COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 32, nº 3 (1976), p. 293-300
<http://www.numdam.org/item?id=CM_1976__32_3_293_0>

COMPOSITIO MATHEMATICA, Vol. 32, Fasc. 3, 1976, pag. 293–300
Noordhoff International Publishing
Printed in the Netherlands

SYMMETRIC BASES IN MINKOWSKI SPACES

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An analog of a well-known property of symmetric bases in infinite dimensional Banach spaces is established for finite dimensional spaces. It is shown that if a finite dimensional space E has a basis with the property that each permutation of indices naturally induces an isomorphism of norm at most λ, then E has a (possibly different) 9λ-unconditional basis. Restated in terms of symmetry parameters this answers a question posed by Gordon [2], which is implicit in the paper of Gurarii, Kadec and Macaev [4]. Some examples are given to show the non-isometric nature of the result.

Let $B = (b_i)_{i\varepsilon I}$ be a basic sequence (finite or countably infinite) in a normed space E. For π a permutation of I with $\pi(i) \neq i$ only finitely often, g_π is the isomorphism of E defined by $g_\pi(b_i) = b_{\pi(i)}$, $i\varepsilon I$; and for $(\varepsilon_i)_{i\varepsilon I}$ a sequence of scalars with $|\varepsilon_i| = 1$ for all $i\varepsilon I$ and $\varepsilon_i \neq 1$ only finitely often, g_ε is the operator defined by $g_\varepsilon(b_i) = \varepsilon_i b_i$. Three symmetry parameters of B are defined as follows:

the *unconditional basis constant* of B is $x(B) = \sup_\varepsilon \|g_\varepsilon\|$;

the *diagonal symmetry constant* of B is $\delta(B) = \sup_\pi \|g_\pi\|$;

and the *total symmetry constant* of B is $t(B) = \sup_{\varepsilon,\pi} \|g_\pi g_\varepsilon\|$.

Clearly $x(B) \leqq t(B)$ and $\delta(B) \leqq t(B)$ for every basis B, and it is known that $x(B) \leqq 2\beta\delta(B)^2$, where β is the basis constant of B (cf. [7]). But also observe that no inequalities of the form $x(B) \leqq f(\delta(B))$ or $\delta(B) \leqq f(x(B))$ are valid for all bases, where f indicates a real function independent of the particular basis. A simple sequence of examples showing that the first relation cannot hold may be given as follows.

* Partially supported by NSF GP-42666

AMS (MOS) subject classification (1970)
Primary 46B15
Secondary 46B05
Key words and phrases, Minkowski space, unconditional constant, symmetry constant, diagonal symmetry constant.

For n odd let B be the unit vector basis of $\mathbb{R}^n$ considered with the norm $\|x\| = \max |\langle x, \varepsilon \rangle|$, where the maximum is taken over all *n-tuples* $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of signs with $\sum_{i \leqq n} \varepsilon_i = 1$. Then $\delta(B) = 1$ and $x(B) = n$.

For p any one of the parameters x, δ or t define the corresponding symmetry parameter of E by $p(E) = \inf_B p(B)$, with the infimum taken over all bases for E.

The Banach Mazur distance between isomorphic spaces E and F is defined as $d(E, F) = \inf \|u\| \|u^{-1}\|$, the infimum being taken over all isomorphisms u between E and F. It is immediate that each of the three symmetry parameters of E defined above is continuous in the sense that $p(E) \leqq d(E, F)p(F)$ holds for all E and F.

Although the diagonal and total symmetry constants of a particular basis in a finite dimensional space may behave quite differently, the diagonal and total symmetry constants of the space itself are equivalent. More precisely,

THEOREM 1: *The relations* $\delta(E) \leqq t(E) \leqq 9\delta(E)$ *hold in every finite dimensional space* E.

The first inequality is obvious. To prove the second it is convenient to first consider the case in which E has a basis $B = (b_i)_{i \leqq n}$ with $\delta(B) = 1$. The coefficient functionals of the basis are denoted by $(b_i')_{i \leqq n}$, and m is the greatest integer satisfying $2m \leqq n$. The group of all permutations of $\{1, 2, \ldots, k\}$ is written S_k. The proof of the special case requires two lemmata, the first of which is given without proof.

LEMMA 1: *Let* $w_1, w_2, \ldots, w_m$ *and* $v_1, v_2, \ldots, v_m$ *be two finite sequences of non-negative reals with* $v_i \geqq v_{i+1}$ *for* $1 \leqq i < m$. *For* $1 \leqq k \leqq m$ *set* $u_{2k} = u_{2k-1} = v_k$, *and* $u_n = 0$ *if* n *is odd. Then*

$$\max_{\pi \in S_n} \sum_{k \leqq m} [u_{\pi(2k)} + u_{\pi(2k-1)}] w_k = 2 \max_{\tau \in S_m} \sum_{k \leqq m} v_{\tau(k)} w_k.$$

LEMMA 2: *Let* $|\ |$ *be any norm on* E *for which* $\delta(B \subset (E, |\ |)) = 1$. *Then*

(a) $q = n^{-1}(\sum_{i \leqq n} b_i') \otimes (\sum_{k \leqq n} b_k)$ *is a norm one projection,*

(b) $p = 2^{-1} \sum_{k \leqq m} (b_{2k}' - b_{2k-1}') \otimes (b_{2k} - b_{2k-1})$ *is a norm one projection,*

(c) $1_E - q = (n-1)m^{-1}(n!)^{-1} \sum_{\pi \in S_n} g_\pi^{-1} p g_\pi$, *and*

(d) $t((b_{2k} - b_{2k-1})_{k \leqq m} \subset (E, |\ |)) = 1$.

PROOF OF LEMMA 2: Part (a) follows from the equality $q = (n!)^{-1} \sum_\pi g_\pi$, and (b) is true since $p = 2^{-1}(1 - g_\tau)$, where $\tau \in S_n$ interchanges $2k$ with

$2k-1$ for each $k = 1, 2, \ldots, m$.

To verify (c) write $w = (n!)^{-1}\sum_\pi g_\pi^{-1} p g_\pi$. Since w commutes with each g_π, $w = s1_E + tq$ for some scalars s and t. Write H for the kernel of q. Then

$$s1_H = (n!)^{-1} \sum_\pi (g_\pi|H)^{-1}(p|H)(g_\pi|H)$$

and $p|H$ is a projection onto an m dimensional subspace of H, so

$$s(n-1) = \text{trace}\,(s1_H) = \text{trace}\,(p|H) = m.$$

Also

$$sn + t = \text{trace}\,(w) = \text{trace}\,(p) = m,$$

so that $s = -t = m(n-1)^{-1}$.

Finally, given signs $\delta_1, \delta_2, \ldots, \delta_m$ and $\tau \in S_m$ let $\pi \in S_n$ be the permutation which maps $\{2k, 2k-1\}$ onto $\{2\tau(k), 2\tau(k)-1\}$, $1 \leqq k \leqq m$, and which satisfies $\pi(2k) = 2\tau(k)$ if $\delta_k = 1$, $\pi(2k) = 2\tau(k)-1$ if $\delta_k = -1$. For each $k = 1, 2, \ldots, m$, $\delta_k(b_{2\tau(k)} - b_{2\tau(k)-1}) = g_\pi(b_{2k} - b_{2k-1})$, which proves (d).

Proof of Theorem 1: Assume $\delta(B) = 1$ and let (( )) be the norm on E defined by

$$((x)) = \max_{\pi, \varepsilon} \|p g_\varepsilon g_\pi(x)\|,$$

where $\|\ \|$ is the given norm on E and the maximum is over all $\pi \in S_n$ and n-tuples of signs ε. Let F denote E under (( )). Notice that each operator $g_\varepsilon g_\pi$ is an isometry of F, and hence $t(F) = \delta(B \subset F) = 1$, so the assumptions of Lemma 2 are satisfied by the basis B in both norms, (( )) and $\|\ \|$.

The first claim is that $((x)) = \|x\|$ for all x in $[b_{2k} - b_{2k-1}]_{k \leqq m}$, the span of the vectors $b_{2k} - b_{2k-1}$. The inequality $((x)) \geqq \|x\|$ is immediate, and for the other direction it is enough, by Lemma 2(d), to consider vectors of the form $x = \sum_{k \leqq m} a_k(b_{2k} - b_{2k-1})$ with $|a_k| \geqq |a_{k+1}|$ for $1 \leqq k < m$. For ε an n-tuple of signs and $\pi \in S_n$, choose $x' \in E'$ so that $\|x'\| = 1$ and

$$\begin{aligned}\|p g_\varepsilon g_\pi^{-1}(x)\| &= |\langle p g_\varepsilon g_\pi^{-1}(x), x'\rangle| \\ &\leqq 2^{-1} \sum_{k \leqq m} [|\langle x, b'_{\pi(2k)}\rangle| + |\langle x, b'_{\pi(2k-1)}\rangle|]|\langle b_{2k} - b_{2k-1}, x'\rangle|\end{aligned}$$

Applying Lemma 1 with $v_k = |a_k|$ and $w_k = |\langle b_{2k} - b_{2k-1}, x'\rangle|$ shows that

$$\begin{aligned}\|pg_\varepsilon g_\pi^{-1}(x)\| &\leqq \max_{\tau\in S_m} \sum_{k\leqq m} |a_{\tau(k)}^{-1}||\langle b_{2k} - b_{2k-1}, x'\rangle| \\ &= \max_{\tau, |\delta|=1} |\langle \sum_{k\leqq m} \delta_k a_k (b_{2\tau(k)} - b_{2\tau(k)-1}), x'\rangle| \leqq \|x\|,\end{aligned}$$

the last by part (d) of Lemma 2.

We next assert that the inequalities

$$\|(1-q)(x)\| \leqq 2((x)) \quad \text{and} \quad (((1-q)(x))) \leqq 2\|x\|$$

hold for all $x \in E$. For the first, applying Lemma 2 with both $\|\ \|$ and $((\ ))$ yields

$$\begin{aligned}\|(1-q)(x)\| &\leqq (n-1)m^{-1}(n!)^{-1} \sum_\pi \|g_\pi^{-1} p g_\pi(x)\| \\ &\leqq 2(n!)^{-1} \sum_\pi \|pg_\pi(x)\| \\ &= 2(n!)^{-1} \sum_\pi ((pg_\pi(x))) \\ &\leqq 2(n!)^{-1} \sum_\pi ((g_\pi(x))) \\ &= 2((x)),\end{aligned}$$

and the other inequality follows by interchanging the rôles of $\|\ \|$ and $((\ ))$.

Now let λ be the constant satisfying

$$\lambda \|\sum_{i\leqq n} b_i\| = ((\sum_{i\leqq n} b_i))$$

and define $u: F \to E$ by $u = 1_E + (\lambda - 1)q$. Since $((q(x))) = \|q(x)\|\lambda$ for all $x \in E$, $\|u(x)\| \leqq \|(1-q)(x)\| + ((q(x))) \leqq 3((x))$ by the preceeding paragraph and Lemma 2, and hence $\|u\| \leqq 3$. But $u^{-1} = 1_E + (\lambda^{-1} - 1)q$ so the same proof gives $((u^{-1}(x))) \leqq 3\|x\|$ and thus $d(E, F) \leqq 9$. Then

$$t(E) \leqq t(F)d(E, F) \leqq 9.$$

More generally for $B \subset E$ any basis let H be E with the norm $|x| = \max_\pi \|g_\pi(x)\|$. Since $\delta(B \subset H) = 1$ and $d(E, H) \leqq \delta(B)$ the special case shows that $t(E) \leqq d(E, H)t(H) \leqq 9\delta(B)$.Taking the infimum over all possible bases for E completes the proof of the theorem.

REMARK 1: In [4] Gurarii, Kadec and Macaev define the symmetry parameter α of a finite dimensional space E by $\alpha(E) = \inf_B x(B)\delta(B)$. The theorem implies that $\delta(E) \leqq \alpha(E) \leqq 9\delta(E)^2$, answering a question raised by Gordon [2] and by Lewis [5].

REMARK 2: Let E be a Banach space with a diagonally symmetric basis $B = (b_i)_{i \geqq 1}$. Then for any $\varepsilon > 0$ and any finite dimensional $F \subset E$ there is finite dimensional W with $F \subset W \subset E$ and $t(W) \leqq (9+\varepsilon)\delta(B)$. This follows from a routine pertubation argument and the fact that $t([b_1, b_2, \ldots, b_n]) \leqq 9\delta(B)$ for all n. Thus, although the unconditional basis constant of B depends on the basis constant of B, the local unconditional structure of E depends only on $\delta(B)$.

Following [1] define the *asymmetry constant* of a finite dimensional space E by

$$s(E) = \inf_G \sup_{g \in G} \|g\|,$$

with the infimum taken over all compact groups G of isomorphisms of E which have the property that only scalar multiples of the identity commute with the elements of G.

It is clear that $s(E) \leqq t(E)$, so the following theorem strongly indicates the non-isometric nature of the relationship between $\delta(E)$ and $t(E)$.

THEOREM 2: *There is a sequence* $(E_n)_{n \geqq 5}$ *of Minkowski spaces with* $\dim E_n = n$, $\delta(E_n) = 1$ *and* $\liminf_n s(E_n) \geqq (2^{-1}+2^{-\frac{1}{2}})^{\frac{1}{2}}$.

PROOF: Let $e_i \in l_\infty^{n+1}$ and $e_i' \in l_1^{n+1}$ be the unit vectors, $1 \leqq \lambda \leqq n$ and $E_n \subset l_\infty^{n+1}$ be the kernel of $\sum_{i \leqq n} e_i' + \lambda e_{n+1}'$ (a sequence of values for λ will be specified later). The basis $b_i = e_i - \lambda^{-1} e_{n+1}$, $1 \leqq i \leqq n$, has diagonal symmetry constant one so $\delta(E_n) = 1$. To estimate $s(E_n)$ from below we use the inequality [1]

$$s(E_n)^2 \geqq n^{-1} \gamma_\infty(E_n) \pi_1(E_n),$$

with $\gamma_\infty(E_n)$ and $\pi_1(E_n)$ denoting, respectively, the projection constant of E_n and the 1-absolutely summing norm [6] of the identity on E_n. Write G for the group of isometries of l_∞^{n+1} of form $g(e_i) = e_{\pi(i)}$ for some $\pi \in S_{n+1}$ with $\pi(n+1) = n+1$.

If w is a projection of l_∞^{n+1} onto E_n with $\|w\| = \gamma_\infty(E_n)$, then $u = |G|^{-1} \sum_{g \in G} g^{-1} w g$ is also a projection onto E_n with norm $\gamma_\infty(E_n)$. Since u commutes with each element of G

$$u = 1-(\sum_{i\leqq n} e_i'+\lambda e_{n+1}') \otimes (t\sum_{i\leqq n} e_i+se_{n+1})$$

for some scalars s and t with $tn+\lambda s = \operatorname{trace}(1-u) = 1$. Thus

$$\begin{aligned}\gamma_\infty(E_n) &= \|u'\| = \max\{|1-t|+(n+\lambda-1)|t|, n|s|+|1-s\lambda|\}\\ &\geqq \inf_t \max\{|1-t|+(n+\lambda-1)|t|, n\lambda^{-1}|1-nt|+n|t|\}\\ &= 2n(n-1)(n^2-2\lambda+\lambda^2)^{-1}.\end{aligned}$$

To estimate $\pi_1(E_n)$ from below, there is by Pietsch's Theorem [6] a measure μ on $\Omega = \{e_i'|E_n: 1 \leqq i \leqq n+1\}$ such that $\|\mu\| = \pi_1(E_n)$ and $\|x\| \leqq \mu(|\langle x, \cdot\rangle|)$ for all $x \in E_n$. Let ν be a measure on Ω given by $\nu(f) = |G|^{-1}\sum_{g\in G}\mu(fog)$, so that $\|\nu\| = \pi_1(E_n)$, $\|x\| \leqq \nu(|\langle x, \cdot\rangle|)$ for $x \in E_n$ and $\nu(f) = \nu(fog)$ for all $f \in C(\Omega)$ and $g \in G$. The last implies that $s = \nu(\{b_i'\})$ is independent of i, $1 \leqq i \leqq n$. Setting $t = \nu(\{b_{n+1}'\})$ gives scalars s and t satisfying $\pi_1(E_n) = sn+t$ and

$$\|x\| \leqq s\sum_{i\leqq n}|\langle x, e_i'\rangle|+t|\langle x, e_{n+i}\rangle|, x \in E_n.$$

Substituting $e_1-\lambda^{-1}e_{n+1}$ and e_1-e_2 in the last inequality shows that $s+t\lambda^{-1} \geqq 1$ and $2s \geqq 1$, so that

$$\pi_1(E_n) = sn+t \geqq 2^{-1}(n+\lambda).$$

Now vary λ with n by taking $\lambda_n = [2n(n-1)]^{\frac{1}{2}}-n$ for $n \geqq 5$. Combining inequalities yields the desired lower estimate.

REMARK 3: As is observed above every Minkowski space satisfies $x(E) \leqq t(E)$ and $s(E) \leqq t(E)$. Some other possible relations between the three parameters x, t and s are known to be false. The space $A_n = l_1^n \oplus l_2^n$ has unconditional basis constant one but $s(A_n)$ and $t(A_n)$ behave asymptotically like $n^{\frac{1}{4}}$ [1], and the tensor product $B_n = l_2^n \hat{\otimes} l_2^n$ has asymmetry constant one but $x(B_n)$ and $t(B_n)$ both act asymptotically like $n^{\frac{1}{2}}$ [3]. Such examples suggest the following problem. Is there a real function f of two variables such that $t(E) \leqq f(x(E), s(E))$ for all finite dimensional E?

The answer to this problem is negative as is shown by the following example due to J. Lindenstrauss.

EXAMPLE: Let

$$E_n = (\sum_{k=1}^{n} E_k^n)_{l_4}$$

where each E_k^n is isometric to an n^2 dimensional Hilbert space. Then $s(E_n) = x(E_n) = 1$ but $t(E_n) \to_{n\to\infty} \infty$. The first two statements are clear so we will prove only the last one. Let us start with the observation that E_n is isometric to a subspace of $L_4[0,1]$.

LEMMA 1: *Let $B = (b_i)_{i\leqq n}$ be a normalized basic sequence in $L_4[0,1]$ with $t(B) = \alpha$. Then* span $\{b_i\}_{i\leqq n}$ *is α-isomorphic to a subspace of $l_2^n \oplus_4 l_4^n$.*

PROOF: The expression

$$\int_0^1 |\sum_{i=1}^{n} \lambda_i b_i|^4$$

is α-equivalent to

$$\frac{1}{2^n n!}\sum_{\varepsilon}\sum_{\sigma}\int_0^1 |\sum_{i=1}^{n} \varepsilon_i \lambda_i b_{\sigma(i)}|^4$$

where $\varepsilon = (\varepsilon_i)_{i=1}^n$ ranges over all 2^n choices of signs and σ over all $n!$ permutations of $\{1,2,\ldots,n\}$. This latter sum is of the form

$$a\sum_{i=1}^{n}\lambda_i^4 + c(\sum_{i=1}^{n}\lambda_i^2)^2$$

for suitable positive a and c.

LEMMA 2: *Let $E \subset l_4^n$ $\dim E \geqq n^{\frac{2}{3}}$ then $d(E, l_2^{\dim E}) \to_{n\to\infty} \infty$.*

This Lemma follows immediately from Corollary 3.1 of [8]. Using those two Lemmas we will estimate $t(E_n)$. Suppose $t(E_n) \leqq C$ for $n = 1,2,3,\ldots$ then E_n embeds uniformly into $l_2^{n^3} + l_4^{n^3}$. Denote

$$\beta_k = \|P \circ \varphi | E_k^n\|$$

where φ is an isomorphic embedding from E_n into $l_2^{n^3} \oplus l_4^{n^3}$ and P is a projection from $l_2^{n^3} \oplus l_4^{n^3}$ onto $l_2^{n^3}$ annihilating $l_4^{n^3}$. Let

$$z_k \in E_k^n, \qquad \|z_k\| = 1, \qquad \|P \circ \varphi(z_k)\| = \beta_k.$$

Then for some ε_k, $|\varepsilon_k| = 1$

$$n^{\frac{1}{4}} = \|\sum_{k=1}^{n} \varepsilon_k z_k\| \geqq \frac{1}{C} \|P \circ \varphi(\sum_{k=1}^{n} \varepsilon_k z_k)\| \geqq \frac{1}{C} (\sum_{k=1}^{n} |\beta_k|^2)^{\frac{1}{2}}.$$

But this implies that for n big enough at least one β_k must be very small. Then an easy perturbation argument implies that $l_4^{n^3}$ contains uniformly $l_2^{n^2}$ which contradicts Lemma 2.

REFERENCES

[1] D. J. H. GARLING, Y. GORDON: Relations between some constants associated with finite dimensional Banach spaces. *Israel J. Math. 9* (1971) 346–361.
[2] Y. GORDON: Asymmetry and projection constants of Banach spaces. *Israel J. Math. 14* (1972) 50–62.
[3] Y. GORDON, D. R. LEWIS: Absolutely summing operators and local unconditional structures, *Acta Math. 133* (1974) 27–48.
[4] V. I. GURARII, M. I. KADEC, V. I. MACAEV: On Banach-Mazur distance between certain Minkowski spaces. *Bull. Acad. Pol. Sci. Ser. Math. Astron. Phy. 13* (1965) 719–722.
[5] D. R. LEWIS: A relation between diagonal and unconditional basis constant. Math. Ann. 218 (1975) 193–198.
[6] A. PIETSCH: Absolut p-summierende Abbildungen in normierten Räumen. *Studia Math. 28* (1967) 333–353.
[7] I. SINGER: *Bases in Banach spaces*. Berlin-New York, Springer Verlag 1970.
[8] A. PELCZYNSKI, H. P. ROSENTHAL: Localization techniques in L_p spaces. *Studia Math. 52* (1975) 263–289.

(Oblatum 1–VII–1975)

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