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HOMOTOPY CLASSIFICATION OF REGULAR SECTIONS

Andrew du Plessis

0. Introduction

Let $E(N) \xrightarrow{p} N$ be a smooth bundle over an n-manifold N; let

$$\begin{tikzcd} & E^r(N) \arrow[d, "p^r"] \\ E(N) \arrow[ur, "p^{r,0}"] \arrow[r, "p"] & N \end{tikzcd}$$

be the bundle of r-jets of local sections of $E(N)$.

If σ is a smooth section $N \to E(N)$, then its r-jet $j^r\sigma$ is a section $N \to E^r(N)$. Conversely, if τ is a section $N \to E^r(N)$, then $p^{r,0}\tau$ is a section $N \to E(N)$.

Clearly $p^{r,0}(j^r\sigma) = \sigma$; however, $j^r(p^{r,0}\tau) \neq \tau$ unless τ is the r-jet of some section $N \to E(N)$. In this case we say τ is *integrable*.

Let Ω be a subset of $E^r(N)$; in this context we shall call Ω a *regularity condition.*

A section $\tau: N \to E^r(N)$ is *Ω-regular* if $\tau(N) \subset \Omega$; and a section $N \to E(N)$ is Ω-regular if its r-jet is.

Let $\Gamma_\Omega E(N), \Gamma_\Omega E^r(N)$ be the spaces of Ω-regular sections of $E(N)$ and $E^r(N)$, equipped respectively with the C^r and compact-open topologies.

We shall say that a regularity condition Ω is *integrable* over the manifold N if

$$j^r: \Gamma_\Omega E(N) \to \Gamma_\Omega E^r(N)$$

is a weak homotopy equivalence (w.h.e.). (Note that if Ω is integrable over a compact manifold N, this weak homotopy equivalence is a genuine homotopy equivalence, for in this case $\Gamma_\Omega E(N)$, $\Gamma_\Omega E^r(N)$ are metrizable and a theorem of Palais [15] may be applied.)

The terminology expresses the irrelevance (up to homotopy) of integrability conditions on Ω-regular sections of $E^r(N)$ in these circumstances.

Hence, if Ω is integrable over N, we have a homotopy classification of Ω-regular sections $N \to E(N)$. This may not be very illuminating; however, in the cases we shall consider, Ω will be a sub-bundle of $E^r(N)$, so that the classification will be in terms of cross-sections of Ω. The use we shall have for this notion of integrability is precisely this reduction of a geometric problem to a well-understood algebraic one.

The regularity conditions we consider will be the *natural* ones; that is, conditions arising naturally from the differentiable structure of the n-manifold N. In the formulation of Wall [26], we shall require that E is a C^∞-bundle defined over smooth n-manifolds, and that Ω is a sub-C^∞-bundle of the C^∞-bundle E^r.

More explicitly, we require that for any n-manifold N, the bundle $E(N)$ has the following properties:

1. If $U \subset N$ is open, then $E(U) = E(N)|U$.
2. For any diffeomorphism $f: U \to V$ of open sets in N, there is a diffeomorphism $E(f): E(U) \to E(V)$ covering f and depending continuously on f s.t.
   a) $E(g) \circ E(f) = E(g \circ f)$, where $g: V \to W$ is another local diffeomorphism of N.
   b) $E(1_U) = 1_{E(U)}$, where 1_U is the identity on U, $1_{E(U)}$ the identity on $E(U)$.

This gives an action on $E(N)$ by local diffeomorphisms of N; these also act on sections of $E(N)$ by

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle f^E(\sigma)}\downarrow & & \downarrow{\scriptstyle \sigma} \\ E(U) & \xrightarrow{E(f)} & E(V) \end{array}$$

so that if $f: U \to V$ is a local diffeomorphism of N, σ a local section $V \to E(V)$, then we have a local section $f^E(\sigma) = E(f)^{-1} \circ \sigma \circ f: U \to E(U)$.

All this induces a similar local diffeomorphism action on $E^r(N)$; we suppose that $\Omega(N)$ is a sub-bundle invariant under this action.

We also suppose that Ω is open in E^r, so that Ω-regularity is a *stable* condition on sections.

Henceforth, all the regularity conditions we consider will be both natural and stable.

The point of all this is the following:

THEOREM A

A (stable, natural) regularity condition is integrable over any open manifold (i.e. a manifold N s.t. $N-\partial N$ has no compact component).

This very powerful general result was proved by Gromov [7] in 1969. To provide some insight we consider the following special case:

Let $E(N)$ be the trivial bundle $N \times P$; then $E^1(N)$ may be identified with Hom (TN, TP). Take Ω to be the set of homomorphisms with rank n everywhere, so that Ω-regular sections are immersions $N \to P$. Suppose N is an open manifold. Then Theorem A implies that a smooth map $f: N \to P$ is homotopic to an immersion $\Leftrightarrow$ there is a monomorphism $TN \to f^*TP$.

Now suppose that $N = S^1$, a closed 1-manifold; let P be the real line $\mathbb{R}$. Since TS^1 is trivial, there is a monomorphism $TS^1 \to f^*T\mathbb{R} \cong S^1 \times \mathbb{R}$ for any map $f: S^1 \to \mathbb{R}^1$. But S^1 cannot be immersed in $\mathbb{R}$.

Thus regularity conditions are not in general integrable over closed manifolds; nevertheless our purpose in this paper is to show that a large class of regularity conditions are.

We will prove

THEOREM B

If $\Omega \subset E^r$ $(r \geqq 1)$ is an extensible regularity condition it is integrable over any closed manifold (and, of course, over any open manifold, by Theorem A).

$\Omega \subset E^r$ $(r \geqq 1)$ is an *extensible* regularity condition if there exist

i) a C^∞-bundle E' over $(n+1)$-manifolds

ii) a C^∞-bundle map over n-manifolds $\pi: i^*E' \to E$

$$(\text{where } i_N: N = N \times 0 \subset N \times \mathbb{R})$$

iii) a (natural, stable) regularity condition $\Omega' \subset E'^r$ s.t.

$$\pi^r(i^*\Omega') = \Omega$$

(where $\pi^r: i^*(E'^r) \to E^r$ is the natural map induced by π).

We call $\Omega' \subset E'^r$ an *extension* for $\Omega \subset E^r$. Motivation for this definition will be given in (1.5). However, some initial explanatory remarks, and some examples, are in order.

Firstly, none of this machinery is redundant in general: as an example, we consider

Singularities of 1*-forms*

A 1-form ω on a smooth n-manifold N has an exterior derivative $d\omega$, whose *rank* at $x \in N$ we define to be the largest integer k s.t. $(d\omega)^k \neq 0$ at x.

The requirement that the rank of a 1-form be nowhere less than some integer c gives a regularity condition $\Omega^c \subset E^1$ (where $E = T^*$ is the cotangent bundle for n-manifolds). This regularity condition is extensible for $2c < n$; the extension is given by $E' = T^*$ (still the cotangent bundle, but this time for $(n+1)$-manifolds), $\Omega' = \Omega^{c+1} \subset (E')^1$, where

$$\pi_N: i^*T^*(N \times \mathbb{R}) \to T^*(N)$$

is the natural projection

$$i^*T^*(N \times \mathbb{R}) = T^*(N) \oplus \mathbb{R} \to T^*(N).$$

This result is best possible both for extensibility and for the integrability result implied; for if $2c = n$, no Ω^c-regular form can exist on a closed n-manifold N. (For suppose ω were such a form. Then $(d\omega)^c = d(\omega \wedge (d\omega)^{c-1})$ is an n-form without zeros on N. Hence it defines an orientation on N; and $\int_N (d\omega)^c \neq 0$. But $\int_N d(\omega \wedge (d\omega)^{c-1}) = 0$ by Stokes' theorem. This contradiction shows that no such ω can exist.) Thus $\Gamma_{\Omega^c}(T^*(N)) = \emptyset$. However, it is easy to see that $\Gamma_{\Omega^c}((T^*)^1)(N)$ has the homotopy type of the space of almost-complex structures on N, so need not be empty (e.g. S^6 has an almost-complex structure).

We now turn our attention to the case where $E(N) = N \times P$ is a trivial C^∞-bundle, so that sections of $E(N)$ may be identified with maps $N \to P$. In this case the situation may be rather simpler.

Let $\Omega(N) \subset E^r(N) = J^r(N, P)$ be a natural, stable regularity condition on maps $N \to P$. We attempt to construct an extension as follows: We define $\tilde{E}(M)$ for $(n+1)$-manifolds M to be the trivial C^∞-bundle $M \times P$, and take $\pi: i^*\tilde{E} \to E$ to be the identity on fibres. We can define a natural stable regularity condition $\tilde{\Omega}(M) \subset J^r(M, P)$ as follows: let F be a germ of map $\mathbb{R}^{n+1} \to P$ near $x \in \mathbb{R}^{n+1}$. Then $j^r_x F \in \tilde{\Omega} \Leftrightarrow j^r_y Fk \in \Omega$ for any germ k of immersion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ near $y \in \mathbb{R}^n$ s.t. $k(y) = x$. It is easy to see that $\pi^r(i^*\tilde{\Omega}) \subset \Omega$ (and indeed that $\tilde{\Omega}$ is the largest (natural, stable) regularity condition in $\tilde{E}^r$ with this property); so for extensibility we need only check in any particular case to see if $\pi^r(i^*\tilde{\Omega}) \supset \Omega$. Of course, this need not be the only way in which an extension for Ω might be constructed; it is, however, sufficient for the examples we give below.

Maps transverse to a field of planes

Let P be a smooth manifold, and let η be a sub-bundle of TP. Let $\gamma = TP/\eta$, with natural projection $\pi\colon TP \to \gamma$. The regularity condition Ω for maps $f\colon N \to P$ transverse to η (so that $\pi \circ Tf\colon TN \to \gamma$ is of maximal rank everywhere) is extensible, hence integrable, if $\dim N < \dim P - \dim \eta$. (This result, and its generalisation to maps of foliations (see Note 3) was first obtained by Gromov [7]; when η is the zero bundle, it gives the classical result of Smale-Hirsch [12] on immersions of closed manifolds).

In this case $\tilde{\Omega}(M)$ may be identified as the bundle of 1-jets of germs of maps $M \to P$ transverse to η.

r^{th}-order non-degenerate immersions

Let P be a Riemannian manifold. Pohl [19] has shown that a symmetric linear connection on P naturally induces a splitting of the exact sequence

$$0 \to T_{r-1}P \to T_rP \to 0^rTP \to 0$$

(a point of T_rP_x ($x \in P$) is a differential operator of order $\leqq r$ acting on germs at x of functions on P) and hence a natural map $\pi_r\colon T_rP \to TP$.

In the case where $P = \mathbb{R}^p$ with the usual connection, and if $f\colon N \to \mathbb{R}^p$ is an immersion, $\operatorname{Im} \pi_r T_r f_x$ is exactly the osculating space of order r to f at $f(x)$ i.e. the image of the linear map

$$df_x \oplus d^2f_x \oplus \ldots \oplus d^rf_x\colon TN \oplus 0^2TN \oplus \ldots \oplus 0^rTN \to \mathbb{R}^p.$$

Thus $\operatorname{Im} \pi_r T_r f$, for any immersion $f\colon N \to P$, gives a natural generalisation of the notion of osculating space. The regularity condition we consider here is that this osculating space is of maximal dimension; and so we say an immersion $f\colon N \to P$ is *r^{th}-order non-degenerate* if the composite $\pi_r T_r f$ is injective on fibres. This defines a regularity condition $\Omega^r \subset J^r(N, P)$, which is extensible if $\dim P$ is sufficiently large; it is an interesting algebraic problem to determine exactly how large it need be.

We define $v(n, r) = n + {}_{n+1}c_2 + \ldots + {}_{n+1}c_r$ (this is fibre $\dim T_rN$ for an n-manifold N). Then $\Omega^r \subset J^r(N, P)$ is extensible if

$$v(\dim N + 1, r) \leqq \dim P,$$

though this is by no means the complete answer. For example, $\Omega^2 \subset J^2(N, P)$ is extensible if

$$\dim P \geqq v(\dim N + 1, 2) - \sigma(\dim N + 1).$$

Here, the number $\sigma(n)$, for any integer n, is defined as follows: write n in the form $(2s+1)2^{4a+b}$ where $-1 \leqq a$ and $1 \leqq b \leqq 4$. Then

$$\sigma(n) = 1+2^{b-1}+8a.$$

($\sigma(n)$ is in fact the maximum number of symmetric orthogonal $n \times n$ matrices all of whose non-zero linear combinations are non-singular).

For example, the following are sufficient for extensibility of Ω^2:

$$\dim N = 1 \quad \text{and} \quad \dim P \geqq 3$$

(The classification up to regular homotopy of curves of everywhere non-vanishing curvature in Riemannian manifolds of dimension $\geqq 3$ implied by this was first obtained by Feldman [4], [5].)

$$\dim N = 2 \quad \text{and} \quad \dim P \geqq 8$$

$$\dim N = 3 \quad \text{and} \quad \dim P \geqq 11$$

The values given for $\dim N = 1$ or $\dim N$ even are best possible for extensibility of Ω^2 via $\tilde{\Omega}^2$. However, they are by no means best possible for integrability. Gromov and Eliashberg [9] have proved, by completely different methods, that Ω^r is integrable whenever $\nu(\dim N, r) < \dim P$.

Immersions with non-vanishing mean curvature

Let P be a Riemannian manifold with given metric and let $f: N \to P$ be an immersion with normal bundle Q. Then the second fundamental form of f at $x \in N$ is the symmetric bilinear map $TN_x \circ TN_x \to Q_x$ which is the projection onto Q_x of the second derivative of f at x w.r.t. any local metric coordinates.

The mean curvature vector of f at x is the trace of this map, calculated on a basis in TN_x orthonormal w.r.t. the inner-product induced via f by the metric on P.

The regularity condition for immersions $N \to P$ with nowhere-vanishing mean curvature vector is extensible if $\dim N+1 < \dim P$.

(The classification of such immersions thus implied was first obtained by Feldman in [6] by rather different means.)

Non-singular maps

We consider smooth maps $f: N^n \to P^p$ without certain types of singularity.

Let $I = (i_1, \ldots, i_r)$ be a non-increasing sequence of r positive integers. We define $\Omega^I \subset J^r(N, P)$ to be the union of Boardman varieties

$\bigcup\{\Sigma^K | K \leqq I$ in lexicographic order$\}$, so that a map $f\colon N \to P$ is Ω^I-regular $\Leftrightarrow \Sigma^{k_1 \cdots k_r}(f) = \emptyset$ for any sequence $(k_1, \ldots, k_r)$ 'greater' than $I = (i_1, \ldots, i_r)$. Ω^I is extensible, hence integrable, if $i_r > \dim N - \dim P - d^I$; here we define $d^I = \sum_{s=1}^{r-1} \alpha_s$, where

$$\alpha_s = \begin{cases} 1 & \text{if } i_s - i_{s+1} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 1$, this result was first obtained by Feit [3].

For $r = 2$, $n = p$, the result is best possible; maps $f\colon N \to P$ s.t. $\bar{\Sigma}^2(f) \cup \Sigma^{11}(f) = \emptyset$ were considered by Eliashberg in [2], where an example is given which shows that this regularity condition $(\Omega^{1,0})$ is *not* integrable.

It is also of interest to consider maps $f\colon N \to P$ s.t. $(j^r f)^{-1}\bar{\Sigma}^I$ is empty; in the case $r = 2$, this regularity condition is always extensible if $\dim N < \dim P$.

(Proofs of the assertions made here will be given in another paper.)

The main body of methods and ideas used in this paper results from the accumulation of expertise of the many mathematicians who have worked in this field. The author gratefully acknowledges the help he has received from their papers; the extent of his debt is made clear in the following brief account of previous work.

The first theorem on integrability was proved by Whitney [27], who considered immersions of S^1 in $\mathbb{R}^2$. His results were extended to immersions of S^1 in an arbitrary Riemannian manifold by Smale [22], who went on to find ([23], [24]) similar theorems for immersions of first S^2 in $\mathbb{R}^p$ $(p \geqq 3)$ and then arbitrary S^n in $\mathbb{R}^p$ $(p > n)$. Using the ideas developed in these theorems, Hirsch [12] obtained the homotopy classification of immersions implied by Theorem B (and (implicitly) that implied by Theorem A).

The methodology of these results was at first unclear, but several expositions (notably those of Thom [25], Hirsch-Palais (unpublished seminar), Haefliger-Poenaru [11] and Poenaru [20]) clarified the situation, and led Phillips to some other special cases of Theorem A (submersions were treated in [16] and maps transverse to a foliation in [17]).

The situation in regard to integrability over closed manifolds was less clear, for the elegant final form of the classification theorem for immersions made use of a special property – the existence of normal bundles (see Phillips [16]). Perhaps as a result, when Gromov [7] proved Theorem A, he found the extension to closed manifolds only for a very restricted class of regularity conditions (which he called 'injective').

Meanwhile, in [3] Feit considered the case of the so-called 'k-mersions' (smooth maps with differential of rank $\geqq k$ everywhere), which do not have 'normal' bundles in any global sense (nor indeed is their regularity condition injective in the sense of Gromov). She was therefore forced to argue locally (with locally-defined 'k-normal' bundles) to obtain Theorem B for this regularity condition. Although we shall use it rather differently, it is this 'localization' which is the key to Theorem B.

The author records here his deep appreciation and gratitude for the inspiration given him by Professor C. T. C. Wall of Liverpool University, without whose valuable advice and patient encouragement this paper would never have been written.

Notes

1. Theorems A and B are false in general if the requirement of stability is relaxed. For example, Phillips [18] has shown that the regularity condition for maps of constant rank $k \neq \dim N$, $\dim P$ is integrable only for N sufficiently highly co-connected.

2. Gromov has announced (in [8]; 3.A.2) the following, as yet unpublished theorem:

let Ω be a (stable, natural) regularity condition on sections of the C^∞-bundle E over n-manifolds. Let $\psi: N_0 \to N$ be a generic smooth map of manifolds, where $\dim N_0 < \dim N = n$. Then this induces a map $\psi^r: \psi^* E^r \to (\psi^* E)^r$.

Gromov's assertion is that $\psi^r \Omega$ is an integrable regularity condition over N_0. In general, $\psi^r \Omega$ is neither natural nor stable; however, when ψ is an immersion it is both, and then this result is a special case of Theorem B.

3. Let $\mathbb{R}^n \times \mathbb{R}^k$ be endowed with the natural n-dimensional foliation whose leaves are $\mathbb{R}^n \times x$, for each $x \in \mathbb{R}^k$.

Let $\mathscr{D}^n_k$ be the pseudogroup of local homeomorphisms of $\mathbb{R}^n \times \mathbb{R}^k$ that take n-dimensional leaves to n-dimensional leaves and restrict on each leaf to a diffeomorphism depending continuously on the leaf. Then a manifold with a $\mathscr{D}^n_k$-structure (in the sense of Wall [26]) is an (n, k)-foliation i.e. an $(n+k)$-manifold with a smooth n-dimensional foliation (of codimension k).

We also have the concept of a $\mathscr{D}^n_k$-bundle; in particular, if E is any C^∞-bundle over n-manifolds, from it we may construct a $\mathscr{D}^n_k$-bundle $\bar{E}$ over (n, k)-foliations s.t. $\bar{E}(M)|L = E(L)$ for each leaf L of the foliation on M.

We define $\bar{j}^r: \Gamma\bar{E}(M) \to \Gamma\bar{E}^r(M)$ by $\bar{j}^r(f)|L = j^r(f|L)$.

Let Ω be a sub $-C^\infty$-bundle of E^r.

Let $\Gamma_{\bar{\Omega}}(\bar{E}(M)) = \{f \in \Gamma\bar{E}(M) | \bar{j}^r(f) \in \bar{\Omega}(M)\}$.

We shall say that $\bar{\Omega}$ is integrable over the (n, k)-foliation M if $\bar{j}^r: \Gamma_{\bar{\Omega}}(\bar{E}(M)) \to \Gamma\bar{\Omega}(M)$ is w.h.e.

From Theorem B and an abstract proposition of Gromov ([7]; 3.4.1) follows for a stable, natural regularity condition Ω:

THEOREM C

If Ω is extensible, $\bar{\Omega}$ is integrable over any (n, k)-foliation.

1. Plan of proof

In this chapter we study the strategy of proof for Theorems A and B together; for the proof of A both forms part of and motivates the extra conditions required for the proof of B.

Our exposition for Theorem A follows closely that of Haefliger [10]; another, equally lucid, account is that of Poenaru [21].

(1.1) On a smooth n-manifold N there is a proper Morse function $f: N \to [0, \infty)$ whose critical points $a_1, a_2, \dots$ may be ordered so that their critical values $c_i = f(a_i)$ are increasing.

For each i, there is a neighbourhood U_i of a_i and local co-ordinates $(x_1, \dots, x_n)$ in U_i so that f has the form

$$f = c_i - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

in U_i (here k is the *index* of a_i).

For each i, let $N_i = f^{-1}[0, c_i + \varepsilon_i]$, where $0 < \varepsilon_i < c_{i+1} - c_i$ and ε_i is small.

Let

$$N'_{i-1} = N_i - \{x = (x_1, \dots, x_n) \in U_i | x_1^2 + \dots + x_k^2 < \tfrac{1}{2}\delta_i\},$$

where δ_i is a small positive number. Then N'_{i-1} is a manifold whose boundary has an edge diffeomorphic to $S^{k-1} \times S^{n-k-1}$; we note that

$$N'_{i-1} = M_{i-1} \cup_{\partial N_{i-1}} \{(x, t) \in \partial N_{i-1} \times [0, 1] | t \leqq g(x)\}$$

where $g: \partial N_{i-1} \to (0, 1]$ is a continuous map. We say that N'_{i-1} is N_{i-1} plus a *collarlike* neighbourhood.

$N_i = N'_{i-1} \cup A_k$, where A_k is diffeomorphic to $D^k \times D^{n-k}$, and $N'_{i-1} \cap A_k$ is diffeomorphic to a collar neighbourhood B_k of $\partial D^k \times D^{n-k}$; we say that

N_i is N'_{i-1} plus a *k-handle*. (For proofs of all these assertions, see Milnor [14].)

In this way N may be represented as the union of an increasing sequence

$$N_1 \subset \ldots \subset N_{i-1} \subset N'_{i-1} \subset N_i \subset \ldots$$

of compact manifolds with boundary.

(1.2) Let E be a C^∞-bundle over n-manifolds, and Ω a (stable, natural) regularity condition on sections of E.

For brevity, we shall write

$$\begin{cases} \Gamma_\Omega(N) & \text{for } \Gamma_\Omega E(N) \\ \Gamma(N) & \text{for } \Gamma_\Omega E^r(N) = \Gamma\Omega(N). \end{cases}$$

In our proof we shall require the following propositions:

PROPOSITION 1: *Ω is integrable over the n-disc D^n.*

PROPOSITION 2: *Let N' be N plus a collarlike neighbourhood. Then the restriction maps*

$$\Gamma_\Omega(N') \to \Gamma_\Omega(N)$$
$$\Gamma(N') \to \Gamma(N)$$

are w.h.e. and Serre fibrations.

PROPOSITION 3: *For any (manifold) pair (A, B), the restriction map*

$$\Gamma(A) \to \Gamma(B)$$

is a Serre fibration.

PROPOSITION 4: *Let $A_k = 2D^k \times D^{n-k}$, $B_k = S^{k-1} \times [1, 2] \times D^{n-k}$, where $k < n$ (where we make the natural identification*

$$S^{k-1} \times [\alpha, \beta] = \beta D^k - \alpha \mathring{D}^k \; (\beta > \alpha > 0)).$$

Then the restriction map

$$\rho_\Omega : \Gamma_\Omega(A_k) \to \Gamma_\Omega(B_k)$$

is a Serre fibration.

PROPOSITION 5: *Let* $A_n = 2D^n$, $B_n = S^{n-1} \times [1, 2]$ *and suppose* Ω *is extensible. Then the restriction map*

$$\rho_\Omega : \Gamma_\Omega(A_n) \to \Gamma_\Omega(B_n)$$

is a Serre fibration.

These results are proved in § 2; from them, the theorems are proved by induction on handle-index, using the following

LEMMA: *Suppose we have a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\bar{g}} & E' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

where p, p' *are Serre fibrations and* g *is w.h.e. Then* $\bar{g}$ *is w.h.e.* $\Leftrightarrow$ *its restriction to each fibre of* E *is w.h.e.*

PROOF: Follows from the homotopy exact sequences of the fibrations and the 5-lemma.

(1.3) First notice that when N is an open manifold, there is a proper Morse function on N with all critical points of index $< n$. Hence by the analysis of (1.1), N may be represented as a union of handles of index $< n$ (for a proof of this, see Phillips [16]).

Thus to prove theorem A (at least for compact manifolds) we need only show that if a compact manifold N is a finite union of handles of index $< n$, Ω is integrable over N. To prove theorem B we must make an extra induction step.

The argument in the two cases is similar, as follows:

Assume inductively that Ω is integrable over compact manifolds N which are unions of handles of index $< k$. (Proposition 1 starts this induction). To make the induction step, we need to show that if Ω is integrable over N, it is integrable over $\bar{N} = N' \cup A_k$, where $N' \cap A_k = B_k$, and N' is N plus a collarlike neighbourhood.

We have the commutative diagram

$$\begin{array}{ccc} \Gamma_\Omega(A_k) & \xrightarrow{jr} & \Gamma(A_k) \\ \Big\downarrow{\scriptstyle \rho_\Omega} & & \Big\downarrow{\scriptstyle \rho} \\ \Gamma_\Omega(B_k) & \xrightarrow{jr} & \Gamma(B_k). \end{array}$$

ρ is a Serre fibration by proposition 3; ρ_Ω is a Serre fibration by proposition 4 if $k < n$, otherwise by proposition 5, when we must assume that Ω is extensible. By proposition1, $j^r: \Gamma_\Omega(A_k) \to \Gamma(A_k)$ is w.h.e. and so also is $j^r: \Gamma_\Omega(B) \to \Gamma(B)$, by the induction hypothesis (B_k is 0-handle plus $(k-1)$-handle).

So the lemma implies j^r is w.h.e. on each fibre.

Consider now the commutative diagram

$$\begin{tikzcd} \Gamma_\Omega(\bar{N}) \arrow[r] \arrow[d, "\rho_\Omega"] & \Gamma(\bar{N}) \arrow[d, "\rho"] \\ \Gamma_\Omega(N') \arrow[r] & \Gamma(N'). \end{tikzcd} \tag{2}$$

Restriction to A and B maps diagram (2) to diagram (1), and the vertical maps of (2) are pull-backs of the fibrations ρ_Ω, ρ in (1). Hence they are also Serre fibrations, and of course j^r restricted to each fibre is w.h.e. So by the induction hypothesis and proposition 2, $j^r: \Gamma_\Omega(N') \to \Gamma(N')$ is w.h.e. Applying the lemma again $j^r: \Gamma_\Omega(\bar{N}) \to \Gamma(\bar{N})$ is w.h.e.; that is, Ω is integrable over $\bar{N}$.

(1.4) This proves Theorem B; to obtain Theorem A for (non-compact) manifolds which are not a union of finitely many handles, we note that

$$\Gamma_\Omega(N) = \varprojlim \Gamma_\Omega(N_i), \qquad \Gamma(N) = \varprojlim \Gamma(N_i)$$

and apply the following

LEMMA: (Phillips [16])

Suppose we have a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & A_i \arrow[r] \arrow[d, "j_i"] & \ldots \arrow[r] & A_1 \arrow[d, "j_1"] \\ \ldots \arrow[r] & B_i \arrow[r] & \ldots \arrow[r] & B_1 \end{tikzcd}$$

where all the horizontal maps are Serre fibrations, and the j_i are w.h.e. Then $\varprojlim j_i: \varprojlim A_i \to \varprojlim B_i$ is w.h.e.

(1.5) We now describe the idea behind our condition of extensibility. As the argument above shows, all we require of it is that it ensures that

$$\rho_\Omega: \Gamma_\Omega(A_n) \to \Gamma_\Omega(B_n)$$

is a Serre fibration; that is, that ρ_Ω has PCHP (see (2.1)).

Consider the following situation: $\bar{E}'$ is a C^∞ bundle over $(n+1)$-manifolds, and Ω' an r^{th} order regularity condition on its sections. Suppose that there are natural maps

$$\Gamma_{\Omega'}(E'(N\times[-1,1])) \underset{i_N}{\overset{p_N}{\rightleftarrows}} \Gamma_\Omega(E(N)) \text{ s.t. } p_N i_N = 1;$$

so that we have a method of constructing Ω'-regular sections of E' from Ω-regular sections of E.

Then a lifting problem

$$\begin{tikzcd} Q \arrow[r, "G_0"] \arrow[d, "1\times 0"'] & \Gamma_\Omega(2D^n) \arrow[d, "\rho_\Omega"] \\ Q\times I \arrow[r, "G"] & \Gamma_\Omega(S^{n-1}\times[1,2]) \end{tikzcd} \qquad (Q \text{ a compact polyhedron})$$

gives a lifting problem

$$\begin{tikzcd} Q \arrow[r, "iG_0"] \arrow[d, "1\times 0"'] & \Gamma_{\Omega'}(2D^n\times[-1,1]) \arrow[d, "\rho_{\Omega'}"] \\ Q\times I \arrow[r, "iG"] & \Gamma_{\Omega'}(S^{n-1}\times[1,2]\times[-1,1]). \end{tikzcd}$$

Since $n < n+1$, proposition 4 shows that $\rho_{\Omega'}$ is a Serre fibration, so that a lift $\bar{G}$ for (iG, iG_0) may be constructed. Then $p\bar{G}$ is a lift for (G, G_0); so ρ_Ω is also a Serre fibration.

(1.6) As an example, we note that from a second-order non-degenerate immersion $f\colon S^1 \to \mathbb{R}^3$ (i.e. df and d^2f are linearly independent) we may 'functorially' construct a surface $F\colon S^1\times\mathbb{R}\to\mathbb{R}^3$ of everywhere non-zero curvature s.t. $F|S^1\times 0 = f$ (this construction was first suggested by Little).

However, no such 'functorial' construction is possible if we replace $\mathbb{R}^3$ by an arbitrary Riemannian 3-manifold. Nevertheless, we will show that it is only necessary for such constructions to be possible 'locally'; here 'locally' means both locally in N and in Γ_Ω.

We show how our notion of extensibility exhibits these properties; first, recall that a regularity condition $\Omega\subset E^r$ is *extensible* if there is

i) a C^∞-bundle E' over $(n+1)$-manifolds

ii) a C^∞-bundle map over n-manifolds $\pi\colon i^*E'\to E$ (where $i_N\colon N = N\times 0\subset N\times\mathbb{R}$)

iii) a regularity condition $\Omega'\subset E'^r$ s.t. $\pi^r(i^*\Omega') = \Omega$ (where $\pi^r\colon i^*(E'^r)\to E^r$ is the natural map induced by π; i.e. $\pi^r_N(j^r_{(x,0)}f) = j^r_x(\pi_N f i_N)$, where f is any local section of $E'(N\times\mathbb{R})$ at $(x,0)$).

Hence, for any smooth n-manifold N we may define a natural map

$$\bar{\pi}_N\colon \Gamma_{\Omega'}(E'(N\times\mathbb{R}))\to\Gamma_\Omega(E(N)) \quad \text{by} \quad \bar{\pi}_N(f)=\pi_N f i_N;$$

this is well-defined since

$$j^r_{(x,0)}f\in\Omega' \Rightarrow j^r_x(\pi f i)\in\pi^r i^*\Omega'=\Omega$$

for any local section f of $E'(N\times\mathbb{R})$ at $(x,0)$.

We make precise in the following lemma the sense in which this map has 'local' left-inverses:

(1.7) LEMMA: *Let $\Omega\subset E^r$ be an extensible regularity condition on n-manifolds; and let $\Omega'\subset E'^r$ be an extension.*

Then, for any n-manifold N, any point $x\in N$, and any $f\in\Gamma_\Omega(N)$, there exists

i) *an open neighbourhood W of x in N*
ii) *an open neighbourhood V of $f(W)$ in $E(W)$*
iii) *a number $\varepsilon>0$*

s.t., for any sub-n-manifold Z of W, there exists a continuous map $\rho_Z\colon H_Z\to\Gamma E'(Z\times\mathbb{R})$

$$(\textit{where } H_Z=\{g\in\Gamma E(Z)|g(Z)\subset V\})$$

s.t.

i) $\pi\cdot\rho_Z(h)\cdot i=h$ for all $h\in H_Z$
ii) $\rho_{Z'}(h|Z')=\rho_Z(h)|Z'\times\mathbb{R}$ *for any sub-n-manifold Z' of Z and any $h\in H_Z$*
iii) $\rho_Z(f|Z)|Z\times[-\varepsilon,\varepsilon]$ *is Ω'-regular.*

Proof: Let $p^{r,s}\colon E^r\to E^s$ $(s\leqq r)$ be the natural map 'forgetting' partial derivatives of order greater than s. This map is the projection of a locally trivial fibre bundle and is therefore an open map. In particular, $p^{r,s}\Omega$ $(s\leqq r)$ is an open sub-bundle of E^s.

Notice also that the diagram

$$\begin{array}{ccc} i^*(E'^r) & \xrightarrow{\pi^r} & E^r \\ \Big\downarrow{\scriptstyle i^*(p'^{r,s})} & & \Big\downarrow{\scriptstyle p^{r,s}} \\ i^*(E'^s) & \xrightarrow{\pi^s} & E^s \end{array} \qquad (s\leqq r)$$

is commutative, so that, since $\Omega'\subset E'^r$ is an extension of $\Omega\subset E^r$, we have

$$p^{r,s}\Omega = \pi^s(i^*(p'^{r,s}\Omega')).$$

In particular, $p^{r,0}\Omega = \pi(i^*(p'^{r,0}\Omega')) \subset E$, so that

$$\pi| : i^*(p'^{r,0}\Omega') \to p^{r,0}\Omega$$

is surjective. Indeed, this map is also a submersion on fibres (and so a fortiori a submersion), which is what we prove now.

For any n-manifold N, let

$$e' \in p'^{r,0}\Omega'(N \times \mathbb{R})|N \times 0;$$

and let $e = \pi_N(e')$, $x = p(e)$, so that $p'(e') = x \times 0$. For a sufficiently small neighbourhood U of x in N, and a sufficiently small real number $\varepsilon > 0$, $\exists$ trivialisations

$$E'(N \times \mathbb{R})|U \times (-\varepsilon, \varepsilon) = U \times (-\varepsilon, \varepsilon) \times F' \text{ (where } F' = E'(N \times \mathbb{R})_{x \times 0})$$

$$E(N)|U = U \times F \text{ (where } F = E(N)_x)$$

(and w.r.t. these trivialisations, π_U becomes a level-preserving smooth map $U \times F' \to U \times F$).

We may therefore consider germs of section $(N \times \mathbb{R}, x \times 0) \to (E', e')$ as germs of map $(U \times (-\varepsilon, \varepsilon), x \times 0) \to (F', e')$; and germs of section $(N, x) \to (E, e)$ as germs of map $(N, x) \to (F, e)$, w.r.t. these trivialisations; and we may also, of course identify $E^1(U)_e$ as $J^1(U, F)_{x,e}$ and hence as $\operatorname{Hom}(TU_x, TF_e)$ via the correspondence $j^1f_x \leftrightarrow df_x$.

Now define $W_e = \{v \in TF_e | v \in \operatorname{Im} \alpha$ for some $\alpha \in p^{r,1}\Omega(u)_e\}$; since $p^{r,1}\Omega$ is an open sub-bundle in $E^1, p^{r,1}\Omega(u)_e$ is identified as an open subset in $\operatorname{Hom}(TU_x, TF_e)$. It follows easily that W_e is open in TF_e.

Now recall that $p^{r,1}\Omega = \pi^1(i^*(p'^{r,1}\Omega'))$, so that

$$p^{r,1}\Omega(U)_e = \pi^1_{e'}(p'^{r,1}\Omega'(U \times (-\varepsilon, \varepsilon))_{e'})$$

(recall $\pi(e') = e$) i.e. for any $e' \in p'^{r,0}\Omega'$, and any Ω-regular germ $f: (U, x) \to (F, e)$, $\exists$ an Ω'-regular germ $f': (U \times (-\varepsilon, \varepsilon), x \times 0) \to (F', e')$ s.t. $j^1f_x = j^1(\pi f' i)_x$ i.e. $df_x = d(\pi_x)_{e'} \cdot df'_{x \times 0} \cdot i_x$ (where $\pi_x = \pi|\{x\} \times F'$).

In particular, then, $\operatorname{Im} d(\pi_x)_{e'} \supset \operatorname{Im} df_x$.

But for each $\alpha \in p^{r,1}\Omega(U)_e$, $\exists$ an Ω-regular germ f s.t. $j^1f_x = \alpha$; so that $\operatorname{Im} d(\pi_x)_{e'} \supset \operatorname{Im} \alpha$ for all $\alpha \in p^{r,1}\Omega(U)_e$, and hence $\operatorname{Im} d(\pi_x)_{e'} \supset W_e$. But $d(\pi_x)_{e'}: TF'_{e'} \to TF_e$ is a linear map; so, since its image contains the open subset W_e of TF_e, it must be surjective. Hence, since this is true for any $e' \in p'^{r,0}\Omega'(N)$, $\pi_N| : p'^{r,0}\Omega'(N \times \mathbb{R})|N \times 0 \to p^{r,0}\Omega(N)$ is a submersion on fibres.

Hence, for any points $e \in p^{r,0}\Omega(N)$, $e' \in \pi_N^{-1}e$, we may choose local co-ordinates

$\{x_1, \ldots, x_n\}$ in a neighbourhood X of $p(e)$

$\{x_1, \ldots, x_n, y_1, \ldots, y_p\}$ in a neighbourhood $X \times Y$ of e

$\{x_1, \ldots, x_n, z_1, \ldots, z_q\}$ in a neighbourhood $X \times Z$ of e'

(where $p = \text{fibre dim } E$, $q = \text{fibre dim } E'$, so $q \geqq p$)

s.t.

$$\pi(x_1, \ldots, x_n, z_1, \ldots, z_q) = (x_1, \ldots, x_n, z_1, \ldots, z_p).$$

Thus we may define a local left-inverse k to π by

$$k(x_1, \ldots, x_n, y_1, \ldots, y_p) = (x_1, \ldots, x_n, y_1, \ldots, y_p, 0, \ldots, 0),$$

s.t. $k(e) = e'$.

Now let $f \in \Gamma_\Omega(N)$, and let $x \in N$. Ω is extensible, so there exists $\tau \in \Omega'_{(x,0)}$ s.t. $\pi^r\tau = j^r_x f$. Choose local co-ordinates X, $X \times Y$, $X \times Z$ as above, near x, $f(x)$, $p^{r,0}\tau$ respectively, and let k be the local left-inverse to π s.t. $kf(x) = p^{r,0}\tau$.

Let P be the r^{th}-order polynomial representative of $\tau - j^r(kf \times 0)_{x \times 0}$ (w.r.t. the chosen local co-ordinates), so that $\pi P i = 0$.

Then there is a neighbourhood B of $x \times 0$ in $U \times \mathbb{R}$ s.t. $P + kf \times 0$ is Ω'-regular in B. Indeed we may choose $B = W \times [-\varepsilon, \varepsilon]$ for some open neighbourhood W of x in N, and some $\varepsilon > 0$.

Now let Z be any sub-n-manifold of W, and let

$$H_Z = \{g \in \Gamma E(Z) | g(Z) \subset W \times Y\};$$

define

$$\rho_Z : H_Z \to \Gamma(E(Z \times \mathbb{R}))$$

by

$$\rho_Z(h) = P + kh \times 0.$$

Then W, $V = W \times Y$, ε, ρ_Z have the required properties.

2. Details of proof

This chapter is devoted to the details of the proofs outlined in § 1; here propositions 1–5 of (1.2) are proved.

We note the following:

i) A map $f: X \to Y$ is a weak homotopy equivalence

$$\Leftrightarrow f_*: [Q, X] \to [Q, Y]$$

is a bijection for any compact polyhedron Q.

ii) A map $p: A \to B$ is a Serre fibration $\Leftrightarrow$ it has the polyhedral covering homotopy property (PCHP) i.e. from any commutative diagram of the following kind:

$$\begin{tikzcd} Q \arrow[r, "G_0"] \arrow[d, "1\times 0"'] & A \arrow[d, "p"] \\ Q\times I \arrow[r, "G"] & B \end{tikzcd} \qquad (Q \text{ a compact polyhedron})$$

(a *lifting problem*) there may be constructed a map $\bar{G}: Q \times I \to A$ s.t. $p\bar{G} = G$ and $G_0(q) = \bar{G}(q, 0)$ for each $q \in Q$; that is the following diagram commutes

$$\begin{tikzcd} Q \arrow[r, "G_0"] \arrow[d, "1\times 0"'] & A \arrow[d, "p"] \\ Q\times I \arrow[r, "G"] \arrow[ur, "\bar{G}"] & B. \end{tikzcd}$$

We say that $\bar{G}$ is a *lift* for (G, G_0).

We note further that the PCHP is equivalent to the local PCHP (that is, for each $b \in B$ there is a neighbourhood U of b in B s.t. $p|p^{-1}U: p^{-1}U \to U$ has the PCHP). Hence a map is a Serre fibration $\Leftrightarrow$ it has the local PCHP (for a proof see Dold ([1]; 4.8)).

(2.1) Proposition 1: *Ω is integrable over the n-disc D^n.*

Proof: Since D^n is contractible, $E(D^n)$ is a product bundle $D^n \times F$ whose sections will be identified with maps $D^n \to F$. The fibre of $E^r(D^n)$ at $0 \in D^n$ may then be identified with $J^r_0(D^n, F)$.

Now the restriction map

$$\rho: \Gamma(D^n) \to \Gamma(0)$$

(where $\Gamma(0)$ is the fibre of $\Omega(D^n)$ at $0 \in D^n$) is a homotopy equivalence, so it will be sufficient to prove that $\rho \circ j^r : \Gamma_\Omega(D^n) \to \Gamma(0)$ is w.h.e.

Consider F as a closed submanifold of some $\mathbb{R}^N$, and let $\pi: W \to F$ be a smooth retraction of a tubular neighbourhood.

Let $f: Q \to \Gamma(0) \subset J^r_0(D^n, F) \subset J^r_0(D^n, \mathbb{R}^N)$ be any map (where Q is a compact polyhedron).

Replacing each jet in $J^r_0(D^n, \mathbb{R}^N)$ by its polynomial representative of degree r, we obtain a map $F: Q \times D^n \to \mathbb{R}^N$ so that for each $q \in Q$ the map $F_q: D^n \to \mathbb{R}^N$ defined by $F_q(x) = F(q, x)$ is smooth, $j^r F_q$ is continuous w.r.t. q, and $j^r F_q(0) = f(q)$.

Let V be a neighbourhood of 0 in D^n s.t. $F(Q \times V) \subset W$, and let $F'_q = \pi F_q | V$.

Since $\Omega(D^n)$ is an open sub-bundle, there is a neighbourhood U of 0 in V s.t. $F'_q | U \in \Gamma_\Omega(U)$.

Now let h be an embedding of D^n in U which is the identity on a neighbourhood of 0.

Then $h^E(F'_q) = g(q)$ defines a map $g: Q \to \Gamma_\Omega(D^n)$ whose r-jet at 0 is f. Thus $(\rho \circ j^r)_*: [Q, \Gamma_\Omega(D^n)] \to [Q, \Gamma(0)]$ is surjective.

Now let $f_0, f_1: Q \to \Gamma_\Omega(D^n)$ be continuous maps, and suppose there exists a map $k: Q \times I \to \Gamma(0) \subset J^r_0(D^n, \mathbb{R}^N)$ s.t.

$$k(q, 0) = j^r_0(f_0(q)),\ k(q, 1) = j^r_0(f_1(q)) \ (q \in Q).$$

Replacing each jet in $J^r_0(D^n, \mathbb{R}^N)$ by its polynomial representative, we obtain a map $K: (Q \times I) \times D^n \to \mathbb{R}^N$ s.t. $j^r_0(K(q, t)) = k(q, t)$.

Let $\tilde{K}: (Q \times I) \times D^n \to \mathbb{R}^N$ be defined by

$$\tilde{K}(q, t)(x) = \begin{cases} (1-3t) f_0(q)(x) + 3t K(q, 0)(x) & (0 \leqq t \leqq \frac{1}{3}) \\ K(q, 3t-1)(x) & (\frac{1}{3} \leqq t \leqq \frac{2}{3}) \\ (3t-2) f_1(q)(x) + (3-3t) K(q, 1)(x) & (\frac{2}{3} \leqq t \leqq 1) \end{cases}$$
$$\text{for all } q \in Q,\ x \in D^n.$$

(regarding $f_0(q)$, $f_1(q)$ as maps $D^n \to \mathbb{R}^N$).

Now let V be a neighbourhood of 0 in D^n, and $v > 0$ a number s.t.

$$\tilde{K}(Q \times ([0, v] \cup [1-v, 1]) \times D^n \cup Q \times I \times V) \subset W.$$

Let

$$\tilde{K}' = \pi \tilde{K} | Q \times ([0, v] \cup [1-v, 1]) \times D^n \cup Q \times I \times V.$$

Since $\Omega(D^n)$ is an open sub-bundle there is a neighbourhood U of 0 in V and a number $\mu \in (0, \nu]$ s.t.

$$\tilde{K}'|Q \times ([0, \mu] \cup [1-\mu, 1]) \times D^n \cup Q \times I \times U$$

is Ω-regular. Now let h be an isotopy of embeddings of D^n in itself s.t. $h_0 = h_1 = 1_{D^n}$, and $h_t(D^n) \subset U$ for $\mu \leqq t \leqq 1-\mu$. Then we may define a homotopy $K: Q \times I \to \Gamma_\Omega(D^n)$ s.t.

$$K(q, 0) = f_0(q), \; K(q, 1) = f_1(q) \; (q \in Q)$$

by $K(q, t) = h_t^E(\tilde{K}'(q, t))$.

Thus $(p \circ j_r)_*: [Q, \Gamma_\Omega(D^n)] \to [Q, \Gamma(0)]$ is $(1-1)$.

So $p \circ j^r$ is w.h.e.

(2.2) PROPOSITION 2: *Let N' be N plus a collarlike neighbourhood. Then the restriction maps*

$$\rho_\Omega: \Gamma_\Omega(N') \to \Gamma_\Omega(N)$$

$$\rho: \Gamma(N') \to \Gamma(N)$$

are w.h.e. and Serre fibrations.

PROOF: Let $G: Q \to \Gamma_\Omega(N)$ be a Q-family of Ω-regular sections (Q a compact polyhedron); we may extend G to obtain a Q-family G' of sections over a neighbourhood of N in N'. Since Ω-regularity is an open condition, there is a neighbourhood U of N in N' over which G' is Ω-regular.

Let h be an embedding of N' in U, which is the identity on a neighbourhood of N. Then $h^E(G')$ is an Ω-regular family extending G.

So $[Q, \Gamma_\Omega(N')] \to [Q, \Gamma_\Omega(N)]$ is surjective.

Now let

$$G_0, G_1: Q \to \Gamma_\Omega(N')$$

be continuous maps, and let

$$K: Q \times I \to \Gamma_\Omega(N)$$

be a homotopy s.t. $K(q, 0) = G_0(q)|N$, $K(q, 1) = G_1(q)|N$ for each $q \in Q$.

Let $\pi: Q \times I \times N' \to N'$ be the obvious projection, and let $\bar{E} = \pi^* E(N')$. Then k, f_0, f_1 together give a section of $\bar{E}|Q \times I \times N \cup Q \times \{0, 1\} \times N'$. This section may be extended to a section of

$$\bar{E}|Q \times I \times U \cup Q \times \{[0,\mu] \cup [1-\mu, 1]\} \times N',$$

for some neighbourhood U of N in N' and some $\mu \in (0, 1]$. Indeed, since Ω-regularity is an open condition we may suppose (by restricting to smaller U, μ) that the maps

$$\tilde{G}_0 : Q \times [0, \mu] \to \Gamma E(N')$$
$$\tilde{G}_1 : Q \times [1-\mu, 1] \to \Gamma E(N')$$
$$\tilde{K} : Q \times I \to \Gamma E(U)$$

this section defines are in fact Ω-regular.

Now let h be an isotopy of embeddings of N' to itself, s.t. $h_0 = h_1 = 1_{N'}$, and $h_t(N') \subset U$ for $\mu \leqq t \leqq 1-\mu$. Define a homotopy $K: Q \times I \to \Gamma_\Omega(N')$ between G_0 and G_1 by

$$K(q,t) = \begin{cases} h_t^E(\tilde{G}_0(q,t)) & (0 \leqq t \leqq \mu) \\ h_t^E(\tilde{K}(q,t)) & (\mu \leqq t \leqq 1-\mu) \\ h_t^E(\tilde{G}(q,t)) & (1-\mu \leqq t \leqq 1). \end{cases} \qquad (q \in Q)$$

Thus $[Q, \Gamma_\Omega(N')] \to [Q, \Gamma_\Omega(N)]$ is $(1-1)$.

Hence $\Gamma_\Omega(N') \to \Gamma_\Omega(N)$ is w.h.e.

Now suppose we have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \Gamma_\Omega(N') \\ \Big\downarrow{\scriptstyle 1\times 0} & & \Big\downarrow{\scriptstyle \rho_\Omega} \\ Q \times I & \xrightarrow{G} & \Gamma_\Omega(N). \end{array}$$

Extend the $Q \times I$-family of sections of $E(N)$ to a neighbourhood of N in N', the extension agreeing with the Q-family G_0 on this neighbourhood. Since Ω-regularity is an open condition, there is a neighbourhood U of N in N' over which this extended $Q \times I$-family G' is Ω-regular.

Let $G'': Q \times I \to \Gamma(E(N'))$ be a $Q \times I$-family of sections s.t.

$$G''(q,t)(x) = \begin{cases} G'(q,t)(x) & x \in \text{in a nhd. } V \text{ of } N \text{ in } U \\ G_0(q)(x) & x \notin U \end{cases}$$

and $G''(q, 0) = G_0(q)$.

Then, since Ω-regularity is an open condition, there is $\mu \in (0, 1]$ s.t. $G''(q,t)$ is Ω-regular for $t \leqq \mu$.

Let $h_t (t \in [0, 1])$ be an isotopy of N' to itself s.t. h_0 is the identity,

h_t is fixed on a neighbourhood of N, and $h_1(N') \subset V$.

Define $\bar{G}: Q \times I \to \Gamma_\Omega(N')$ by

$$\bar{G}(q, t) = \begin{cases} h^E_{t/\mu}(G''(q, t)) & (t \leqq \mu) \\ h^E_1(G'(q, t)) & (t \geqq \mu). \end{cases}$$

This is the required lift for (G, G_0).

So $\Gamma_\Omega(N') \to \Gamma_\Omega(N)$ is a Serre fibration. The proofs for $\Gamma(N') \to \Gamma(N)$ may be made similarly (or more simply!)

(2.3) PROPOSITION 3: *For any (manifold) pair (A, B), the restriction map*

$$\Gamma(A) \xrightarrow{\rho} \Gamma(B)$$

is a Serre fibration.

PROOF: Suppose we have a lifting problem

$$\begin{tikzcd} Q \arrow[r, "G_0"] \arrow[d, "1 \times 0"'] & \Gamma(A) \arrow[d, "\rho"] \\ Q \times I \arrow[r, "G"] & \Gamma(B) \end{tikzcd} \qquad (Q \text{ a compact polyhedron})$$

We may rearrange this as follows

$$\begin{tikzcd} Q \times I \times B \cup Q \times 0 \times A \arrow[r, "G \cup G_0"] \arrow[d] & \Omega(A) \arrow[d, "\text{projection}"] \\ Q \times I \times A \arrow[r, "0 \times 0 \times 1"] & A \end{tikzcd}$$

$\Omega(A) \to A$ is a locally trivial bundle, and hence has the polyhedral covering homotopy extension property (PCHEP) (see Hu [14]; pp 62 and 65). In particular it has PCHEP for the pair $(Q \times A, Q \times B)$. The lift which this provides gives the required lift $\bar{G}$ for (G, G_0).

Hence $\Gamma(A) \to \Gamma(B)$ is a Serre fibration.

(2.4) PROPOSITION 4: *Let $A_k = 2D^k \times D^{n-k}$, $B_k = S^{k-1} \times [1, 2] \times D^{n-k}$; where $k < n$. Then the restriction map*

$$\rho_\Omega: \Gamma_\Omega(A_k) \to \Gamma_\Omega(B_k)$$

is a Serre fibration.

PROOF: Suppose we have a lifting problem

$$\begin{tikzcd} Q \arrow[r, "G_0"] \arrow[d, "1\times 0"'] & \Gamma_\Omega(A_k) \arrow[d, "\rho\Omega"] \\ Q\times I \arrow[r, "G"] & \Gamma_\Omega(B_k) \end{tikzcd} \qquad (Q \text{ a compact polyhedron})$$

We construct a lift $\bar{G}$ in three stages:

1) Extend the $Q \times I$-family G of sections of $E(B_k)$ to a neighbourhood of B_k in A_k, the extension agreeing with the Q-family G_0 on this neighbourhood. Since Ω-regularity is an open condition, there is a neighbourhood of B_k in A_k over which this extended $Q \times I$-family G' is Ω-regular. In particular G' is regular over $S^{k-1} \times [\alpha, 2] \times D^{n-k}$ for some $\alpha < 1$.

2) For each $s \in [0, 1]$ we construct maps

$$\mu_s: Q \times [s, 1] \to \Gamma E(U)$$

(where U is a neighbourhood of $S^{k-1} \times [\alpha, 2] \times D^{n-k}$) s.t.

$$\mu_s(q, t)(x, y) = \begin{cases} G'(q, t)(x, y) & (t = s; \text{ or } x \text{ lies in a nhd. of } S^{k-1} \times [1, 2]) \\ G'(q, s)(x, y) & (x \text{ lies in a nhd. of } S^{k-1} \times \alpha). \end{cases}$$

Indeed these constructions may be made uniformly w.r.t. s.

Since Ω-regularity is an open condition, $\mu_s(q, t)$ is Ω-regular for $|t-s|$ sufficiently small. $[0, 1]$ is compact, so there is an increasing sequence $0 = t_0 < \ldots < t_l = 1$ s.t. $\mu_i(q, t)(= \mu_{t_i}(q, t))$ is Ω-regular for $t_i \leqq t \leqq t_{i+1}$.

3) We construct a lift G_1 on $Q \times [0, t_1]$ by

$$G_1(q, t)(x, y) = \begin{cases} \mu_0(q, t)(x, y) & (x \in S^{k-1} \times [\alpha, 2]) \\ G_0(q)(x, y) & (\text{otherwise}). \end{cases}$$

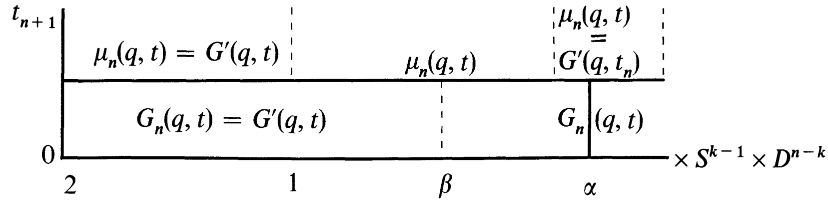
This is Ω-regular since μ_0, G_0 are.

Now suppose inductively that we have constructed a lift

$$G_n: Q \times [0, t_n] \to \Gamma_\Omega(A)$$

s.t. $G_n = G'$ on a neighbourhood of $S^{k-1} \times [\beta, 2] \times D^{n-k}$, where $\alpha < \beta < 1$.

Then we have the following situation:



We hope to construct the extension by defining $G_{n+1}(q, t) = G_n(q, t_n)$ for $t_n \leqq t \leqq t_{n+1}$ and $x \in \gamma D^k$, for some $\gamma > \beta$. But μ_n is not constant w.r.t. t at γ so using μ_n and this G_{n+1} will not be continuous. But μ_n is constant w.r.t. t on a neighbourhood of $S^{k-1} \times \alpha \times 0$; we shall find deformations $\tilde{\mu}_n, \tilde{G}_n$ off a neighbourhood of B s.t. $\tilde{\mu}_n$ is constant w.r.t. t on a nhd. of $S^{k-1} \times \gamma \times 0$, and so that $\tilde{\mu}_n = \tilde{G}_n$ on a neighbourhood of $S^{k-1} \times [\beta, 2] \times 0$ at $t = t_n$.

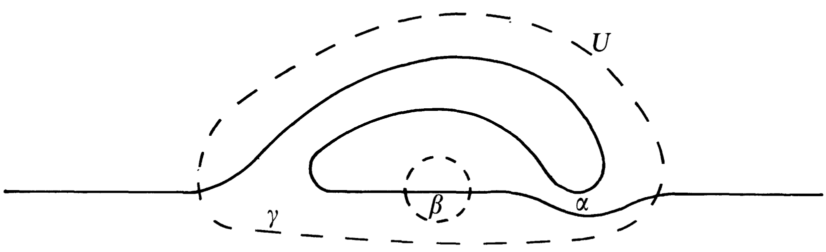
To do this, we require the following:

let $U \subset A$ be a neighbourhood of $S^{k-1} \times [\alpha, \beta] \times 0$ on which μ_n and G' are both defined, and s.t. $U \cap B = \emptyset$.

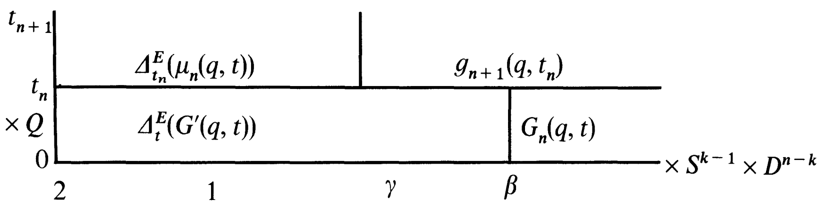
Since $k < n$, there is an isotopy $\Delta_t\{0 \leqq t \leqq t_n\}$ of A s.t.

1) Δ_t is the identity outside U and on a nhd. of $S^{k-1} \times \beta \times 0$; and Δ_t is the identity for $t \leqq \frac{1}{2} t_n$.

2) $\Delta_{t_n}(S_\gamma \times 0) = S_\alpha \times 0$, for some γ s.t. $\beta < \gamma < 1$.



Let $C = (2D^k \times 0) \cup B$, and let V be a small neighbourhood of C in A. Define $g_{n+1}: Q \times [0, t_{n+1}] \to \Gamma_\Omega(V)$ by



(the neighbourhood V is determined by the requirements

$$\Delta_t(\beta, y) = (\beta, y) \forall t \qquad \text{for } (\beta, y) \in V$$

$\Delta_{t_n}(\gamma, y) \subset$ nhd. of $S^{k-1} \times \alpha$ in which $\mu_n(q, t) = \mu_n(q, t_n)$ for $(\gamma, y) \in V$) that is, by

$$g_{n+1}(q, t)(x, y) = \begin{cases} G_n(q, t)(x, y) & (\|x\| \leqq \beta, 0 \leqq t \leqq t_n) \\ \Delta_t^E(G'(q, t))(x, y) & (\beta \leqq \|x\| \leqq 2, 0 \leqq t \leqq t_n) \\ \Delta_{t_n}^E(\mu_n(q, t))(x, y) & (\gamma \leqq \|x\| \leqq 2, t_n \leqq t \leqq t_{n+1}) \\ g_{n+1}(q, t_n)(x, y) & (\|x\| \leqq \gamma, t_n \leqq t \leqq t_{n+1}). \end{cases}$$

Now let h_t be an isotopy of embeddings of A into itself s.t. h_0 is the identity, h_t is the identity on a neighbourhood of B in A, and $h_t(A) \subset V$ for $t \geqq \frac{1}{2}t_n$.

Then a lift $G_{n+1}: Q \times [0, t_{n+1}] \to \Gamma_\Omega(A)$ is defined by

$$G_{n+1}(q, t) = h_t^E(g_{n+1}(q, t)).$$

(2.5) PROPOSITION 5: *If Ω is extensible, the restriction map*

$$\rho_\Omega: \Gamma_\Omega(2D^n) \to \Gamma_\Omega(S^{n-1} \times [1, 2])$$

is a Serre fibration.

PROOF: We will show that ρ_Ω has the local PCHP. Let

$$f \in \Gamma_\Omega(S^{n-1} \times [1, 2]),$$

and let f' be an Ω-regular extension of f on $S^{n-1} \times [1-a, 2]$, for some $a \in (0, 1)$ (such an extension exists because Ω-regularity is an open condition).

Let $\Omega' \subset E'^r$ be an extension of Ω. Then, by lemma (1.7), for each point $x \in S^{n-1}$ there exist open neighbourhoods W of x in $S^{n-1} \times [1-a, 2]$, V of $f'(W)$ in $E(W)$ and a number $\varepsilon > 0$ s.t., for any sub-n-manifold $Z \subset W$, there is a map

$$\rho_Z: H_Z \to \Gamma(E'(Z \times \mathbb{R}))$$

(where $H_Z = \{g \in \Gamma(E(Z)) | g(Z) \subset V\}$) s.t.

i) $\rho_{Z'}(h|Z') = \rho_Z(h)|Z' \times \mathbb{R}$ for any sub-n-manifold Z' of Z and any $h \in H_Z$.

ii) $\rho_Z(f'|Z)|Z \times [-\varepsilon, \varepsilon]$ is Ω'-regular.

We may take Z to be of the form $N \times [1-c, 1+c]$, where N is an open neighbourhood of x in S^{n-1} and $c \in (0, a]$. S^{n-1} is compact, so there is a finite covering $\{N_i\}$ of S^{n-1} by such neighbourhoods. Let $c = \min\{c_i\}$, $\varepsilon = \min\{\varepsilon_i\}$.

Let $\mathscr{G}$ be a neighbourhood of f in $\Gamma_\Omega(S^{n-1} \times [1, 2])$ s.t., for any $g \in \mathscr{G}$

$$\left.\begin{array}{l} \text{i) } g(N_i \times [1, 1+c]) \subset V_i, \text{ so that} \\ \qquad g|N_i \times [1, 1+c] \in H_{N_i \times [1, 1+c]} \\ \text{ii) } \rho_{N_i \times [1, 1+c]}(g|N_i \times [1, 1+c])|N_i \times [1, 1+c] \times [-\varepsilon, \varepsilon] \\ \text{is } \Omega'\text{-regular} \end{array}\right\} \text{For each } i.$$

Now suppose we have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \rho_\Omega^{-1}\mathscr{G} \\ \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle \rho_\Omega} \\ Q \times I & \xrightarrow{G} & \mathscr{G} \end{array} \qquad (Q \text{ a compact polyhedron})$$

Local PCHP will be established if we construct a lift for (G, G_0). Since Q is compact, there exists a $b \in (0, \frac{1}{2}c]$ s.t., for each $q \in Q$

$$\left.\begin{array}{l} \text{i) } G_0(q)(N_i \times [1-2b, 1+c]) \subset V_i, \text{ so that} \\ \qquad G_0(q)|N_i \times [1-2b, 1+c] \subset H_{N_i \times [1-2b, 1+c]} \\ \text{ii) } \rho_{N_i \times [1-2b, 1+c]}(G_0(q)|N_i \times [1-2b, 1+c])| \\ N_i \times [1-2b, 1+c] \times [-\varepsilon, \varepsilon] \text{ is } \Omega'\text{-regular} \end{array}\right\} \text{For each } i.$$

Define

$$G'(q, t)(x) = \begin{cases} G(q, t)(x) & (x \in S^{n-1} \times [1, 2]) \\ G_0(q)(x) & (x \in (1-b)D^n). \end{cases}$$

For convenience, we write

$$X = S^{n-1} \times \{[1-2b, 1-b] \cup [1, 1+c]\}$$
$$Y = S^{n-1} \times [1-2b, 1+c].$$

It remains to lift $(G'|X, G_0|Y)$.

Let K be a triangulation of S^{n-1} each of whose $(n-1)$-simplexes $|A|$

lies in one of the neighbourhoods N_i, say N_A.

Then for each simplex $C \in K$, $|C| \subset N_A$ for each $(n-1)$-simplex A s.t. $C < A$.

For some open neighbourhood $N(C)$ of $|C| \times [1-2b, 1+c]$ in Y we define

$$\begin{cases} {}^AH_0^C\colon Q \to \Gamma_{\Omega'}(N(C)\times[-\varepsilon,\varepsilon]) \text{ by } {}^AH_0^C(q) = \rho_{N(C)}(G_0(q)) \\ {}^AH^C\colon Q\times I \to \Gamma_{\Omega'}(N(C)\cap X\times[-\varepsilon,\varepsilon]) \text{ by } {}^AH^C(q,t) = \rho_{N(C)\cap X}(G'(q,t)) \end{cases}$$

for each simplex C and each $(n-1)$-simplex A s.t. $C < A$.

Let K^l be the l-skeleton of K, and suppose inductively that we have constructed the following:

$\bar{N}(C)$ is a neighbourhood of $|C| \times [1-2b, 1+c]$ in $N(C)$ for each $(j-1)$-simplex C.

$N(K^{j-1}) = X \cup \bigcup_C \bar{N}(C)$ is {a neighbourhood of $|K^{j-1}| \times [1-2b, 1+c]$ in Y} $\cup X$.

$G^{j-1}\colon Q\times I \to \Gamma_\Omega(N(K^{j-1}))$ lifts $(G', G_0|N(K^{j-1}))$

${}^AH^C\colon Q\times I \to \Gamma_{\Omega'}(\bar{N}(C)\times[-\varepsilon,\varepsilon])$ lifts $({}^AH^C|\bar{N}(C)\cap X, {}^AH_0^C|\bar{N}(C))$

$$\text{s.t.}\begin{cases} \pi\,{}^AH^C(q,t)i = G^{j-1}(q,t) \\ {}^AH^C(q,t) = {}^AH^{C'}(q,t) \text{ on } (\bar{N}(C)\cap\bar{N}(C'))\times[-\varepsilon,\varepsilon], \\ \qquad\qquad \text{where } C, C' \text{ are } (j-1) \text{ simplexes facing } A \end{cases}$$

for each $(j-1)$-simplex C and each $(n-1)$-simplex A s.t. $C < A$.

(This induction 'starts' with $j = 0$, $K^{-1} = \phi$).

Let $N'(K^{j-1})$ be a neighbourhood of $X \cup |K^{j-1}| \times [1-2b, 1+c]$ in $N(K^{j-1})$, and for each j-simplex E let $\bar{N}(E)$ be a neighbourhood of $E \times [1-2b, 1+c]$ in $N(E)$ s.t.

$$\bar{N}(E) \cap N'(K^{j-1}) \subset \cup\,\{\bar{N}(C) | C \text{ a } (j-1)\text{-simplex facing } E\}$$

and s.t. there is a diffeomorphism

$$(\bar{N}(E), \bar{N}(E)\cap N'(K^{j-1})) \cong (2D^{j+1}\times D^{n-j-1}, S^j\times[1,2]\times D^{n-j-1})$$

Then for each E we have a $(j+1)$-handle lifting problem

$$\begin{tikzcd} Q \arrow[r, "G_0|"] \arrow[d, "1\times 0"] & \Gamma_\Omega(\bar{N}(E)) \arrow[d, "\rho_\Omega"] \\ Q\times I \arrow[r, "G_{j-1}|"] & \Gamma_\Omega(\bar{N}(E)\cap N'(K^{j-1})) \end{tikzcd}$$

and, for each E and each $(n-1)$-simplex A s.t. $E < A$, a $(j+1)$-handle lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{{}^AH_0^E|} & \Gamma_{\Omega'}(\bar{N}(E)\times[-\varepsilon,\varepsilon]) \\ \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle \rho_{\Omega'}} \\ Q\times I & \xrightarrow{{}^A\bar{H}^E} & \Gamma_{\Omega'}(\bar{N}(E)\cap N'(K^{j-1})\times[-\varepsilon,\varepsilon]) \end{array}$$

(where ${}^A\bar{H}^E(q,t) = {}^AH^C(q,t)$ in $\bar{N}(C)\times[-\varepsilon,\varepsilon]$)
s.t. $\pi^A H_0^E(q)i = G_0(q)|$ and $\pi^A\bar{H}^E(q,t)i = G^{j-1}(q,t)|$.

In order to make the induction step, we distinguish two cases:

1) $j+1 < n$; use Proposition 6 (see (2.6) below) to find lifts ${}^AH^E$ for $({}^A\bar{H}^E, {}^AH_0^E|)$ s.t. $\pi^A H^E(q,t)i = \pi^{A'}H^E(q,t)i$ for $(n-1)$-simplexes A, A' faced by E.

Then define G^j on $N(K^j) = X \cup_E^{\cup} \bar{N}(E)$ by

$$G^j(q,t)(x) = \begin{cases} \pi^A H^E(q,t)i(x) & \text{on } \bar{N}(E) \\ G^{j-1}(q,t)(x) & \text{on } N'(K^{j-1}) \end{cases}$$

2) $j+1 = n$; use Proposition 4 to find lifts ${}^AH^A$ for $({}^A\bar{H}^A, {}^AH_0^A|)$.
Then define G^{n-1} on $N(K^{n-1}) = X \cup_A^{\cup} \bar{N}(A) = Y$ by

$$G^{n-1}(q,t)(x) = \begin{cases} \pi^A H^A(q,t)i(x) & \text{on } \bar{N}(A) \\ G^{n-2}(q,t)(x) & \text{on } N'(K^{n-2}) \end{cases}$$

G^{n-1} is the required lift for $(G'|X, G_0|Y)$.

(2.6) Proposition 6: *Let* $A = 2D^k\times D^{n-k}$, $B = S^{k-1}\times[1,2]\times D^{n-k}$, $J = [-1,1]$, *where* $k < n$.

Suppose we have a lifting problem

$$\begin{array}{ccc} Q & \xrightarrow{G_0} & \Gamma_{\Omega}(A) \\ \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle \rho_{\Omega}} \\ Q\times I & \xrightarrow{G} & \Gamma_{\Omega}(B) \end{array}$$

and lifting problems

$$\begin{array}{ccc} Q & \xrightarrow{H_0^j} & \Gamma_{\Omega'}(A\times J) \\ \downarrow{\scriptstyle 1\times 0} & & \downarrow{\scriptstyle \rho_{\Omega'}} \\ Q\times I & \xrightarrow{H^j} & \Gamma_{\Omega'}(B\times J) \end{array} \qquad (\text{for } j = 1,\ldots,r)$$

$$\text{s.t.} \begin{cases} \pi H_0^j(q)i = G_0(q) \ \textit{for all} \ q \in Q \ \textit{and each} \ j = 1, \ldots, r \\ \pi H^j(q,t)i = G(q,t) \ \textit{for all} \ (q,t) \in Q \times I \ \textit{and each} \ j = 1, \ldots, r. \end{cases}$$

Then there is a lift $\bar{G}$ *of* (G, G_0) *and lifts* $\bar{H}^j$ *of* (H^j, H_0^j) *s.t.* $\pi\bar{H}^j(q,t)i = \bar{G}(q,t)$ *for all* $(q,t) \in Q \times I$ *and each* $j = 1, \ldots, r$.

PROOF: The method follows closely that of Proposition 4.

1) Extend the $Q \times I$-family G of sections of $p^{r,0}\Omega(A)|B$ to a neighbourhood U of B in A, the extension G' agreeing with G_0 in U; and then extend the $Q \times I$-families H^j of sections of $p'^{r,0}\Omega'(A \times J)|B \times J$ to a neighbourhood U' of $B \times J$ in $U \times J$, the extensions H'^j agreeing with the Q-families H_0^j in U', and being s.t. $\pi H'^j i = G'$.

(These extensions H'^j are obtained as follows: first extend each i^*H^j, considered as a $Q \times I$-family of partial sections over $G'(B)$ of

$$\pi| : \pi^{-1}G'(U) \cap i^*(p'^{r,0}\Omega'(A \times J)) \to G'(U),$$

to $G'(\bar{U})$, where $\bar{U}$ is some neighbourhood of B in U, the extensions $\bar{H}^j$ agreeing with $i^*H_C^j$ on $G'(\bar{U})$. ($\pi|$ may not be quite a fibration, but it is a surjective submersion on fibres (by the Lemma (1.7)) so that partial sections over neighbourhood retracts do have local extensions). Now extend each $H^j \cup \bar{H}^j$, considered as a section of

$$p'^{r,0}\Omega'(A \times J)|B \times J \cup \bar{U} \times \{0\},$$

to some neighbourhood U' of $B \times J$ in $\bar{U} \times J$, the extensions H'^j agreeing with H_0^j on U').

Since Ω'-regularity is an open condition, there is a neighbourhood of $B \times J$ in $A \times J$ over which these extended $Q \times I$-families H'^j are Ω'-regular; in particular they are Ω'-regular over $S^{k-1} \times [\alpha, 2] \times D^{n-k} \times J$, for some $\alpha \in (0,1)$. Of course, since $G' = \pi H'^j i, G'$ is then Ω-regular over $S^{k-1} \times [\alpha, 2] \times D^{n-k}$.

2) Let V be a neighbourhood of $S^{k-1} \times [\alpha, 2] \times D^{n-k}$ s.t. $V \times J \subset U'$.

For each $s \in [0,1]$, find a suitably close approximation ${}_sG''$ to G' s.t. the restriction of the Q-family

$${}_sG''(q, \ ) : V \times I \to p^{r,0}\Omega(V) \times I \ \ (q \in Q)$$

to $S^{k-1} \times [\frac{1}{3} + \frac{2}{3}\alpha, \frac{2}{3} + \frac{1}{3}\alpha] \times D^{n-k} \times I$ is smooth; and s.t. ${}_sG''(q, \ )$ is equal to $G'(q, \ )$ on $V \times \{s\}$ and in a neighbourhood of

$$S^{k-1} \times (\{\alpha\} \cup [1,2]) \times D^{n-k} \times I.$$

It is clear that such approximations may be found uniformly w.r.t. s. Now find suitably close approximations ${}_sH''^j$ to H'^j s.t. the restrictions of the Q-families

$${}_sH''^j(q,\ ): V \times J \times I \to p'^{r,0}\Omega'(V \times J) \times I \ (q \in Q)$$

to $S^{k-1} \times [\frac{1}{3}+\frac{2}{3}\alpha, \frac{2}{3}+\frac{1}{3}\alpha] \times D^{n-k} \times J \times I$ are smooth, and s.t. ${}_sH''^j(q,\ )$ is equal to $H'(q,\ )$ on $V \times J \times \{s\}$ and near

$$S^{k-1} \times (\{\alpha\} \cup [1,2]) \times D^{n-k} \times J \times I;$$

and s.t.

$$\pi_s H''^j(q,t)i = {}_sG''(q,t)((q,t) \in Q \times I).$$

(The argument showing that construction of ${}_sH''^j$ is possible is similar to 1); again the key is that $\pi| \times 1_I$ is 'nearly' a fibration; so that a family of sections over ${}_sG''(V \times I)$ (which each H'^j provides, since ${}_sG''$ is close (hence homotopic) to G') may be arbitrarily closely approximated by a family of smooth sections).

Again it is clear that the approximations ${}_sH''^j$ may be constructed uniformly w.r.t. s.

Now let $\rho: (0,2] \to [0,1]$ be a smooth function s.t. $\rho(0, \frac{1}{3}+\frac{2}{3}\alpha] = \{0\}$, $\rho[\frac{2}{3}+\frac{1}{3}\alpha, 2] = \{1\}$, and define

$$v_s^j: Q \times [s,1] \to \Gamma E'(V \times J) \ (j = 1, \ldots, r; s \in [0,1])$$

by

$$v_s^j(q,t)((p,u),y,r) = {}_sH''^j(q, s+(t-s)\rho(u))((p,u),y,r)$$

(for $(q,t) \in Q \times I$, $((p,u),y,r) \in (S^{k-1} \times (0,2] \times D^{n-k} \times J) \cap (V \times J)$ so that

$$v_s^j(q,t)(x,y,r) = \begin{cases} H'^j(q,t)(x,y,r)(t = s; \text{ or } x \text{ near } S^{k-1} \times [1,2]) \\ H'^j(q,s)(x,y,r)(x \text{ near } S^{k-1} \times \alpha) \end{cases}$$

v_s^j is, of course, constructed uniformly w.r.t. s.

Now, since Ω'-regularity is an open condition, $v_s^j(q,t)$ is Ω'-regular for $|t-s|$ sufficiently small. So, since I is compact, there exists an increasing sequence $0 = t_0 < \ldots < t_l = 1$ s.t. $v_k^j(q, t) = v_{t_k}^j(q, t)$ is Ω'-regular for $t_k \leqq t \leqq t_{k+1}$, and each $j = 1, \ldots, r$.

Define maps $\mu_k \colon Q \times [t_k, t_{k+1}] \to \Gamma_\Omega(U)$ by

$$\mu_k(q, t) = \pi v_k^j(q, t) i.$$

3) Now construct lifts H_1^j for (H^j, H_0^j) on $Q \times [0, t_1]$ by

$$H_1^j(q, t)(x, y, r) = \begin{cases} v_0^j(q, t)(x, y, r) & (x \in S^{k-1} \times [\alpha, 2]) \\ H_0^j(q)(x, y, r) & (\text{otherwise}). \end{cases}$$

Then G_1 defined by $G_1(q, t) = \pi H_1^j(q, t) i$ is a lift for (G, G_0) over $Q \times [0, t_1]$.

Now construct a lift for (G, G_0) over $Q \times I$ using the inductive method of Proposition 4. The deformations Δ_t, h_t used, in conjunction with the μ_k, for each inductive step provide deformations $\Delta_t \times 1_J$, $h_t \times 1_J$ which may be used, in conjunction with the v_k^j, to construct lifts for (H^j, H_0^j). These lifts, composed with π and i, will be the lift already constructed for G.

Appendix

Let $E(N)$ be the trivial bundle $N \times P$, so that its sections may be identified with maps $N \to P$, and $E^r(N)$ may be identified with $J^r(N, P)$.

Let Ω_P be a natural, stable regularity condition on smooth maps of an n-manifold into P (so that $\Omega_P(N)$ is an open sub $-C^\infty$-bundle of $J^r(N, P) \to N$). Let W be an (open) submanifold of P; then we denote by Ω_W the regularity condition defined by $\Omega_W(N) = J^r(N, W) \cap \Omega_P(N)$. This condition is natural (because the local diffeomorphism action is trivial) and stable, and it is extensible if Ω_P is. We may exploit these facts to prove the following.

Approximation Theorem: *Let Ω be an extensible regularity condition on smooth maps $N \to P$, and suppose there is an Ω-regular section $\sigma \colon N \to J^r(N, P)$ covering the map $f \colon N \to P$ (i.e. $f = p^{r,0}\sigma$). Then f may be fine C^0-approximated by Ω-regular maps whose r-jets are homotopic to σ as sections of $\Omega(N)$.*

Proof: Let ρ be any smooth metric on P, and let $\alpha \colon N \to (0, \infty)$ be any smooth function. We shall show that there exists an Ω-regular map $g \colon N \to P$ s.t. $j^r g$ is Ω-regularly homotopic to σ (i.e. σ and $j^r g$ are homotopic as sections of $\Omega(N)$) and s.t. $\rho(f(x), g(x)) < \alpha(x)$ for each $x \in N$ (we say g is an *α-approximation* to f).

For each $x \in N$, let W_x be an open convex co-ordinate neighbourhood of $f(x)$ contained in $\{p \in P | \rho(f(x), p) < \frac{1}{4}\alpha(x)\}$, and let

$$U_x = \{y \in N | \alpha(y) > \tfrac{1}{2}\alpha(x)\} \cap f^{-1}W_x.$$

The sets $\{U_x\}$ form an open covering of N. Choose a C^∞-triangulation of N so fine that each n-simplex $|A|$ lies in one of the open sets U_x, say U_A.

Suppose inductively that we have constructed a neighbourhood N_{j-1} of the $(j-1)$-skeleton in N, and an Ω-regular map $g_{j-1}: N_{j-1} \to P$ s.t. $j^r g_{j-1}$ is Ω-regularly homotopic to $\sigma | N_{j-1}$ and s.t. $g_{j-1}(N_{j-1} \cap U_A) \subset W_A$ for each n-simplex A (these constructions may clearly be made for $j = 1$).

Now let N'_{j-1} be a neighbourhood of the $(j-1)$-skeleton in N_{j-1}, and for each j-simplex E let $N(E)$ be a neighbourhood of $|E|$ in $W_E = \cap \{W_A | E < A\}$ s.t. the $N(E) - N'_{j-1}$ are disjoint and

$$(N(E), N(E) \cap N'_{j-1}) \cong (2D^j \times D^{n-j}, S^{j-1} \times [1, 2] \times D^{n-j})$$

Consider the following commutative diagram

$$\begin{array}{ccc}
C^\infty_\Omega(N(E), W_E) & \xrightarrow{\quad j^r \quad} & J^r_\Omega(N(E), W_E) \\
\Big\downarrow {\scriptstyle \rho_\Omega} & & \Big\downarrow {\scriptstyle \rho} \\
C^\infty_\Omega(N(E) \cap N'_{j-1}, W_E) & \xrightarrow{j^r} & J^r_\Omega(N(E) \cap N'_{j-1}, W_E)
\end{array}$$

whose vertical maps are fibrations (by propositions 3, 4, 5) and whose horizontal maps are w.h.e. by Theorem A. $j^r g_{j-1} | N(E) \cap N'_{j-1}$ is Ω-regularly homotopic to $\sigma | N(E) \cap N'_{j-1}$, and so there is a section τ in the fibre over $j^r g_{j-1} | N(E) \cap N'_{j-1}$ Ω-regularly homotopic to $\sigma | N(E)$; and there is an Ω-regular map $g_E: N(E) \to W_E$ in the fibre over $g_{j-1} | N(E) \cap N'_{j-1}$ (using the lemma of (1.2)) s.t. $j^r g_E$ is Ω-regularly homotopic to τ and hence σ.

Define $N_j = \cup_E N(E)$, and $g_j: N_j \to P$ by $g_j(x) = g_E(x)$ for $x \in N(E)$. This completes the induction step.

Hence we may construct, by this method, an Ω-regular map $g: N \to P$ s.t. $j^r g$ is Ω-regularly homotopic to σ and s.t. $g(U_A) \subset W_A$ for each n-simplex A. This map g is in fact the required α-approximation to f, as we can see by the following argument:

suppose $y \in U_A$, and that U_A was originally defined with reference to the point x_A. $f(y), g(y) \in W_A$, and so

$$\rho(f(y), g(y)) \leqq \rho(f(y), f(x_A)) + \rho(g(y), f(x_A)) < \tfrac{1}{4}\alpha(x_A) + \tfrac{1}{4}\alpha(x_A) = \tfrac{1}{2}\alpha(x_A).$$

But $y \in U_A$, so $\alpha(y) > \frac{1}{2}\alpha(x_A)$.

Thus $\rho(f(y), g(y)) < \alpha(y)$.
This completes the proof.

REMARKS:

1. The requirement that Ω is extensible is necessary even for open manifolds N; for this result adds detail to the classifications of Theorems A and B by the remark that the homotopy class of an Ω-regular section of $J^r(N, P)$ covering a proper map may be represented by the jet of an Ω-regular proper map.

Hence it is essentially the non-extensibility of the regularity condition for submersions that makes the construction of foliations with compact leaves so difficult (the inverse images of points by a submersion gives a foliation of the source manifold).

2. A more general theorem may be proved: consider smooth maps from an (n, k)-foliation M into P; such a map is $\bar{\Omega}$-regular if its restriction to each leaf is Ω-regular (see § 1, note 3).

Suppose $f: M \to P$ is a map s.t. for each leaf L, $f|L$ is a smooth map covered by an Ω-regular section $\sigma_L: L \to J^r(L, P)$ depending continuously on the leaf. Then f may be fine C°-approximated by Ω-regular maps $M \to P$ whose restrictions to each leaf L have r-jets Ω-regularly homotopic to σ_L.

The proof of this is exactly as above; to show that the diagram corresponding to (1) has the required properties, we use the abstract proposition 3.4.1 of Gromov's paper [7].

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