a factor of a free module F. Thus $F \to P \to 0$ is exact, and if P is strictly projective, the sequence splits. Hence $F \simeq P \oplus Q$ for some Q and by the usual argument ([6]) a direct summand of a free module is projective.

The question arises as to the existence of strictly projective modules. Maxson shows [10, p. 53] that a free group A is not strictly projective as follows: Let G be the free group sum of A with itself. Then there is a s.e.s. $0 \to K \to G \xrightarrow{f} A \to 0$ where K is the normal subgroup of G generated by A. Since A is free it is projective. If the splitting $g : A \to G$ were normal we would have $G = \text{Im}\, g \oplus \text{ke}\, f$. But a group cannot be both a free sum and a direct sum. Hence A is not strictly projective.

Now since free groups are the only projective groups [5], we conclude by Prop. 2.2 that there are no strictly projective groups.

Dualizing definition 2.3 we have

DEFINITION 2.4: A module I is *loosely injective* if every s.e.s. $0 \to I \to M \to N \to 0$ splits.

Finally, it is useful to distinguish two other forms of "injectivity."

DEFINITION 2.5(a) I is *n-injective* (for "normal-injective") if for every diagram

$$\begin{array}{ccccc} & & I & & \\ & & \uparrow{\scriptstyle f} & & \\ 0 & \to & A & \xrightarrow{g} & B \end{array}$$

of R-modules and R-module homomorphisms in which g is normal, there exists $h: B \to I$ such that $hg = f$.
(b) I is called *almost injective* if every s.e.s. $0 \to I \to M \to N \to 0$ almost splits.

PROPOSITION 2.3: (1) *I is injective* $\Rightarrow$ (2) *I is n-injective* $\Rightarrow$ (3) *I is loosely injective* $\Rightarrow$ (4) *I is almost injective.*

PROOF: (1)$\Rightarrow$(2) and (3)$\Rightarrow$(4) follow directly from the definitions. (2)$\Rightarrow$(3) If $0 \to I \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a s.e.s., then f is normal since $\operatorname{Im} f = \ker g$. Since I is n-injective there exists $h: M \to I$ such that $hf = 1_I$.

Examples: In the category of groups, (regarded as Z-near-ring modules)
1. There are no non-trivial injectives [5].
2. If G is n-injective then every diagram of the following type

$$\begin{tikzcd} & G & \\ 0 \arrow[r] & nZ \arrow[u, "f"] \arrow[r, "g"'] & Z \end{tikzcd}$$

can be completed. Thus for every $n \in Z$ and $x \in G$ there is an $h: Z \to G$ such that $h(n) = x$ i.e. $h(1)^n = x$. Therefore G is a divisible group.
3. G is loosely injective iff G is a direct summand of every group in which G is a normal subgroup. Equivalently G is a complete group [1, Thm. 1] i.e. G has trivial center and every automorphism is inner. Since S_3 is complete but not divisible, this class properly contains the class of n-injectives.
4. G is almost injective iff G has trivial center and $0 \to \operatorname{Inn} G \to \operatorname{Aut} G \to K \to 0$ almost splits [11, Thm. 2.7].

In general we say the module M is the *semi-direct sum* of its R-subgroups A and B, and write $M = A \dot{+} B$ if A is an R-submodule, $M = A + B$ and $A \cap B = (0)$. In this case B is called a semi-direct summand of M. Then every m can be expressed uniquely as $a + b$ for some $a \in A$, $b \in B$. Moreover the canonical projection $p: M \to B$ given by $p(a + b) = b$ is an R-homomorphism since $p(a + b + a_1 + b_1) = p(a + b + a_1 - b + b + b_1)$

$$= p(a + x + b + b_1) \quad \text{where } x = b + a_1 - b \in A$$

$$= b + b_1 = p(a + b) + p(a_1 + b_1).$$

Also $r(a+b) = r(b+a_1)$ since A is a normal subgroup and $r(b+a_1) - rb \in A$ since A is an R-submodule so by uniqueness, $r(b+a_1) = a' + rb$ for some a' and therefore $p(r(a+b)) = p(a'+rb) = rb = rp(a+b)$.

PROPOSITION 2.4: *(a) If $M = A \dot{+} B$, and M is n-injective then so is B. (b) If $M = A \oplus B$ and M is loosely injective, then so are A and B.*

PROOF: (a) A diagram

$$\begin{array}{c} B \\ {\scriptstyle f}\uparrow \\ 0 \to X \underset{g}{\to} Y \end{array} \quad \text{where } g \text{ is normal}$$

can be embedded in a diagram

$$\begin{array}{c} M \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle i} \\ B \\ {\scriptstyle f}\uparrow \\ 0 \to X \underset{g}{\to} Y \end{array}$$

where p and i are the canonical projection and injection associated with $M = A \dot{+} B$, so $pi = 1_B$. Since M is n-injective there exists $h: Y \to M$ such that $hg = if$ and ph is the required map $Y \to B$.
(b) Consider a s.e.s.

$$0 \to A \xrightarrow{f} P \xrightarrow{g} Q \to 0. \qquad \text{Define a s.e.s.}$$

$$0 \to A \oplus B \xrightarrow{\bar{f}} P \times B \to K \to 0 \quad \text{where}$$

$\bar{f}(a+b) = (f(a), b)$ and K is the quotient module to make the sequence exact (the normality of f yields the normality of $\bar{f}$). Since this splits by hypothesis, $P \times B = A \oplus B \oplus K$. Identifying P with $P^* = P \times 0$ in $P \times B$ every $p \in P$ can be expressed uniquely as $p = a + b + k$. Hence $-a + p = b + k \in P^* \cap (K \oplus B) = Y$ say. Thus $P^* = A \oplus Y$ and so A is a direct summand of P as required. The argument is symmetric for B.

COROLLARY: *A direct summand of a complete group is complete.*

LEMMA 2.5: *If the s.e.s. $0 \to L \to M \xrightarrow{f} N \to 0$ almost splits then $M \simeq L \dot{+} N$.*

PROOF: This is almost identical to the proof that (a) $\Rightarrow$ (b) in lemma 2.1. This time however the splitting map $g: N \to M$ is not necessarily normal so $N \simeq g(N)$ is in general only an R-subgroup of M and the sum $L + N$ is only semi-direct.

We now drop the requirement that modules be unital (though R may be) and show there are no injectives in this category. To do this we need to embed any group G in a simple group of arbitrarily high cardinality. The procedure of Baer [5] for embedding a group in a simple group does not suffice but there is an alternate method due to Karrass and Solitar [8]. We may assume G is infinite for otherwise it can be embedded by Cayley's Theorem in a finite symmetric group and hence in an infinite symmetric group.

If p is an infinite cardinal whose successor is denoted p^*, let $S(p)$ denote the full symmetric group on a set X of cardinality p, and $S_p = \{\sigma \in S(p) | \# \{x | \sigma(x) \neq x\} < p\}$. Then ([8]) S_p^*/S_p is a simple group of order 2^p which contains copies of all groups of order p. Iterating this procedure gives a distinct chain of simple groups (distinct because the orders are strictly increasing) containing a group of order p.

Now let R be a near-ring with underlying group $(R, +)$. If G is a group containing R, G has a natural R-module structure given by

$$\begin{aligned} rg &= r \cdot g \quad &&\text{if } g \in R \\ &= r &&\text{if } g \in G - R. \end{aligned}$$

THEOREM 2.6: *There are no injective R-modules for any near-ring R.*

PROOF: Suppose I is injective. We reach a contradiction by showing I has to contain simple R-modules of arbitrarily high cardinality. Let G be any simple group containing $(R, +)$ and endow it with its R-module structure. It remains simple as an R-module, for any R-submodule is a normal subgroup. If $x \neq 0$ is arbitrarily chosen in I the map $f: R \to I$ given by $f(r) = rx$ is an R-homomorphism well-defined on the cyclic R-subgroup R of G. Since I is injective, this can be extended to all of G. Since G is simple the extension map has no kernel so G is an R-subgroup of I. Since we can find such simple G of arbitrarily high cardinality, the proof is complete.

The theorem is false in general for categories of unital R-modules as shown in [2]. However that example is a category of (R, S)-modules with $S =$ all distributive elements of R, and the existence question remains open in the general (unital) d.g. case as well as the (unital) non d.g. case.

3. Semi-simplicity

There are several equivalent conditions for a semi-simple module.

PROPOSITION 3.1 ([3]): *The following are equivalent* (*M is called semi-simple*).

(*a*) *Every R-submodule of M is a direct summand.*

(*b*) *M is a sum of simple submodules.*

(*c*) *M is a direct sum of simple submodules.*

DEFINITION: The near-ring R is *semi-simple* if ${}_RR$ is a semi-simple module.

PROOF: The proof parallels the usual ring theoretic proof as found e.g. in [9, p. 59–61].

Similarly the following proposition is easily verified following ring theoretic proofs.

PROPOSITION 3.2: (*a*) [3, *Cor.* 4.14] *A factor module of a semi-simple module is semi-simple.*

(*b*) [3, *p.* 63] *An external direct sum of simple modules is semi-simple.*

THEOREM 3.3 (*cf.* [10, *Thm. II* 2.9]): *The following are equivalent for a near-ring R.*

(*a*) *Every M is n-injective.*

(*b*) *Every M is loosely injective.*

(*c*) *Every M is strictly projective.*

(*d*) *Every s.e.s. of R-modules splits.*

(*e*) *Every M is semi-simple.*

Each of these implies

(*f*) *R is semi-simple.*

PROOF: (a)$\Rightarrow$(b) by Prop. 2.3 (c)$\Leftrightarrow$(d)$\Rightarrow$(e)$\Rightarrow$(f) by [10, Thm. II, 2.9].

(b)$\Rightarrow$(c) If $M \xrightarrow{f} P \to 0$ is exact, then so is the s.e.s. $0 \to \ker f \to M \to P \to 0$ which, by (b), splits.

(e)$\Rightarrow$(a) Given

$$\begin{array}{ccccc} & & M & & \\ & & {\scriptstyle f}\uparrow & & \\ 0 & \to & A & \underset{g}{\to} & B \end{array} \quad \text{with } g \text{ normal}$$

then by (e) $B = A \oplus C$ for some C so we can define $h : B \to M$ by $h(a + c) = f(a)$.

It remains an open question whether R semi-simple $\Rightarrow$ every R-module is semi-simple. We do have the following partial results.

PROPOSITION 3.4: *(a) R is semi-simple iff every direct sum of cyclic R-modules is semi-simple.*
(b) If every R-subgroup of M is an R-submodule then R semi-simple $\Rightarrow$ M semi-simple.

PROOF: (a) ($\Leftarrow$) Since $1 \in R$, R is cyclic.
($\Rightarrow$) Let Ann $m = \{r \in R / rm = 0\}$. If $M = Rm$ then $M \simeq R/\text{Ann}\, m$ so by Prop. 3.2(a), R semi-simple $\Rightarrow M$ semi-simple. By Prop. 3.2(b) every direct sum of semi-simple modules is semi-simple.
(b) (Cf. [3, p. 74]) Since R is semi-simple, $R = \bigoplus_{1}^{n} J_i$ where the J_i are simple R-submodules of R. If $m \in M$ then $J_i m$ is an R-subgroup of M which by hypothesis is an R-submodule. The mapping $\psi : J_i \to J_i m$ given by $\psi(j) = jm$ is an R-module homomorphism and since J_i is simple, either $J_i m = J_i$ or $J_i m = 0$. Thus M is the direct sum of the nonzero $J_i m$ which are simple R-submodules of M.

Recently Choudhary and Tewari [4] have given several equivalent conditions for a near-ring to be strictly semi-simple. The above results yield a comparable theorem in the semi-simple case. (Thm. 3.6) For comparison we first rephrase their theorem in our terminology.

THEOREM 3.5 [4]: *If R is a near-ring with right identity the following are equivalent (R is called strictly semi-simple).*

(a) *R is a direct sum of irreducible left ideals.*
(b) *Every R-subgroup of a cyclic R-module is a semi-direct summand.*
(c) *Every R-subgroup of R is a semi-direct summand.*
(d) *If L_1, L_2 are R-subgroups of R and $0 \to L_1 \to L_2$ is exact, then it splits.*
(e) *If L_1, L_2 are R-subgroups of R with $L_1 \subset L_2$ then every R-homomorphism of L_1 into any M can be lifted to L_2.*
(f) *R has the d.c.c. on left ideals and $J(R) = 0$.*
(g) *R has the d.c.c. on R-subgroups and no nilpotent nonzero R-subgroup.*

Here $J(R) = \cap$ {left ideals which are maximal R-subgroups}.

THEOREM 3.6: *If R is a unitary near-ring, the following are equivalent:*

(a) *R is a direct sum of simple left ideals.*

(*b*) *Every R-submodule of a cyclic R-module is a direct summand.*
(*c*) *Every left ideal of R is a direct summand.*
(*d*) *For every left ideal L and R-subgroup L_1 of R the exact sequence $0 \to L \to L_1$ splits.*
(*e*) *If L is a left ideal and L_1 an R-subgroup of R such that $L \subset L_1$, then every R-homomorphism of L into any M can be lifted to L_1.*
(*f*) *R has the d.c.c. on left ideals and $N(R) = \cap$ {maximal left ideals}.* *Here $N(R) = \cap$ {maximal left ideals}.*

PROOF: (a)$\Rightarrow$(b) By Prop. 3.4(a) every cyclic R-module is semi-simple so by Prop. 3.1 every R-submodule of a cyclic R-module is a direct summand.
(b)$\Rightarrow$(c) R is a cyclic R-module in the unital case so by (b) every R-submodule (i.e. left ideal) is a direct summand.
(c)$\Rightarrow$(a) This is part of Prop. 3.1.
(c)$\Rightarrow$(d) Since L is a direct summand of R, $R = L \oplus K$ with canonical projection $p: R \to L$. The restriction of p to L_1 is the required splitting.
(d)$\Rightarrow$(e) If $f: L \to M$ is an R-homomorphism and $g: L_1 \to L$ is the splitting for $0 \to L \xrightarrow{i} L_1$ where i is the inclusion map, then fg is the required lifting since for all $x \in L$ $fg(x) = fgi(x) = f(x)$.
(e)$\Rightarrow$(c) If L is any left ideal, the identity map on L can be lifted (by (e)) to all of R. Thus $0 \to L \xrightarrow{i} R$ splits and so by lemma 2.1 L is a direct summand of R.

(a)$\Rightarrow$(f) $R = \bigoplus_1^n J_i$, J_i simple $\Rightarrow R$ has d.c.c. on left ideals. Put $S_j = \bigoplus_{i \neq j} J_i$. Then $N(R) \subseteq \cap S_j = 0$.
(f)$\Rightarrow$(a) (Cf. [3, p. 87]) Let S be the family of all finite intersections of maximal left ideals. By d.c.c. let C be a minimal element, so for every maximal left ideal B, $B \cap C = C$ i.e. $C \subseteq B$. Therefore $C \subseteq N(R) = 0$. Write $0 = C = \bigcap_1^n S_i$ where the S_i are maximal left ideals such that $C_i = \bigcap_{j \neq i} S_j \neq 0$ for all i. By maximality, $R = C_i \oplus S_i$ so by [3, p. 86] $R = \bigoplus_i^n C_i$. Since $C_i \simeq R/B_i$, the C_i are simple.

Example Let G be a finite nonabelian group and let $E(G)$ be the unital near-ring d.g. by the endomorphisms of G. Then $E(G)$ is finite so has d.c.c. on left ideals. Johnson [7] has shown that if G is the sum of its minimal fully invariant subgroups then $N(E(G)) = 0$. Thus for such G, $E(G)$ is semi-simple.

The question arises as to when a semi-simple near-ring is strictly semi-simple (the converse clearly is always true). It is known (see [3]) that a semi-simple module is strictly semi-simple if every maximal R-submodule is a maximal R-subgroup.

PROPOSITION 3.7: *A semi-simple near-ring R is strictly semi-simple if every non-zero R-subgroup contains a non-zero R-submodule (left ideal).*

PROOF: Let $L_1 \subset L_2$ be R-subgroups and $f: L_1 \to M$. By hypothesis there is a left ideal $L \subset L_1$. f can be restricted to L and by Theorem 3.6(e) extended to L_2. Thus by Theorem 3.5(e) R is strictly semi-simple.

REMARK: For rings, a module M is injective iff it satisfies Baer's condition, namely every homomorphism $f: L \to M$ for L a left ideal of R, can be extended to R. For near-rings, this is not apparently true for n-injectives, but Theorem 3.6 shows that R is semi-simple iff every M satisfies Baer's condition.

In this connection it can be noted that the Theorem of [13] to the effect that in the d.g. case a module M is injective iff it satisfies Baer's condition, is false. (This is also noted in [2]) A counter-example is provided by the category of groups which has no injectives but in which the divisible groups satisfy Baer's condition.

4. Essential extensions

DEFINITION 4.1 [10]: The R-module M is an *essential* (resp. *strictly essential*) extension of its R-submodule A if $K \cap A \neq (0)$ for all non-zero R-submodules (resp. R-subgroups) K of M.

In standard module theory the injective modules are precisely those with no proper essential extensions. The analogous results for near-rings are:

PROPOSITION 4.1: *If A is loosely injective* (*resp. almost injective*) *then A has no proper essential* (*resp. strictly essential*) *extensions.*

PROOF: Suppose M is an essential extension of A, so we have a s.e.s. $0 \to A \to M \to B \to 0$. This splits (resp. almost splits) by hypothesis so by lemma 2.1 M is a direct sum (resp. by lemma 2.5 M is a semi-direct sum) of A and B. Thus $A \cap B = (0)$ where B is an R-submodule (resp. R-subgroup) of M.

COROLLARY: *If A is n-injective, it has no essential extensions.*

As a pseudo converse to this corollary we have

PROPOSITION 4.2: *If A is an R-submodule of an n-injective module I and A has no essential extension, then A is n-injective.*

PROOF: (Cf. [9, Prop. 8, p. 91]) Let A' be an R-submodule of I maximal with respect to the property $A \cap A' = (0)$. Then $A + A'/A' \simeq A$ is an R-submodule of I/A' and we show this is an essential extension. It will follow that $I/A' = A + A'/A'$ so $I = A \oplus A'$. Since I is n-injective, so is A by Prop. 2.4(a).

Now if there is an R-submodule K with $A' \subset K \subset I$ such that $K/A' \cap A + A'/A' = (0)$, then $K \cap (A + A') \subset A'$ so $K \cap A \subset A' \cap A = (0)$ so by the maximality of A', $K = A'$. Thus I/A' is essential over A as claimed.

In view of this proposition, it would be nice to know if every A is an R-submodule of an n-injective module (for some near-rings R). In the case of groups, this means knowing whether every group can be embedded as a normal subgroup of an n-injective group. Now, the class of n-injective groups is not known, but an n-injective group is both divisible and complete. It is known that every group is a subgroup of a divisible group [12, p. 99] and of a complete group, namely a full symmetric group. This suggests trying to drop the normality requirement in the proposition. This can be done, but at the expense of changing the essentiality condition. The revised result is

PROPOSITION 4.3: *If A is an R-subgroup of an n-injective module I, and A is not essential in any R-module containing it as a proper R-subgroup, then A is n-injective.*

PROOF: Similar to Prop. 4.2. This time $A + A'/A'$ is an R-subgroup instead of an R-submodule of I/A'. Then $I = A' \dotplus A$ so Proposition 2.4(a) can still be used.

Because Proposition 2.4(b) was not proven for semi-direct sums, but only direct sums, it is not possible to give such a rewording of the next proposition.

PROPOSITION 4.4: *If A is an R-submodule of a loosely injective module I and A has no proper essential extension then A is loosely injective.*

PROOF: As for Proposition 4.2, but invoke Proposition 2.4(b) in place of Proposition 2.4(a).

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