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PRIMITIVE IDEMPOTENTS AND THE SOCLE IN GROUP RINGS OF PERIODIC ABELIAN GROUPS

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Abstract

Let $K$ be a field and $G$ a periodic abelian group containing no elements of order $p$ if $\text{char } K = p > 0$. We establish necessary and sufficient conditions for the group ring $KG$ to contain primitive idempotents. We also characterize the socle of $KG$, and show that when the socle is non-zero the ascending socular series reaches $KG$ after a finite number of steps.

1. Introduction

Let $K$ be a field and $G$ a periodic abelian group containing no elements of order $p$ if $\text{char } K = p > 0$. We shall investigate the circumstances under which the group ring $KG$ contains primitive idempotents. We find (Lemma 3.1 and Theorem 3.4) that the following three conditions are necessary and sufficient:

(a) $G$ is almost locally cyclic (i.e. has a locally cyclic subgroup of finite index);

(b) $G$ satisfies the minimum condition on subgroups; and

(c) $|k(G) \cap K : k| < \infty$, where $k$ is the prime field of $K$, and $k(G)$ is a certain algebraic extension of $k$, to be defined in Section 2.

Note that (a) and (b) hold if and only if $G$ has the form

$$G \cong F \times C_{p_1^{e_1}} \times \ldots \times C_{p_m^{e_m}},$$

where $F$ is a finite abelian group and the $C_{p_i^{e_i}}$ are Prüfer $p_i$-groups for distinct primes $p_i$. To foreshadow the significance of (c), we remark that (c) always holds if $G$ is finite or $K$ is a finite extension of $k$, but if $K$ is algebraically closed then (c) holds only if $G$ is finite.
For groups $G$ satisfying (a) and (b), we consider the connection between primitive idempotents in $KG$ and irreducible $KG$-modules. When (c) holds, there is a one-to-one onto correspondence between primitive idempotents in $KG$ and isomorphism classes of irreducible $KG$-modules with finite centralizer (i.e. finite kernel in $G$); moreover there are only finitely many non-isomorphic such modules having any fixed finite subgroup of $G$ as centralizer (Theorem 3.4). But if (c) fails to hold the situation is quite different: there are no primitive idempotents in $KG$, but given any finite subgroup $C$ of $G$ such that $G/C$ is locally cyclic, there exist $2^{\aleph_0}$ non-isomorphic irreducible $KG$-modules with centralizer $C$ (Theorem 3.3).

In Section 4 we characterize the socle of $KG$: it is zero if (c) fails, and otherwise it is the intersection of certain maximal ideals of $KG$ (Theorem 4.2). When (a), (b) and (c) hold we find that the ascending socular series of $KG$ reaches $KG$ after a finite number of steps, i.e. that $KG$ has a finite series with completely reducible factors. The number of steps is one plus the number of primes involved in the maximal divisible subgroup of $G$ (Theorem 4.3).

When $G$ is a locally cyclic group with Min, it is convenient to consider a condition equivalent both to (c) and to the existence of primitive idempotents in $KG$: namely, the existence of $K$-inductive subgroups in $G$. We call a finite subgroup $H$ of $G$ $K$-inductive if every irreducible $KH$-module faithful for $H$ remains irreducible when induced up to $G$. It is with the study of $K$-inductive subgroups that we commence.

Special cases of some of the results have been obtained in papers of Hartley [2], Berman [1], and Müller [4]; more detailed references will be given in the sequel. The author is deeply indebted to Dr Brian Hartley for his aid and encouragement in the writing of this paper.

2. $K$-Inductive subgroups

Let $G$ be a periodic abelian group; $\pi(G)$ the set of primes $p$ such that $G$ has elements of order $p$, and $K$ a field with $\text{char } K \notin \pi(G)$. Let $KG$ be the group ring of $G$ over $K$. Let $\bar{K}$ be an algebraic closure of $K$, and $\bar{K}^*$ its multiplicative group. We denote by $K(G)$ the $K$-subalgebra of $\bar{K}$ generated by all images of homomorphisms $G \to \bar{K}^*$; $K(G)$ is in fact a subfield of $\bar{K}$. Since the torsion subgroup of $\bar{K}^*$ is a direct product of Prüfer groups, one for each prime not equal to char $K$, if $G$ is locally cyclic then $\bar{K}^*$ has exactly one subgroup isomorphic to $G$; the elements of this subgroup generate $K(G)$ as a $K$-algebra, for any quotient of $G$ is isomorphic (albeit unnaturally) to a subgroup of $G$. 
**Lemma (2.1):** Let $H$ be a finite cyclic group and $K$ a field with $\text{char } K \notin \pi(H)$. Then there exist irreducible $KH$-modules faithful for $H$, and all such modules have dimension $|K(H): K|$ over $K$.

**Proof:** $K(H)^*$ has a unique subgroup isomorphic to $H$, so we may choose a monomorphism $\theta: H \to K(H)^*$. Then $K(H)$ becomes a $KH$-module with $H$-action given by

$$v \cdot h = vh^\theta, \quad v \in K(H), \ h \in H.$$ 
If $0 \neq v \in K(H)$ then $v \cdot KH = vK(H) = K(H)$, so $K(H)$ is an irreducible $KH$-module; it is faithful for $H$ as $\theta$ is one-to-one.

Let $V$ be any irreducible $KH$-module faithful for $H$. Then $V$ is isomorphic to $KH/M$ for some maximal ideal $M$ of $KH$. Now $KH/M$ is a field, containing (since $V$ is faithful) a multiplicative subgroup isomorphic to $H$ which generates it over $K$. It follows that $KH/M$ is algebraic over $K$, and thence isomorphic to the field $K(H)$. Thus

$$\dim_k V = \dim_k KH/M = |K(H): K|,$$
completing the proof.

If $G$ is a periodic abelian group, we will denote by $\Omega(G)$ the subgroup generated by all elements of prime order in $G$. This subgroup is finite if and only if $G$ satisfies Min, the minimum condition on subgroups. If $K$ is a field and $V$ a $KG$-module, we write

**Lemma (2.2):** Let $G$ be a periodic abelian group, $H$ a subgroup of $G$ containing $\Omega(G)$, and $K$ a field with $\text{char } K \notin \pi(G)$. Let $V$ be an irreducible $KH$-module faithful for $H$, and $W$ a non-zero submodule of the induced module $V^G = V \otimes_{KH} KG$. Then $W$ is faithful for $G$.

**Proof:** Since $G$ is abelian, the restriction $V^G|_H$ of $V^G$ to $H$ is a direct sum of copies of $V$. As $V$ is irreducible, $W_H$ is also a direct sum of copies of $V$. Suppose $1 \neq g \in C_G(W)$. There exists an integer $n$ such that $1 \neq g^n \in \Omega(G) \leq H$. But then $1 \neq g^n \in C_H(W_H) = C_H(V)$, a contradiction as $V$ is faithful for $H$. Hence $W$ is faithful for $G$.

Let $K$ be a field and $G$ a locally cyclic group with Min such that $\text{char } K \notin \pi(G)$. A finite subgroup $H$ of $G$ will be called $K$-inductive in $G$ if whenever $V$ is an irreducible $KH$-module faithful for $H$, the induced module $V^G$ is an irreducible $KG$-module.
**Lemma (2.3):** A finite subgroup $H$ of $G$ is $K$-inductive if and only if the following two conditions are satisfied:

(a) $H$ contains $\Omega(G)$;
(b) whenever $L$ is a finite subgroup of $G$ containing $H$, we have


**Proof:** Suppose $H$ is $K$-inductive in $G$. By Lemma 2.1 there exists an irreducible $KH$-module $V$ faithful for $H$; then $V^G$ is irreducible.

(a) Suppose $H \not\supseteq \Omega(G)$; then there exists a finite non-trivial subgroup $L$ of $G$ with $HL = H \times L$. Now $V^{H \times L}$ is reducible: indeed

$$\{ \sum_{x \in L} v \otimes x : v \in V \}$$

is a proper submodule. A fortiori $V^G$ is reducible, a contradiction. So $H \supseteq \Omega(G)$.

(b) Let $L$ be a finite subgroup of $G$ containing $H$. Then $V^L$ like $V^G$ is irreducible; by (a) and Lemma 2.2 $V^L$ is faithful for $L$. Hence using Lemma 2.1,

$$|K(L): K(H)| = |K(L): K|/|K(H): K|$$

$$= \dim_K V^L / \dim_K V$$

$$= |L: H|,$$

since $V^L = V \otimes_{KH} KL$.

Now suppose (a) and (b) hold. We may express $G$ as the union of a chain

$$H = H_0 \leq H_1 \leq H_2 \leq \ldots \leq G$$

of finite subgroups. Let $V$ be any irreducible $KH$-module faithful for $H$. By (a) and Lemma 2.2, any irreducible submodule of $V^{H_i}$ is faithful for $H_i$, so has dimension $|K(H_i): K|$ by Lemma 2.1. But by (b) and Lemma 2.1,

$$|K(H_i): K| = |K(H_i): K(H)| |K(H): K|$$

$$= |H_i: H| \dim_K V$$

$$= \dim_K V^{H_i}.$$  

Hence $V^{H_i}$ is itself irreducible. Now $V^G$ may be regarded as the union of the $V^{H_i}$, so is also irreducible. Thus $H$ is $K$-inductive in $G$. 

COROLLARY (2.4): A finite subgroup $H$ of $G$ is $K$-inductive if and only if there exists an irreducible $KH$-module $V$ faithful for $H$ such that $V^G$ is irreducible.

PROOF: If such a $V$ exists then by the first half of the proof of Lemma 2.3 $H$ satisfies (a) and (b); then by the second half $H$ is $K$-inductive. The converse follows from Lemma 2.1.

Note also that if $H \leq L \leq G$ and $L$ is finite then in any case we have

$$|K(L): K(H)| \leq |L: H|.$$ 

For if $m = |L: H|$ and the subgroup of $K(L)^*$ isomorphic to $L$ is generated by $\xi$, then $\xi^m \in K(H)$, so the polynomial $f(X) = X^m - \xi^m$ has degree $m$ over $K(H)$ and $\xi$ as a root. Hence $|K(L): K(H)| = |K(\xi): K(H)| \leq m$.

LEMMA (2.5): Let $F$ and $K$ be subfields of some field. Then

$$|KF: F| \leq |K: K \cap F|.$$  

(Here the ring $KF$ may or may not be a field.)

PROOF: Any basis of $K$ over $K \cap F$ also spans $KF$ over $F$.

THEOREM (2.6): Let $G$ be a locally cyclic group with $\text{Min}$, and $K$ a field with char $K \notin \pi(G)$. If there exists any $K$-inductive subgroup in $G$, then there exists a unique minimal $K$-inductive subgroup in $G$.

PROOF: Since $K$-inductive subgroups are finite, it is sufficient to show that if $H_1$ and $H_2$ are $K$-inductive in $G$, then so is $H_1 \cap H_2$. But let $H_1$ be $K$-inductive, and $H_2$ any subgroup of $G$. Then

$$\Omega(H_2) \leq \Omega(G) \cap H_2 \leq H_1 \cap H_2.$$ 

Moreover, if $L$ is a finite subgroup of $H_2$ containing $H_1 \cap H_2$, then $H_1 \cap H_2 = H_1 \cap L$, so

$$|K(L): K(H_1 \cap H_2)| = |K(L): K(H_1 \cap L)|$$

$$\geq |K(L): K(H_1) \cap K(L)|$$

$$\geq |K(H_1)K(L): K(H_1)|$$

by Lemma 2.5. Since $LH_1$ is cyclic, we have $K(H_1)K(L) = K(LH_1)$. So as
$H_1$ is $K$-inductive in $G$, 

$$|K(G): K(H_1 \cap H_2)| \geq |K(LH_1): K(H_1)|$$

$$= |LH_1 : H_1|$$

$$= |L : H_1 \cap L|$$

$$= |L : H_1 \cap H_2|.$$ 

But $|K(L): K(H_1 \cap H_2)| \leq |L : H_1 \cap H_2|$ by the remark following Corollary 2.4, so by Lemma 2.3 $H_1 \cap H_2$ is $K$-inductive in $H_2$. If now $H_2$ is also $K$-inductive in $G$, it easily follows that $H_1 \cap H_2$ is $K$-inductive in $G$. This completes the proof.

We shall now investigate more closely the conditions under which a locally cyclic group with Min contains inductive subgroups for various fields.

**Lemma (2.7):** Let $G$ be a locally cyclic group with Min. Then $\Omega(G)$ is $Q$-inductive in $G$.

**Proof:** Suppose $L$ is a finite subgroup of $G$ containing $H = \Omega(G)$, and let $\varepsilon$ be a primitive $|L|$-th root of unity. Then

$$|\Omega(L): \Omega| = |\Omega(\varepsilon): \Omega| = \varphi(|L|),$$

where $\varphi$ is the Euler function. Thus

$$|\Omega(L): \Omega(H)| = \varphi(|L|)/\varphi(|H|)$$

$$= \varphi(|L : H||H|)/\varphi(|H|)$$

$$= |L : H|,$$

for $\pi(L) = \pi(H)$ and if $p$ is a prime dividing an integer $m$, then $\varphi(pm) = p\varphi(m)$. Hence $\Omega(G) = H$ is $Q$-inductive in $G$ by Lemma 2.3.

If $m$ and $n$ are positive integers, their highest common factor is denoted by $(m, n)$. If $(m, n) = 1$, we will denote by $o(m, n)$ the order of $m$ modulo $n$, i.e. the smallest positive integer $r$ such that $nm^r - 1$. If $G$ is a locally cyclic group with Min, say

$$G \cong C_{p_1^{n_1}} \times \ldots \times C_{p_k^{n_k}},$$

where the $p_i$ are distinct primes and $1 \leq n_i \leq \infty$, then $N = p_1^{n_1} \ldots p_k^{n_k}$
will be called the Steinitz number associated with $G$. Evidently the concepts of divisibility and highest common factor extend to Steinitz numbers.

The following is a slightly strengthened form of Lemma 2.2 in [2].

**Lemma (2.8):** Let $G$ be a locally cyclic group with $\text{Min}$, and $\mathbb{F}_{p^d}$ a finite field of order $p^d$, with $p \not\in \pi(G)$. Let $N$ be the Steinitz number associated with $G$, and put

$$n = (N, 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots),$$

$$r = o(p^d, n),$$

$$m = (N, p^{rd} - 1).$$

Then the unique subgroup $H$ of order $m$ in $G$ is $\mathbb{F}_{p^d}$-inductive in $G$.

**Proof:** Since $n|p^{rd} - 1$, we have $n|m$, whence $\Omega(G) \leq H$. Let $L$ be a finite subgroup of $G$ containing $H$. Then $L$ is cyclic and $\mathbb{F}_{p^d}(L)$ is the smallest extension of $\mathbb{F}_{p^d}$ such that $L$ may be embedded in $\mathbb{F}_{p^{rd}}$, i.e. such that $l = |L|$ divides $|\mathbb{F}_{p^d}^*| = p^{dt} - 1$. Hence $t$ is the smallest positive integer such that $l|p^{dt} - 1$, so we have

$$|\mathbb{F}_{p^d}(L): \mathbb{F}_{p^d}| = t = o(p^d, l).$$

By Lemma 2.3, to show that $H$ is $\mathbb{F}_{p^d}$-inductive in $G$ it is sufficient to prove that $|\mathbb{F}_{p^d}(L): \mathbb{F}_{p^d}(H)| = |L: H|$, i.e. that if $m|l|N$ then

$$\frac{o(p^d, l)}{o(p^d, m)} = \frac{l}{m}.$$

Note that $o(p^d, m) = r$, for since $n|m$, $r = o(p^d, n)|o(p^d, m)$, while as $m|p^{rd} - 1$, $o(p^d, m)|r$. We will prove by induction on $l/m$ (more precisely, on the sum of the exponents in the prime power factors of $l/m$) that if $o(p^d, l) = t$ and $p^{dt} - 1 = kl$, then $(k, N/m) = 1$, and $t/r = l/m$.

Firstly, let $l = m$, so $t = r$. Write $p^{rd} - 1 = km$. Then

$$(km, N) = (p^{rd} - 1, N) = m,$$

so $(k, N/m) = 1$. Also $t/r = 1 = l/m$.

Now suppose that $m|l|q|N$, where $q$ is a prime. Let $t = o(p^d, l)$ and $p^{dt} - 1 = kl$. By induction we may assume that $(k, N/m) = 1$ and that $t/r = l/m$. We then have
Let $q_1 | N$ be prime. If $q_1 \neq q$ then as $qq_1 | l$ we have

$$p^{aq} \equiv 1 + qkl \pmod{qq_1}.$$ 

If $q_1 = q$ we have $q | l$ so (since $q | l^s$ for $s = 2, \ldots, q - 1$)

$$p^{aq} \equiv 1 + qkl + (kl)^q \pmod{lq^2},$$

whence

$$p^{aq} \equiv 1 + qkl \pmod{lq^2}$$

provided $q > 2$. But if $q = 2$ then $2^2 | lq | N$ whence $2^2 | n | m | l$, and again we obtain

$$p^{aq} \equiv 1 + qk l \pmod{lq^2}.$$ 

In particular we see that $lq | p^{aq} - 1$, so $t' = o(p^d, lq) | tq$. Moreover, $l | lq$, so $t = o(p^d, l) | t'$. If $lq | p^{aq} - 1 = kl$, then $q | k$. But $m | lq | N$, so $q | (N/m)$, a contradiction as $(k, N/m) = 1$. Hence $lq \nmid p^{aq} - 1$. Thus $t | t' | tq$, but $t \neq t'$, so $o(p^d, lq) = t' = tq$. We have

$$t'/r = tq/r = lq/m.$$ 

Now write $p^{aq} - 1 = k'lq$. By the above congruences, if $q_1$ is any prime divisor of $N$, we have

$$k'lq \equiv klq \pmod{lqq_1},$$

whence

$$k' \equiv k \pmod{q_1}.$$ 

Thus if $q_1 | (k', N/m)$ then $q_1 | (k, N/m) = 1$, a contradiction. Hence $(k', N/m) = 1$. This completes the induction, and the proof.

It can be shown that $H$ is the minimal $F_{p^d}$-inductive subgroup of $G$ unless $|O_2(G)| = 4$ and $p^d \equiv 3 \pmod{4}$, in which case the subgroup of index 2 in $H$ is minimal inductive.

**Lemma (2.9):** Let $D$ and $E$ be subfields of some field, and suppose that $E$
is a finite normal extension of $D \cap E$. Then
(a) $D$ and $E$ are linearly disjoint over $D \cap E$;
(b) if $F$ is a subfield of $E$ containing $D \cap E$ then $FD \cap E = F$.

**Proof:**
(a) $E$ is the splitting field of some monic irreducible polynomial $f$ over $D \cap E$. In fact $f$ is still irreducible over $D$. For if $f = gh$, where $g$ and $h$ are monic polynomials over $D$, then the roots of $g$ and $h$ are roots of $f$, so all lie in $E$. The coefficients of $g$ and $h$ are (plus or minus) elementary symmetric functions in the roots, so lie in $D \cap E$. But $f$ is irreducible over $D \cap E$, so over $D$ too.

Let $n$ be the degree of $f$, and $\xi$ one of its roots. Then $\{1, \xi, \ldots, \xi^{n-1}\}$ is a basis of $E$ over $D \cap E$, consisting of elements which are linearly independent over $D$. So $D$ and $E$ are linearly disjoint over $D \cap E$.

(b) Let $\omega_i$ be a basis of $D$ over $D \cap E$, with $\omega_1 = 1$. Then $FD = \sum F\omega_i$.

By (a), the $\omega_i$ are linearly independent over $E$ (see Chapter IV Section 5 of [3]). Suppose

$$\beta = \sum \alpha_i \omega_i \in FD \cap E \quad (\alpha_i \in F).$$

Then

$$(\alpha_1 - \beta)\omega_1 + \sum_{i \neq 1} \alpha_i \omega_i = 0 \quad (\alpha_1 - \beta, \alpha_i \in E)$$

so $\beta = \alpha_1 \in F$. Thus $FD \cap E = F$.

**Theorem (2.10):** Let $K$ be any field, $k$ its prime field, and $G$ a locally cyclic group satisfying Min with char $k \notin \pi(G)$. Then $G$ has a $K$-inductive subgroup if and only if

$$|k(G) \cap K : k| < \infty.$$  

(Here $k(G) \cap K$ is a subfield of $\bar{K}$, in which $\bar{k}$ and $k(G)$ are embedded.)

**Proof:** Suppose $H$ is a $K$-inductive subgroup of $G$, and that $L$ is a finite subgroup of $G$ containing $H$. Then by the remark following Corollary 2.4 we have $|k(L) : k(H)| \leq |L : H| = |K(L) : K(H)|$ (as $H$ is $K$-inductive). Now $K(L) = k(L) \cdot K(H)$, so by Lemma 2.5

$$|K(L) : K(H)| = |k(L) \cdot K(H) : K(H)| \leq |k(L) : k(L) \cap K(H)| \leq |k(L) : k(H)|$$
(as $k(H) \leq k(L) \cap K(H)$). We now have $|k(L) : k(L) \cap K(H)| = |k(L) : k(H)|$,
whence

$$k(L) \cap K \leq k(L) \cap K(H) = k(H).$$

As $G$ is locally finite it follows that $k(G) \cap K \leq k(H)$. Hence

$$|k(G) \cap K : k| \leq |k(H) : k| \leq |H| < \infty.$$

Conversely, suppose that $|k(G) \cap K : k| < \infty$; say $k(G) \cap K = k(\gamma)$. By
Lemma 2.7 or 2.8, as $k$ is a prime field, $G$ contains a $k$-inductive subgroup $H_1$. Since $G$ is locally finite, there exists a finite subgroup $H$ of $G$ containing $H_1$ and such that $\gamma \in k(H)$. Then

$$k(G) \cap K = k(\gamma) \leq k(H).$$

We will show that $H$ is $K$-inductive in $G$. Note first that $H \supseteq H_1 \supseteq \Omega(G)$ by Lemma 2.3.

Let $L$ be a finite subgroup of $G$ containing $H$. Then the cyclotomic field $k(L)$ is a finite normal extension of $k(L) \cap K$; moreover

$$k(L) \cap K \leq k(G) \cap K \leq k(H).$$

Hence by Lemma 2.9(b), with $D = K$, $E = k(L)$, and $F = k(H)$, we have

$$K(H) \cap k(L) = (K \cdot k(H)) \cap k(L) = k(H).$$

By Lemma 2.9(a), $K(H) (= D)$ and $k(L) (= E)$ are linearly disjoint over their intersection $k(H)$. Hence a basis for $k(L)$ over $k(H)$ also constitutes a basis for $K(L) = K(H) \cdot k(L)$ over $K(H)$. Thus

$$|K(L) : K(H)| = |k(L) : k(H)|$$

$$= |k(L) : k(H_1)|/|k(H) : k(H_1)|$$

$$= |L : H_1|/|H : H_1|$$

$$= |L : H|$$

as $H_1$ is $k$-inductive. By Lemma 2.3, $H$ is $K$-inductive in $G$.

**Corollary (2.11):** Let $K$ be any field, $k$ its prime field, and $G$ a periodic abelian group with $\text{char } k \notin \pi(G)$. Suppose that
Then every locally cyclic quotient of $G$ satisfying Min contains a $K$-inductive subgroup.

**Proof:** If $G$ is any quotient of $G$, every image of $G$ in $k^*$ is also an image of $G$, and therefore $k(G) \leq k(G)$. Now apply Theorem 2.10.

### 3. Primitive idempotents in $KG$

Let $G$ be an abelian group and $K$ a field. If $\alpha = \sum g \in KG$, we denote by $\text{supp} \, \alpha$ the finite set $\{g \in G : \alpha(g) \neq 0\}$. We will write

$$C_\alpha(\alpha) = \{g \in G : \alpha g = \alpha\}.$$

Since $G$ is abelian, $C_\alpha(\alpha)$ is in fact the centralizer $C_\alpha(\alpha KG)$ in $G$ of $\alpha KG$ considered as a $KG$-module. If $e$ is an idempotent in $KG$, we say $e$ is **faithful** (for $G$) if $C_\alpha(e) = 1$.

**Lemma (3.1):** Let $G$ be a periodic abelian group and $K$ a field with $\text{char} \, K \neq \pi(G)$. Suppose $KG$ contains a primitive idempotent $e$. Then $G$ satisfies Min and is almost locally cyclic (i.e. has a locally cyclic subgroup of finite index). If $e$ is faithful, $G$ is locally cyclic, and $\langle \text{supp} \, e \rangle$ is $K$-inductive in $G$.

**Proof:** Let $H = \langle \text{supp} \, e \rangle$, a finite subgroup of $G$. Then $eKH$ is an irreducible $KH$-module, and $eKH[0] = eKG$ is an irreducible $KG$-module (for otherwise $G$ would contain a finite subgroup $L \geq H$ with $eKL$ reducible; but $e$ is primitive in $KL$). As in the proof of Lemma 2.3, it follows that $H \geq \Omega(G)$, whence $\Omega(G)$ is finite and $G$ satisfies Min. If $e$ is faithful for $G$ so for $H$, then $H$ is $K$-inductive in $G$ by Corollary 2.4.

The group $C = C_\alpha(e)$ is finite, since it acts as a group of permutations on the finite set $\text{supp} \, e$. The irreducible $KG$-module $eKG$, considered as a ring, is actually a $g$-ring. The homomorphism $G \to F^*, g \mapsto eg$ has kernel $C$. Hence $G/C$ embeds in $F^*$ so is locally cyclic. Let $|C| = m$. Since $G$ is abelian, $G^m = \{g^m : g \in G\}$ is a quotient of $G$ and indeed of $G/C$, as $C = 1$. Thus $G^m$ is locally cyclic. But $G/G^m$ has finite exponent and satisfies Min, so is finite. Hence $G$ is almost locally cyclic. If $e$ is faithful then $m = 1$ and $G$ itself is locally cyclic. This completes the proof.

We shall now investigate the circumstances under which $KG$ contains primitive idempotents faithful for $G$, given that $G$ is locally cyclic and satisfies Min. We shall need:
LEMMA (3.2): Let $G$ be a periodic abelian group, $K$ a field with 
$\text{char } K \notin \pi(G)$, and $H_0 \subseteq H_1 \subseteq \ldots \subseteq G$ a chain of finite subgroups with union $G$. For each $i$, let $e_i$ be a primitive idempotent in $KH_i$, such that
$e_i e_{i+1} = e_{i+1}$. Then there exists a maximal ideal $M$ of $KG$ such that:

(a) for each $i$, $1-e_i \in M$ and $e_i \notin M$;
(b) $C_O(KG/M) = \bigcup_{i=0}^{\infty} C_O(e_i)$.

PROOF: For each $i$, write

$$KH_i = e_i KH_i \oplus M_i,$$

where $M_i = (1-e_i)KH_i$ is a maximal ideal of $KH_i$. We have

$$(1-e_{i+1})(1-e_i) = 1-e_i,$$

whence

$$M_i = (1-e_i)KH_i \subseteq (1-e_i)KH_{i+1} \subseteq (1-e_{i+1})KH_{i+1} = M_{i+1}.$$

Since $G = \bigcup_{i=0}^{\infty} H_i$, $M = \bigcup_{i=0}^{\infty} M_i$ is an ideal of $KG$. Moreover $e_0 \notin M$, for if $e_0 \in M_i$ then $e_0 e_i = 0$, but then $e_i = e_i e_{i-1} \ldots e_1 e_0 = 0$. Thus $M$ is a proper ideal of $KG$; furthermore it is maximal since $M \cap KH_i = M_i$ for each $i$. For each $i$, $1-e_i \in M_i \subseteq M$, so as $1 \notin M$, $e_i \notin M$. Thus we have (a).

Let $x \in C_O(e_i)$ and $\alpha \in KG$. Choose $j \geq i$ such that $x, \alpha \in KH_j$. Since $e_j = e_j e_{j-1} \ldots e_0$, we have $x \in C_O(e_j)$. Thus $(\alpha x - \alpha) e_j = 0$, whence $\alpha x - \alpha \in (1-e_j)KH_j = M_j \subseteq M$, i.e. $(\alpha + M)x = \alpha + M$. It follows that

$$\bigcup_{i=0}^{\infty} C_O(e_i) \subseteq C_O(KG/M).$$

Conversely let $x \in C_O(KG/M)$, so that $x-1 \in M$. Choose $i$ so that $x \in H_i$. Then $x-1 \in M \cap KH_i = M_i$ (as $M_i$ is maximal in $KH_i$). Thus $e_i(x-1) = 0$, so $e_i x = e_i$ and $x \in C_O(e_i)$. This completes the proof of (b).

THEOREM (3.3): Let $G$ be a locally cyclic group with $Min$ and $K$ a field with $\text{char } K \notin \pi(G)$. Then the following are equivalent:

(a) $KG$ contains a faithful primitive idempotent;
(b) $G$ contains a $K$-inductive subgroup;
(c) there are only finitely many non-isomorphic irreducible $KG$-modules faithful for $G$;
(d) there do not exist $2^{\aleph_0}$ non-isomorphic irreducible $KG$-modules faithful for $G$;
(e) $|k(G) \cap K : k| < \infty$, where $k$ is the prime field of $K$. 
Furthermore, when (a)–(e) hold, there is a one-to-one onto correspondence between faithful primitive idempotents of \( KG \) and isomorphism classes of irreducible \( KG \)-modules faithful for \( G \).

**Proof:** (a) implies (b) by Lemma 3.1, and (b) is equivalent to (e) by Theorem 2.10.

Now suppose \( H \) is a \( K \)-inductive subgroup of \( G \), and \( V \) is an irreducible \( KG \)-module faithful for \( G \). Since \( H \) is finite, \( V_H \) is completely reducible, so it contains an irreducible \( KH \)-submodule \( W \) say. Then \( V_H = \sum_{x \in G} W_x \), and \( W_x \cong W \) as \( KH \)-modules since \( G \) is abelian. Hence

\[
C_H(W) = C_H(V_H) = 1.
\]

So as \( H \) is \( K \)-inductive, \( W^G \) is irreducible. But there is a non-zero \( KG \)-map \( W^G \to V \), \( w \otimes x \mapsto wx \), so \( V \cong W^G \). Thus every irreducible \( KG \)-module faithful for \( G \) is isomorphic to \( W^G \) for some irreducible \( KH \)-module \( W \) faithful for \( H \). (Note that \( W \cong eKH \) and \( V \cong eKG \) for some idempotent \( e \) in \( KH \) which is faithful and primitive in \( KG \).) There are only finitely many non-isomorphic such \( W \), and therefore only finitely many non-isomorphic irreducible \( KG \)-modules faithful for \( G \). Hence (b) implies (c). Trivially (c) implies (d).

The last part of the Theorem now also follows. For if \( e \) is a faithful primitive idempotent in \( KG \), then \( eKG \) is an irreducible \( KG \)-module faithful for \( G \); as we have just shown, every such module arises in this way. If \( e \) and \( f \) are idempotents in \( KG \) and \( eKG \cong f KG \), then if \( \theta : eKG \to f KG \) is an isomorphism, we have \( \theta(e) = f \theta(e) = \theta(e)f \); applying \( \theta^{-1} \) we obtain \( e = ef \). Similarly \( f = fe \), so \( e = f \).

To prove that (d) implies (a), we shall assume that \( KG \) contains no faithful primitive idempotent, and exhibit \( 2^{\aleph_0} \) non-isomorphic irreducible \( KG \)-modules faithful for \( G \). Let

\[
\Omega(G) = L_0 \leq L_1 \leq L_2 \leq \ldots \leq G
\]

be a chain of finite subgroups with union \( G \).

For \( n = 0, 1, 2, \ldots \) let \( T_n \) denote the set of all \( n \)-tuples with each entry either 0 or 1. By induction we will construct for each integer \( n \) a finite subgroup \( H_n \) of \( G \) and for each \( \varphi \in T_n \) a faithful primitive idempotent \( e_\varphi \) in \( KH_n \). Firstly, let \( H_0 = L_0 = \Omega(G) \). By Lemma 2.1, \( KH_0 \) contains a faithful primitive idempotent \( e \).

Now suppose inductively that we have constructed \( H_n \) and \( \{ e_\varphi : \varphi \in T_n \} \). By Lemma 2.2 each \( e_\varphi \) is faithful for \( G \), so by hypothesis is not primitive in \( KG \). Hence we may choose a finite subgroup \( H_{n+1} \) of \( G \) containing \( L_{n+1} \) and such that for each \( \varphi \in T_n \), \( e_\varphi \) decomposes in \( KH_{n+1} \); say
where $e(\varphi, 0)$ and $e(\varphi, 1)$ are primitive idempotents in $KH_{n+1}$. By Lemma 2.2, since $e(\varphi KH_{n+1}) = e(\varphi KH_n)|H_{n+1}$, $e(\varphi, 0)$ and $e(\varphi, 1)$ are faithful for $H_{n+1}$. Thus we have chosen $e(\varphi)$ for each $\varphi' \in T_{n+1}$. This completes the inductive construction. Note that

$$\bigcup_{i=0}^{\infty} H_i = \bigcup_{i=0}^{\infty} L_i = G.$$  

Let $\varphi = (a_1, a_2, a_3, \ldots)$ be an infinite sequence of 0's and 1's. Write $e_0(\varphi) = e$ and $e_n(\varphi) = e(a_1, \ldots, a_n)$ ($n = 1, 2, 3, \ldots$). By Lemma 3.2 there is a maximal ideal $M = M(\varphi)$ of $KG$ with $1 - e_n(\varphi) \in M(\varphi)$ and $e_n(\varphi) \notin M(\varphi)$ for all $n$, and

$$C_G(KG/M(\varphi)) = \bigcup_{n=0}^{\infty} C_G(e_n(\varphi)) = 1.$$  

Thus $V(\varphi) = KG/M(\varphi)$ is an irreducible $KG$-module faithful for $G$.

If $\varphi \neq \psi$ then $V(\varphi)$ and $V(\psi)$ are not $KG$-isomorphic. For if $\varphi$ and $\psi$ differ first in the $n$-th place, then $e_n(\varphi)e_n(\psi) = 0$; hence

$$e_n(\psi) = e_n(\psi)(1 - e_n(\varphi)) \in M(\varphi),$$  

so $e_n(\psi)$ annihilates $V(\varphi)$. But $1 - e_n(\psi) \in M(\psi)$, so $e_n(\psi)$ acts as the identity on $V(\psi)$. This completes the proof of the Theorem.

In Lemma 2.12 of \cite{1}, S. D. Berman proves a result related to part of Theorem 3.3, for the special case of abelian $p$-groups. Note that a field $K$ with prime field $k$ is "of the first kind for $p$", in Berman's terminology, if and only if $|k(Cp^\infty) \cap K| : k| \leq \infty$.

The following corollary to Theorem 3.3 generalizes Lemma 2.5 of \cite{2}.

THEOREM (3.4): Let $K$ be a field, $k$ its prime field, and $G$ an abelian almost locally cyclic group with $Min$ such that $\text{char } k \notin \pi(G)$. If $|k(G) \cap K| : k| = \infty$, then $KG$ contains no primitive idempotents. Suppose that $|k(G) \cap K| : k| < \infty$. If $C$ is any finite subgroup of $G$ such that $G/C$ is locally cyclic, then $KG$ contains a non-zero finite number of primitive idempotents $e$ with $C_G(e) = C$, and there is a one-to-one onto correspondence between such idempotents and isomorphism classes of irreducible $KG$-modules $V$ with $C_G(V) = C$.

PROOF: Let $C$ be any finite subgroup of $G$. We may write
where \( cG \) is the ideal of \( KG \) generated by the augmentation ideal \( c \) of \( KC \), and \( v \) is the idempotent

\[
\frac{1}{|C|} \sum_{x \in C} x.
\]

It is easily deduced that the canonical group ring projection

\[
KG \to K[G/C] \quad (\cong KG/cG \cong vKG)
\]
determines a one-to-one map from the set of primitive idempotents \( e \) in \( KG \) with \( C_{G}(e) = C \) onto the set of faithful primitive idempotents in \( K[G/C] \). (Both these sets might be empty.)

Suppose \( KG \) contains a primitive idempotent \( e \); we will show that

\[
|k(G) \cap K: k| < \infty.
\]

Let \( C = C_{G}(e) \). By the above the image of \( e \) in \( K[G/C] \) is a primitive idempotent faithful for \( G/C \). Thus \( G/C \) is locally cyclic, and by Theorem 3.3

\[
|k(G/C) \cap K: k| < \infty.
\]

Since every image of \( G/C \) is an image of \( G \), we have \( k(G/C) \leq k(G) \). Now let

\[
F = k\left(\prod O_{p}(G)\right)
\]

where the product is taken over those primes \( p \) such that \( O_{p}(G) \) is finite. Then \( |F: k| < \infty \) since \( G \) satisfies Min. Moreover \( k(G) = F \cdot k(G/C) \). For \( k(G) \) is determined by the exponents of the primary components of \( G \), and since \( C \) is finite, if \( \exp O_{p}(G) = \infty \) then \( \exp O_{p}(G/C) = \infty \). Hence by Lemma 2.5,

\[
|k(G): k(G/C)| = |F \cdot k(G/C): k(G/C)| \leq |F: k| < \infty.
\]

Now \( k(G/C) \) is a union of finite normal extensions of \( k \), so also of \( k(G/C) \cap K \); Lemma 2.9(a) together with a local argument shows that \( k(G/C) \) and \( K \) are linearly disjoint over \( k(G/C) \cap K \). In particular, any subset of \( k(G) \cap K \) which is linearly independent over \( k(G/C) \cap K \) is a subset of \( k(G) \) which is linearly independent over \( k(G/C) \). Hence

\[
|k(G) \cap K: k(G/C) \cap K| \leq |k(G): k(G/C)| < \infty.
\]

We now have
Now suppose that $|k(G) \cap K : k| = |k(G) \cap K : k(G/C) \cap K||k(G/C) \cap K : k| < \infty$.

Now suppose that $|k(G) \cap K : k| < \infty$, and that $C$ is a finite subgroup of $G$ such that $G/C$ is locally cyclic. Since $k(G/C) \leq k(G)$ we also have $|k(G/C) \cap K : k| < \infty$. In view of the one-to-one correspondence mentioned in the first paragraph of this proof, an application of Theorem 3.3 to $K[G/C]$ yields the remaining statements of Theorem 3.4.

4. The socular series of $KG$

If $V$ is a module recall that the socle $So(V)$ of $V$ is the sum of all irreducible submodules of $V$. We define the ascending socular series of $V$ by

$$
So_0(V) = 0
$$

$$
So_1(V) = So(V)
$$

$$
\frac{So_{n+1}(V)}{So_n(V)} = So \left( \frac{V}{So_n(V)} \right), \quad n = 1, 2, 3, \ldots
$$

In particular if $A$ is a commutative ring, we obtain an ascending socular series of $A$ considered as an $A$-module.

**Lemma (4.1):** Let $G$ be a locally finite group and $K$ a field with $\text{char } K \notin \pi(G)$. Then the socle of $KG$ (considered as left or right $KG$-module) contains and is generated by all primitive idempotents in $KG$. 
PROOF: We consider the right module case; the proof for the left module case is analogous. If \( e \) is a primitive idempotent in \( KG \) then \( eKG \) is irreducible, for otherwise as \( G \) is locally finite there exists a finite subgroup \( H \) of \( G \) with \( e \in KH \) such that \( eKH \) is reducible, a contradiction as \( KH \) is completely reducible and \( e \) is primitive in \( KH \). Hence \( e \in eKG \leq So(KG_{kg}) \).

Let \( N \) be a minimal right ideal of \( KG \). Since \( G \) is locally finite there exists a finite subgroup \( H \) of \( G \) with \( KH \cap N \neq 0 \). As \( KH \) is completely reducible, \( KH \cap N \) contains an idempotent \( e \). Then \( N = eKG \), so \( e \) is primitive in \( KG \). Hence \( So(KG_{kg}) \) is generated as a right ideal by the primitive idempotents of \( KG \).

THEOREM (4.2): Let \( K \) be a field with prime field \( k \), and \( G \) a periodic abelian group such that \( \text{char } k \notin \pi(G) \). If \( |k(G) \cap K : k| = \infty \), then the socle of \( KG \) is zero. If \( |k(G) \cap K : k| < \infty \), then the socle of \( KG \) is the intersection \( T \) of the maximal ideals \( M \) of \( KG \) such that \( C_G(KG/M) \) is infinite.

PROOF: If \( |k(G) \cap K : k| = \infty \), then by Lemma 3.1 and Theorem 3.4, \( KG \) contains no primitive idempotents. Hence \( So(KG) = 0 \) by Lemma 4.1. Now assume that \( |k(G) \cap K : k| < \infty \).

Suppose that \( N \) is a minimal ideal of \( KG \), \( M \) is a maximal ideal, and \( N \leq M \). Then \( KG = N \oplus M \), so \( C_G(KG/M) = C_G(N) \). Let \( 0 \neq \alpha \in N \); then \( C_G(N) \) is contained in \( C_G(\alpha) \), which is finite since it acts as a group of permutations on \( \text{supp } \alpha \). Hence \( C_G(KG/M) \) is finite. It follows that \( So(KG) \leq T \).

To show that \( T \leq So(KG) \), suppose \( 0 \neq \alpha \in T \). Let \( H = \langle \text{supp } \alpha \rangle \), and write

\[
\alpha = \alpha e_1 + \ldots + \alpha e_m
\]

where the \( e_i \) are orthogonal primitive idempotents in \( KH \), and \( \alpha e_i \neq 0 \) for each \( i \). Since \( e_iKH \) is irreducible, \( \alpha e_iKH = e_iKH \), so there exists \( \beta \in KH \) such that \( e_i = \alpha e \beta_i \); thus \( e_i \in T \). Hence it is sufficient to show that if \( H \) is a finite subgroup of \( G \), \( e \) is a primitive idempotent in \( KH \), and \( e \in T \), then \( e \in So(KG) \), i.e. if \( e \notin So(KG) \) then \( e \notin T \).

If \( C_G(KG/M) \) is infinite for all maximal ideals \( M \) of \( KG \), then \( T \) is the Jacobson radical of \( KG \). But \( KG \) is semisimple (see Theorem 18.7 of [5]), so \( T = 0 \leq So(KG) \) as required. Hence we may assume that there exists a maximal ideal \( M \) of \( KG \) with \( C = C_G(KG/M) \) finite. Then \( G/C \) embeds in the multiplicative subgroup of the field \( KG/M \), so is locally cyclic whence countable. Thus \( G \) is also countable. Hence there
exists a chain

\[ H = H_0 \leq H_1 \leq \ldots \leq G \]

of finite subgroups with union \( G \).

Assume first that \( G \) does not satisfy \( \text{Min} \). Then by Lemmas 3.1 and 4.1 \( \text{So}(KG) = 0 \), so the condition that \( e \notin \text{So}(KG) \) is vacuous; in effect we must show that \( T \) is also zero. We shall construct by induction a subchain \( H_{n_0} \leq H_{n_1} \leq \ldots \) of \( H_0 \leq H_1 \leq \ldots \) and for each \( i \) a primitive idempotent \( e_i \) in \( KH_{n_i} \) such that \( e_i e_{i+1} = e_{i+1} \). Firstly, let \( n_0 = 0 \) and \( e_0 = e \). Suppose we have already found \( n_i \) and \( e_i \). Since \( G \) does not satisfy \( \text{Min} \) and \( C_G(e_i) \) is finite, \( \Omega(G) \) is not contained in \( C_G(e_i) \), so there exists a non-trivial finite subgroup \( L_i \) of \( G \) with \( C_G(e_i) \cap L_i = 1 \). Choose \( n_{i+1} \) such that \( H_{n_{i+1}} \geq L_i H_{n_i} \). Let

\[ n_i = \frac{1}{|L_i|} \sum_{x \in L_i} x \]

be the trivial primitive idempotent in \( KL_{n_i} \) and choose a primitive idempotent \( e_{i+1} \) in \( KH_{n_{i+1}} \) such that \( (e_i e_{i+1}) e_{i+1} = e_{i+1} \); then also \( e_i e_{i+1} = e_{i+1} \). Now \( L_i \leq C_G(e_{i+1}) \), so \( C_G(e_i) \nsubseteq C_G(e_{i+1}) \). By Lemma 3.2 there exists a maximal ideal \( M \) of \( KG \) such that \( e = e_0 \notin M \), and

\[ C_G(KG/M) = \bigcup_{i=0}^{\infty} C_G(e_i), \]

which by construction is infinite. Thus \( e \notin T \) as required. Hence we may assume that \( G \) satisfies \( \text{Min} \).

If \( f \) is a primitive idempotent in \( KH_n \) for some \( n \geq 0 \), consider the set of all sequences \( (f_{n'}, f_{n'+1}, \ldots) \) such that

(i) \( f_i \) is a primitive idempotent in \( KH_i \) for all \( i \geq n \);
(ii) \( f_n = f \);
(iii) \( f_i f_{i+1} = f_{i+1} \) for all \( i \geq n \).

If \( m \geq 0 \) we shall say that \( f \) is \( m \)-stationary if for all such sequences \( (f_{n'}, f_{n'+1}, \ldots) \) and all \( i \geq 0 \) we have \( f_{n+m} = f_{n'+m+i} \). Note that if

\[ f = f'_1 + \ldots + f'_l \]

where the \( f'_j \) are orthogonal primitive idempotents in \( KH_{n+1} \), then \( f \) is \( m \)-stationary (for \( m \geq 1 \)) if and only if each \( f'_j \) is \( (m-1) \)-stationary. Moreover \( f \) is 0-stationary if and only if it is primitive in \( KG \). Hence if \( f \) is \( m \)-stationary and we write \( f \) as a sum of orthogonal primitive
idempotents in $KH_{n+m}$, then each such idempotent will be 0-stationary; thus by Lemma 4.1 we have $f \in So(KG)$.

Now let $e$ be a primitive idempotent in $KH$ with $e \notin So(KG)$. Then $e = e_0$ is not $m$-stationary for any $m$. Hence among the finitely many orthogonal primitive idempotents in $KH_1$ whose sum is $e_0$, there must exist one, say $e_1$, which is not $m$-stationary for any $m$. Similarly we may choose a primitive idempotent $e_2$ in $KH_2$ which satisfies $e_1 e_2 = e_2$ and is not $m$-stationary for any $m$, and so on. In this way we obtain a sequence $e_0 = e, e_1, e_2, \ldots$ such that $e_i$ is a primitive idempotent in $KH_i$, and $e_i e_{i+1} = e_{i+1}$.

Consider the chain of subgroups $C_{G}(e_0) \supseteq C_{G}(e_1) \supseteq \ldots$, and suppose that $C = \bigcup_{i=0}^{\infty} C_{G}(e_i)$ is finite; then $C = C_{G}(e_n)$ for some $n$. For $i \geq n$, $e_i KH_i$ is an irreducible module faithful for $H_i/C$, so $H_i/C$ is cyclic; hence $G/C$ is locally cyclic. Also $|k(G/C) \cap K : k| \leq |k(G) \cap K : k| < \infty$, so by Theorem 2.10 $G/C$ contains a $K$-inductive subgroup. Thus we may choose $s \geq n$ so that $H_s/C$ is $K$-inductive in $G/C$. But $e_s$ is a primitive idempotent in $KH_s$ with $C_{G}(e_s) = C$, so $e_s$ is primitive in $KG$, i.e. 0-stationary, a contradiction. It follows that $\bigcup_{i=0}^{\infty} C_{G}(e_i)$ is infinite, whence by Lemma 3.2 there is a maximal ideal $M$ of $KG$ such that $e_0 \not\in M$ and $C_{G}(KG/M) = \bigcup_{i=0}^{\infty} C_{G}(e_i)$ is infinite. Hence $e \notin T$. This completes the proof of the theorem.

As an example we may take $G$ to be a Prüfer group and $K$ any field satisfying the hypotheses of Theorem 4.2. Then the augmentation ideal $g$ of $KG$ is the only maximal ideal $M$ such that $C_{G}(KG/M)$ is infinite. Hence $So(KG) = g$, a result obtained by W. Müller in [4] in the case where $K$ is a subfield of the field of complex numbers. But $KG/g$ is the trivial irreducible $KG$-module, so $So_2(KG) = KG$. The next theorem generalizes this observation.

**Theorem (4.3):** Let $K$ be a field with prime field $k$, and $G$ an abelian almost locally cyclic group with $Min$ such that $\text{char } k \notin \pi(G)$ and $|k(G) \cap K : k| < \infty$. Let $m$ be the number of factors in a decomposition of the maximal divisible subgroup of $G$ as a direct product of Prüfer groups. Then the ascending socular series of $KG$ reaches $KG$ after exactly $m+1$ steps, i.e. $So_m(KG) \neq KG = So_{m+1}(KG)$.

**Proof:** We may write

$$G = F \times \prod_{i=1}^{m} P_i,$$

where $F$ is finite and for $i = 1, \ldots, m$ $P_i$ is a Prüfer $p_i$-group, where the
\( p_i \) are distinct primes. We proceed by induction on \( m \). If \( r_* = 0 \) then \( G \) is finite, so \( KG \) is completely reducible and \( \text{So}(KG) = KG \).

Suppose \( m \geq 1 \). Let \( \varphi_i : KG \to K[G/P_i] \) be the canonical projection of group rings, and define a \( KG \)-homomorphism \( \theta \) by the commutativity of the diagrams

\[
\begin{array}{ccc}
KG & \longrightarrow & \bigoplus_{i = 1}^{m} K[G/P_i] \\
\downarrow \varphi_i & & \downarrow \\
K[G/P_i] & & \\
\end{array}
\]

Then

\[
\ker \theta = \bigcap_{i = 1}^{m} \ker \varphi_i = \bigcap_{i = 1}^{m} p_i G,
\]

where \( p_i G \) is the ideal of \( KG \) generated by the augmentation ideal \( p_i \) of \( KP_i \).

Since \( KG/p_i G \cong K[G/P_i] \) and \( K[G/P_i] \) is semisimple, it follows that \( p_i G \) is the intersection of the maximal ideals \( M \) of \( KG \) containing it. But if \( M \supseteq p_i G \) then \( C_G(KG/M) \) contains \( P_i \) so is infinite. Thus \( \ker \theta \) is the intersection of certain maximal ideals \( M \) with \( C_G(KG/M) \) infinite, so by Theorem 4.2 \( \ker \theta \supseteq \text{So}(KG) \). On the other hand if \( M \) is any maximal ideal of \( KG \) with \( C_G(KG/M) \) infinite, then \( C_G(KG/M) \) contains \( P_i \) for some \( i \), whence \( \ker \theta \leq p_i G \leq M \). Thus by Theorem 4.2 again we have \( \ker \theta \leq \text{So}(KG) \). Therefore \( \ker \theta = \text{So}(KG) \).

Hence \( \theta \) induces a \( KG \)-monomorphism

\[
\frac{KG}{\text{So}(KG)} \to B = \bigoplus_{i = 1}^{m} K[G/P_i].
\]

By induction, the ascending socular series of \( K[G/P_i] \) (as \( K[G/P_i] \)-module) reaches \( K[G/P_i] \) after exactly \( m \) steps. Thus the ascending socular series of \( B \) (as \( KG \)-module) reaches \( B \) after \( m \) steps, i.e. \( \text{So}_m(B_{KG}) = B \).

Hence

\[
\frac{\text{So}_{m+1}(KG)}{\text{So}(KG)} = \text{So}_m \left( \frac{\frac{KG}{\text{So}(KG)}}{KG} \right) = \frac{KG}{\text{So}(KG)},
\]

whence \( \text{So}_{m+1}(KG) = KG \). If \( \text{So}_m(KG) = KG \) then we would have


\[ S_{n-1} \left( \frac{KG}{So(KG)} \right) = \frac{KG}{So(KG)}, \]

a contradiction as \( K[G/P_i] \) is a quotient of \( KG/So(KG) \) but

\[ S_{n-1}(K[G/P_i]) \neq K[G/P_i]. \]

This completes the proof of the theorem.

Despite Theorem 4.3 the group rings we have been studying do not seem to satisfy any form of the Jordan-Hölder Theorem. In fact, if \( K \) and \( G \) satisfy the hypotheses of Theorem 4.3 and \( G \) is infinite, we may enumerate the primitive idempotents of \( KG \), say as \( e_1, e_2, e_3, \ldots \). Then \( KG \) has a descending composition series

\[ KG = V_0 > V_1 > V_2 > \ldots \]

of type \( \omega \), where for \( n \geq 1 \)

\[ V_n = (1 - \sum_{i=1}^{n} e_i)KG. \]

(Since \( \bigcap_{n=0}^{\infty} V_n \) contains no primitive idempotents it is disjoint from \( So(KG) \) by Lemma 4.1, whence zero by Theorem 4.3.) For each \( n \geq 0 \) the factor \( V_n/V_{n+1} \) is isomorphic to \( e_{n+1}KG \), so \( C_0(V_n/V_{n+1}) \) is finite. Hence for example the trivial irreducible \( KG \)-module does not occur as a factor in the composition series.

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