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PRIMITIVE IDEMPOTENTS AND THE SOCLE IN GROUP RINGS OF PERIODIC ABELIAN GROUPS

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Abstract

Let K be a field and G a periodic abelian group containing no elements of order p if $\operatorname{char} K = p > 0$. We establish necessary and sufficient conditions for the group ring KG to contain primitive idempotents. We also characterize the socle of KG, and show that when the socle is non-zero the ascending socular series reaches KG after a finite number of steps.

1. Introduction

Let K be a field and G a periodic abelian group containing no elements of order p if char $K = p > 0$. We shall investigate the circumstances under which the group ring KG contains primitive idempotents. We find (Lemma 3.1 and Theorem 3.4) that the following three conditions are necessary and sufficient:

(a) G is almost locally cyclic (i.e. has a locally cyclic subgroup of finite index);

(b) G satisfies the minimum condition on subgroups; and

(c) $|k(G) \cap K : k| < \infty$, where k is the prime field of K, and $k(G)$ is a certain algebraic extension of k, to be defined in Section 2.

Note that (a) and (b) hold if and only if G has the form

$$G \cong F \times C_{p_1^\infty} \times \ldots \times C_{p_m^\infty},$$

where F is a finite abelian group and the $C_{p_i^\infty}$ are Prüfer p_i-groups for distinct primes p_i. To foreshadow the significance of (c), we remark that (c) always holds if G is finite or K is a finite extension of k, but if K is algebraically closed then (c) holds only if G is finite.

For groups G satisfying (a) and (b), we consider the connection between primitive idempotents in KG and irreducible KG-modules. When (c) holds, there is a one-to-one onto correspondence between primitive idempotents in KG and isomorphism classes of irreducible KG-modules with finite centralizer (i.e. finite kernel in G); moreover there are only finitely many non-isomorphic such modules having any fixed finite subgroup of G as centralizer (Theorem 3.4). But if (c) fails to hold the situation is quite different: there are no primitive idempotents in KG, but given any finite subgroup C of G such that G/C is locally cyclic, there exist $2^{\aleph_0}$ non-isomorphic irreducible KG-modules with centralizer C (Theorem 3.3).

In Section 4 we characterize the socle of KG: it is zero if (c) fails, and otherwise it is the intersection of certain maximal ideals of KG (Theorem 4.2). When (a), (b) and (c) hold we find that the ascending socular series of KG reaches KG after a finite number of steps, i.e. that KG has a finite series with completely reducible factors. The number of steps is one plus the number of primes involved in the maximal divisible subgroup of G (Theorem 4.3).

When G is a locally cyclic group with Min, it is convenient to consider a condition equivalent both to (c) and to the existence of primitive idempotents in KG: namely, the existence of *K-inductive subgroups* in G. We call a finite subgroup H of G K-inductive if every irreducible KH-module faithful for H remains irreducible when induced up to G. It is with the study of K-inductive subgroups that we commence.

Special cases of some of the results have been obtained in papers of Hartley [2], Berman [1], and Müller [4]; more detailed references will be given in the sequel. The author is deeply indebted to Dr Brian Hartley for his aid and encouragement in the writing of this paper.

2. *K*-Inductive subgroups

Let G be a periodic abelian group, $\pi(G)$ the set of primes p such that G has elements of order p, and K a field with char $K \notin \pi(G)$. Let KG be the group ring of G over K. Let $\bar{K}$ be an algebraic closure of K, and $\bar{K}^*$ its multiplicative group. We denote by $K(G)$ the K-subalgebra of $\bar{K}$ generated by all images of homomorphisms $G \to \bar{K}^*$; $K(G)$ is in fact a subfield of $\bar{K}$. Since the torsion subgroup of $\bar{K}^*$ is a direct product of Prüfer groups, one for each prime not equal to char K, if G is locally cyclic then $\bar{K}^*$ has exactly one subgroup isomorphic to G; the elements of this subgroup generate $K(G)$ as a K-algebra, for any quotient of G is isomorphic (albeit unnaturally) to a subgroup of G.

LEMMA (2.1): *Let H be a finite cyclic group and K a field with char $K \notin \pi(H)$. Then there exist irreducible KH-modules faithful for H, and all such modules have dimension $|K(H): K|$ over K.*

PROOF: $K(H)^*$ has a unique subgroup isomorphic to H, so we may choose a monomorphism $\theta: H \to K(H)^*$. Then $K(H)$ becomes a KH-module with H-action given by

$$v \cdot h = vh^{\theta}, \qquad v \in K(H),\ h \in H.$$

If $0 \neq v \in K(H)$ then $v \cdot KH = vK(H) = K(H)$, so $K(H)$ is an irreducible KH-module; it is faithful for H as θ is one-to-one.

Let V be any irreducible KH-module faithful for H. Then V is isomorphic to KH/M for some maximal ideal M of KH. Now KH/M is a field, containing (since V is faithful) a multiplicative subgroup isomorphic to H which generates it over K. It follows that KH/M is algebraic over K, and thence isomorphic to the field $K(H)$. Thus

$$\dim_K V = \dim_K KH/M = |K(H): K|,$$

completing the proof.

If G is a periodic abelian group, we will denote by $\Omega(G)$ the subgroup generated by all elements of prime order in G. This subgroup is finite if and only if G satisfies Min, the minimum condition on subgroups. If K is a field and V a KG-module, we write

$$C_G(V) = \{g \in G: vg = v \text{ for all } v \in V\}.$$

LEMMA (2.2): *Let G be a periodic abelian group, H a subgroup of G containing $\Omega(G)$, and K a field with char $K \notin \pi(G)$. Let V be an irreducible KH-module faithful for H, and W a non-zero submodule of the induced module $V^G = V \otimes_{KH} KG$. Then W is faithful for G.*

PROOF: Since G is abelian, the restriction $V^G|_H$ of V^G to H is a direct sum of copies of V. As V is irreducible, W_H is also a direct sum of copies of V. Suppose $1 \neq g \in C_G(W)$. There exists an integer n such that $1 \neq g^n \in \Omega(G) \leqq H$. But then $1 \neq g^n \in C_H(W_H) = C_H(V)$, a contradiction as V is faithful for H. Hence W is faithful for G.

Let K be a field and G a locally cyclic group with Min such that char $K \notin \pi(G)$. A finite subgroup H of G will be called *K-inductive in G* if whenever V is an irreducible KH-module faithful for H, the induced module V^G is an irreducible KG-module.

LEMMA (2.3): *A finite subgroup H of G is K-inductive if and only if the following two conditions are satisfied*:
(a) *H contains $\Omega(G)$*;
(b) *whenever L is a finite subgroup of G containing H, we have*

$$|K(L): K(H)| = |L: H|.$$

PROOF: Suppose H is K-inductive in G. By Lemma 2.1 there exists an irreducible KH-module V faithful for H; then V^G is irreducible.

(a) Suppose $H \ngeqq \Omega(G)$; then there exists a finite non-trivial subgroup L of G with $HL = H \times L$. Now $V^{H \times L}$ is reducible: indeed

$$\{\sum_{x \in L} v \otimes x : v \in V\}$$

is a proper submodule. A fortiori V^G is reducible, a contradiction. So $H \geqq \Omega(G)$.

(b) Let L be a finite subgroup of G containing H. Then V^L like V^G is irreducible; by (a) and Lemma 2.2 V^L is faithful for L. Hence using Lemma 2.1,

$$\begin{aligned} |K(L): K(H)| &= |K(L): K|/|K(H): K| \\ &= \dim_K V^L/\dim_K V \\ &= |L: H|, \end{aligned}$$

since $V^L = V \otimes_{KH} KL$.

Now suppose (a) and (b) hold. We may express G as the union of a chain

$$H = H_0 \leqq H_1 \leqq H_2 \leqq \dots \leqq G$$

of finite subgroups. Let V be any irreducible KH-module faithful for H. By (a) and Lemma 2.2, any irreducible submodule of V^{H_i} is faithful for H_i, so has dimension $|K(H_i): K|$ by Lemma 2.1. But by (b) and Lemma 2.1,

$$\begin{aligned} |K(H_i): K| &= |K(H_i): K(H)||K(H): K| \\ &= |H_i: H| \dim_K V \\ &= \dim_K V^{H_i}. \end{aligned}$$

Hence V^{H_i} is itself irreducible. Now V^G may be regarded as the union of the V^{H_i}, so is also irreducible. Thus H is K-inductive in G.

Corollary (2.4): *A finite subgroup H of G is K-inductive if and only if there exists an irreducible KH-module V faithful for H such that V^G is irreducible.*

Proof: If such a V exists then by the first half of the proof of Lemma 2.3 H satisfies (a) and (b); then by the second half H is K-inductive. The converse follows from Lemma 2.1.

Note also that if $H \leqq L \leqq G$ and L is finite then in any case we have

$$|K(L): K(H)| \leqq |L: H|.$$

For if $m = |L: H|$ and the subgroup of $K(L)^*$ isomorphic to L is generated by ξ, then $\xi^m \in K(H)$, so the polynomial $f(X) = X^m - \xi^m$ has degree m over $K(H)$ and ξ as a root. Hence $|K(L): K(H)| = |K(\xi): K(H)| \leqq m$.

Lemma (2.5): *Let F and K be subfields of some field. Then*

$$|KF: F| \leqq |K: K \cap F|.$$

(Here the ring KF may or may not be a field.)

Proof: Any basis of K over $K \cap F$ also spans KF over F.

Theorem (2.6): *Let G be a locally cyclic group with Min, and K a field with char $K \notin \pi(G)$. If there exists any K-inductive subgroup in G, then there exists a unique minimal K-inductive subgroup in G.*

Proof: Since K-inductive subgroups are finite, it is sufficient to show that if H_1 and H_2 are K-inductive in G, then so is $H_1 \cap H_2$. But let H_1 be K-inductive, and H_2 any subgroup of G. Then

$$\Omega(H_2) \leqq \Omega(G) \cap H_2 \leqq H_1 \cap H_2.$$

Moreover, if L is a finite subgroup of H_2 containing $H_1 \cap H_2$, then $H_1 \cap H_2 = H_1 \cap L$, so

$$\begin{aligned} |K(L): K(H_1 \cap H_2)| &= |K(L): K(H_1 \cap L)| \\ &\geqq |K(L): K(H_1) \cap K(L)| \\ &\geqq |K(H_1)K(L): K(H_1)| \end{aligned}$$

by Lemma 2.5. Since LH_1 is cyclic, we have $K(H_1)K(L) = K(LH_1)$. So as

H_1 is K-inductive in G,

$$\begin{aligned}|K(L)\colon K(H_1 \cap H_2)| &\geqq |K(LH_1)\colon K(H_1)| \\ &= |LH_1 : H_1| \\ &= |L\colon H_1 \cap L| \\ &= |L\colon H_1 \cap H_2|.\end{aligned}$$

But $|K(L)\colon K(H_1 \cap H_2)| \leqq |L\colon H_1 \cap H_2|$ by the remark following Corollary 2.4, so by Lemma 2.3 $H_1 \cap H_2$ is K-inductive in H_2. If now H_2 is also K-inductive in G, it easily follows that $H_1 \cap H_2$ is K-inductive in G. This completes the proof.

We shall now investigate more closely the conditions under which a locally cyclic group with Min contains inductive subgroups for various fields.

LEMMA (2.7): *Let G be a locally cyclic group with Min. Then $\Omega(G)$ is $\mathbb{Q}$-inductive in G.*

PROOF: Suppose L is a finite subgroup of G containing $H = \Omega(G)$, and let ε be a primitive $|L|$-th root of unity. Then

$$|\mathbb{Q}(L)\colon \mathbb{Q}| = |\mathbb{Q}(\varepsilon)\colon \mathbb{Q}| = \varphi(|L|),$$

where φ is the Euler function. Thus

$$\begin{aligned}|\mathbb{Q}(L)\colon \mathbb{Q}(H)| &= \varphi(|L|)/\varphi(|H|) \\ &= \varphi(|L\colon H||H|)/\varphi(|H|) \\ &= |L\colon H|,\end{aligned}$$

for $\pi(L) = \pi(H)$ and if p is a prime dividing an integer m, then $\varphi(pm) = p\varphi(m)$. Hence $\Omega(G) = H$ is $\mathbb{Q}$-inductive in G by Lemma 2.3.

If m and n are positive integers, their highest common factor is denoted by (m, n). If $(m, n) = 1$, we will denote by $o(m, n)$ the order of m modulo n, i.e. the smallest positive integer r such that $n|m^r - 1$. If G is a locally cyclic group with Min, say

$$G \cong C_{p_1^{n_1}} \times \dots \times C_{p_k^{n_k}}$$

where the p_i are distinct primes and $1 \leqq n_i \leqq \infty$, then $N = p_1^{n_1} \dots p_k^{n_k}$

will be called the Steinitz number associated with G. Evidently the concepts of divisibility and highest common factor extend to Steinitz numbers.

The following is a slightly strengthened form of Lemma 2.2 in [2].

LEMMA (2.8): *Let G be a locally cyclic group with Min, and $\mathbb{F}_{p^d}$ a finite field of order p^d, with $p \notin \pi(G)$. Let N be the Steinitz number associated with G, and put*

$$n = (N, 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots),$$
$$r = o(p^d, n),$$
$$m = (N, p^{dr} - 1).$$

Then the unique subgroup H of order m in G is $\mathbb{F}_{p^d}$-inductive in G.

PROOF: Since $n | p^{dr} - 1$, we have $n|m$, whence $\Omega(G) \leqq H$. Let L be a finite subgroup of G containing H. Then L is cyclic and $\mathbb{F}_{p^d}(L)$ is the smallest extension $\mathbb{F}_{p^{dt}}$ of $\mathbb{F}_{p^d}$ such that L may be embedded in $\mathbb{F}^*_{p^{dt}}$, i.e. such that $l = |L|$ divides $|\mathbb{F}^*_{p^{dt}}| = p^{dt} - 1$. Hence t is the smallest positive integer such that $l | p^{dt} - 1$, so we have

$$|\mathbb{F}_{p^d}(L) : \mathbb{F}_{p^d}| = t = o(p^d, l).$$

By Lemma 2.3, to show that H is $\mathbb{F}_{p^d}$-inductive in G it is sufficient to prove that $|\mathbb{F}_{p^d}(L) : \mathbb{F}_{p^d}(H)| = |L : H|$, i.e. that if $m|l|N$ then

$$\frac{o(p^d, l)}{o(p^d, m)} = \frac{l}{m}.$$

Note that $o(p^d, m) = r$, for since $n|m$, $r = o(p^d, n) | o(p^d, m)$, while as $m | p^{dr} - 1$, $o(p^d, m) | r$. We will prove by induction on l/m (more precisely, on the sum of the exponents in the prime power factors of l/m) that if $o(p^d, l) = t$ and $p^{dt} - 1 = kl$, then $(k, N/m) = 1$, and $t/r = l/m$.

Firstly, let $l = m$, so $t = r$. Write $p^{dr} - 1 = km$. Then

$$(km, N) = (p^{dr} - 1, N) = m,$$

so $(k, N/m) = 1$. Also $t/r = 1 = l/m$.

Now suppose that $m|l|lq|N$, where q is a prime. Let $t = o(p^d, l)$ and $p^{dt} - 1 = kl$. By induction we may assume that $(k, N/m) = 1$ and that $t/r = l/m$. We then have

$$\begin{aligned} p^{dtq} &= (1+kl)^q \\ &= 1+qkl+\tfrac{1}{2}q(q-1)(kl)^2+\ldots+(kl)^q. \end{aligned}$$

Let $q_1|N$ be prime. If $q_1 \neq q$ then as $qq_1|l$ we have

$$p^{dtq} \equiv 1+qkl \qquad (\mathrm{mod}\ lqq_1).$$

If $q_1 = q$ we have $q|l$ so (since $q|\binom{q}{s}$ for $s = 2, \ldots, q-1$)

$$p^{dtq} \equiv 1+qkl+(kl)^q \qquad (\mathrm{mod}\ lq^2),$$

whence

$$p^{dtq} \equiv 1+qkl \qquad (\mathrm{mod}\ lq^2)$$

provided $q > 2$. But if $q = 2$ then $2^2|lq|N$ whence $2^2|n|m|l$, and again we obtain

$$p^{dtq} \equiv 1+qkl \qquad (\mathrm{mod}\ lq^2).$$

In particular we see that $lq|p^{dtq}-1$, so $t' = o(p^d, lq)|tq$. Moreover, $l|lq$, so $t = o(p^d, l)|t'$. If $lq|p^{dt}-1 = kl$, then $q|k$. But $m|l|lq|N$, so $q|(N/m)$, a contradiction as $(k, N/m) = 1$. Hence $lq \nmid p^{dt}-1$. Thus $t|t'|tq$, but $t \neq t'$, so $o(p^d, lq) = t' = tq$. We have

$$t'/r = tq/r = lq/m.$$

Now write $p^{dt'}-1 = k'lq$. By the above congruences, if q_1 is any prime divisor of N, we have

$$k'lq \equiv klq \qquad (\mathrm{mod}\ lqq_1),$$

whence

$$k' \equiv k \qquad (\mathrm{mod}\ q_1).$$

Thus if $q_1|(k', N/m)$ then $q_1|(k, N/m) = 1$, a contradiction. Hence $(k', N/m) = 1$. This completes the induction, and the proof.

It can be shown that H is the minimal $\mathbb{F}_{p^d}$-inductive subgroup of G unless $|O_2(G)| = 4$ and $p^d \equiv 3 \pmod 4$, in which case the subgroup of index 2 in H is minimal inductive.

LEMMA (2.9): *Let D and E be subfields of some field, and suppose that E*

is a finite normal extension of $D \cap E$. Then
(a) *D and E are linearly disjoint over $D \cap E$;*
(b) *if F is a subfield of E containing $D \cap E$ then $FD \cap E = F$.*

PROOF:
(a) E is the splitting field of some monic irreducible polynomial f over $D \cap E$. In fact f is still irreducible over D. For if $f = gh$, where g and h are monic polynomials over D, then the roots of g and h are roots of f, so all lie in E. The coefficients of g and h are (plus or minus) elementary symmetric functions in the roots, so lie in $D \cap E$. But f is irreducible over $D \cap E$, so over D too.

Let n be the degree of f, and ξ one of its roots. Then $\{1, \xi, \ldots, \xi^{n-1}\}$ is a basis of E over $D \cap E$, consisting of elements which are linearly independent over D. So D and E are linearly disjoint over $D \cap E$.

(b) Let ω_i be a basis of D over $D \cap E$, with $\omega_1 = 1$. Then $FD = \sum F\omega_i$. By (a), the ω_i are linearly independent over E (see Chapter IV Section 5 of [3]). Suppose

$$\beta = \sum \alpha_i \omega_i \in FD \cap E \qquad (\alpha_i \in F).$$

Then

$$(\alpha_1 - \beta)\omega_1 + \sum_{i \neq 1} \alpha_i \omega_i = 0 \qquad (\alpha_1 - \beta, \alpha_i \in E)$$

so $\beta = \alpha_1 \in F$. Thus $FD \cap E = F$.

THEOREM (2.10): *Let K be any field, k its prime field, and G a locally cyclic group satisfying Min with char $k \notin \pi(G)$. Then G has a K-inductive subgroup if and only if*

$$|k(G) \cap K : k| < \infty.$$

(Here $k(G) \cap K$ is a subfield of $\bar{K}$, in which $\bar{k}$ and $k(G)$ are embedded.)

PROOF: Suppose H is a K-inductive subgroup of G, and that L is a finite subgroup of G containing H. Then by the remark following Corollary 2.4 we have $|k(L): k(H)| \leqq |L: H| = |K(L): K(H)|$ (as H is K-inductive). Now $K(L) = k(L) \cdot K(H)$, so by Lemma 2.5

$$\begin{aligned} |K(L): K(H)| &= |k(L) \cdot K(H): K(H)| \\ &\leqq |k(L): k(L) \cap K(H)| \\ &\leqq |k(L): k(H)| \end{aligned}$$

(as $k(H) \leqq k(L) \cap K(H)$). We now have $|k(L)\colon k(L) \cap K(H)| = |k(L)\colon k(H)|$, whence

$$k(L) \cap K \leqq k(L) \cap K(H) = k(H).$$

As G is locally finite it follows that $k(G) \cap K \leqq k(H)$. Hence

$$|k(G) \cap K\colon k| \leqq |k(H)\colon k| \leqq |H| < \infty.$$

Conversely, suppose that $|k(G) \cap K\colon k| < \infty$: say $k(G) \cap K = k(\gamma)$. By Lemma 2.7 or 2.8, as k is a prime field, G contains a k-inductive subgroup H_1. Since G is locally finite, there exists a finite subgroup H of G containing H_1 and such that $\gamma \in k(H)$. Then

$$k(G) \cap K = k(\gamma) \leqq k(H).$$

We will show that H is K-inductive in G. Note first that $H \geqq H_1 \geqq \Omega(G)$ by Lemma 2.3.

Let L be a finite subgroup of G containing H. Then the cyclotomic field $k(L)$ is a finite normal extension of $k(L) \cap K$; moreover

$$k(L) \cap K \leqq k(G) \cap K \leqq k(H).$$

Hence by Lemma 2.9(b), with $D = K$, $E = k(L)$, and $F = k(H)$, we have

$$K(H) \cap k(L) = (K \cdot k(H)) \cap k(L) = k(H).$$

By Lemma 2.9(a), $K(H)$ $(= D)$ and $k(L)$ $(= E)$ are linearly disjoint over their intersection $k(H)$. Hence a basis for $k(L)$ over $k(H)$ also constitutes a basis for $K(L) = K(H) \cdot k(L)$ over $K(H)$. Thus

$$\begin{aligned} |K(L)\colon K(H)| &= |k(L)\colon k(H)| \\ &= |k(L)\colon k(H_1)|/|k(H)\colon k(H_1)| \\ &= |L\colon H_1|/|H\colon H_1| \\ &= |L\colon H| \end{aligned}$$

as H_1 is k-inductive. By Lemma 2.3, H is K-inductive in G.

Corollary (2.11): *Let K be any field, k its prime field, and G a periodic abelian group with char $k \notin \pi(G)$. Suppose that*

$$|k(G) \cap K : k| < \infty.$$

Then every locally cyclic quotient of G satisfying Min contains a K-inductive subgroup.

PROOF: If $\bar{G}$ is any quotient of G, every image of $\bar{G}$ in $\bar{k}^*$ is also an image of G, and therefore $k(\bar{G}) \leqq k(G)$. Now apply Theorem 2.10.

3. Primitive idempotents in KG

Let G be an abelian group and K a field. If $\alpha = \sum \alpha_g g \in KG$, we denote by supp α the finite set $\{g \in G : \alpha_g \neq 0\}$. We will write

$$C_G(\alpha) = \{g \in G : \alpha g = \alpha\}.$$

Since G is abelian, $C_G(\alpha)$ is in fact the centralizer $C_G(\alpha KG)$ in G of αKG considered as a KG-module. If e is an idempotent in KG, we say e is *faithful* (for G) if $C_G(e) = 1$.

LEMMA (3.1): *Let G be a periodic abelian group and K a field with char $K \notin \pi(G)$. Suppose KG contains a primitive idempotent e. Then G satisfies Min and is almost locally cyclic (i.e. has a locally cyclic subgroup of finite index). If e is faithful, G is locally cyclic, and $\langle \mathrm{supp}\, e \rangle$ is K-inductive in G.*

PROOF: Let $H = \langle \mathrm{supp}\, e \rangle$, a finite subgroup of G. Then eKH is an irreducible KH-module, and $eKH|^G = eKG$ is an irreducible KG-module (for otherwise G would contain a finite subgroup $L \geqq H$ with eKL reducible; but e is primitive in KL). As in the proof of Lemma 2.3, it follows that $H \geqq \Omega(G)$, whence $\Omega(G)$ is finite and G satisfies Min. If e is faithful for G so for H, then H is K-inductive in G by Corollary 2.4.

The group $C = C_G(e)$ is finite, since it acts as a group of permutations on the finite set supp e. The irreducible KG-module eKG, considered as a ring, is actually a field F. The homomorphism $G \to F^*$, $g \mapsto eg$ has kernel C. Hence G/C embeds in F^* so is locally cyclic. Let $|C| = m$. Since G is abelian, $G^m = \{g^m : g \in G\}$ is a quotient of G and indeed of G/C, as $C^m = 1$. Thus G^m is locally cyclic. But G/G^m has finite exponent and satisfies Min, so is finite. Hence G is almost locally cyclic. If e is faithful then $m = 1$ and G itself is locally cyclic. This completes the proof.

We shall now investigate the circumstances under which KG contains primitive idempotents faithful for G, given that G is locally cyclic and satisfies Min. We shall need:

LEMMA (3.2): *Let G be a periodic abelian group, K a field with char $K \notin \pi(G)$, and $H_0 \leqq H_1 \leqq \ldots \leqq G$ a chain of finite subgroups with union G. For each i, let e_i be a primitive idempotent in KH_i, such that $e_i e_{i+1} = e_{i+1}$. Then there exists a maximal ideal M of KG such that:*

(a) *for each i, $1-e_i \in M$ and $e_i \notin M$;*
(b) $C_G(KG/M) = \bigcup_{i=0}^{\infty} C_G(e_i)$.

PROOF: For each i, write

$$KH_i = e_i KH_i \oplus M_i,$$

where $M_i = (1-e_i)KH_i$ is a maximal ideal of KH_i. We have

$$(1-e_{i+1})(1-e_i) = 1-e_i,$$

whence

$$M_i = (1-e_i)KH_i \leqq (1-e_i)KH_{i+1} \leqq (1-e_{i+1})KH_{i+1} = M_{i+1}.$$

Since $G = \bigcup_{i=0}^{\infty} H_i$, $M = \bigcup_{i=0}^{\infty} M_i$ is an ideal of KG. Moreover $e_0 \notin M$, for if $e_0 \in M_i$ then $e_0 e_i = 0$, but then $e_i = e_i e_{i-1} \cdots e_1 e_0 = 0$. Thus M is a proper ideal of KG; furthermore it is maximal since $M \cap KH_i = M_i$ for each i. For each i, $1-e_i \in M_i \subseteq M$, so as $1 \notin M$, $e_i \notin M$. Thus we have (a).

Let $x \in C_G(e_i)$ and $\alpha \in KG$. Choose $j \geqq i$ such that $x, \alpha \in KH_j$. Since $e_j = e_j e_{j-1} \cdots e_i$, we have $x \in C_G(e_j)$. Thus $(\alpha x - \alpha)e_j = 0$, whence $\alpha x - \alpha \in (1-e_j(KH_j = M_j \subseteq M$, i.e. $(\alpha + M)x = \alpha + M$. It follows that $\bigcup_{i=0}^{\infty} C_G(e_i) \leqq C_G(KG/M)$.

Conversely let $x \in C_G(KG/M)$, so that $x-1 \in M$. Choose i so that $x \in H_i$. Then $x - 1 \in M \cap KH_i = M_i$ (as M_i is maximal in KH_i). Thus $e_i(x-1) = 0$, so $e_i x = e_i$ and $x \in C_G(e_i)$. This completes the proof of (b).

THEOREM (3.3): *Let G be a locally cyclic group with Min and K a field with char $K \notin \pi(G)$. Then the following are equivalent:*

(a) *KG contains a faithful primitive idempotent;*

(b) *G contains a K-inductive subgroup;*

(c) *there are only finitely many non-isomorphic irreducible KG-modules faithful for G;*

(d) *there do not exist $2^{\aleph_0}$ non-isomorphic irreducible KG-modules faithful for G;*

(e) $|k(G) \cap K : k| < \infty$, *where k is the prime field of K.*

Furthermore, when (a)–(e) hold, there is a one-to-one onto correspondence between faithful primitive idempotents of KG and isomorphism classes of irreducible KG-modules faithful for G.

PROOF: (a) implies (b) by Lemma 3.1, and (b) is equivalent to (e) by Theorem 2.10.

Now suppose H is a K-inductive subgroup of G, and V is an irreducible KG-module faithful for G. Since H is finite, V_H is completely reducible, so it contains an irreducible KH-submodule W say. Then $V_H = \sum_{x \in G} Wx$, and $Wx \cong W$ as KH-modules since G is abelian. Hence

$$C_H(W) = C_H(V_H) = 1.$$

So as H is K-inductive, W^G is irreducible. But there is a non-zero KG-map $W^G \to V$, $w \otimes x \mapsto wx$, so $V \cong W^G$. Thus every irreducible KG-module faithful for G is isomorphic to W^G for some irreducible KH-module W faithful for H. (Note that $W \cong eKH$ and $V \cong eKG$ for some idempotent e in KH which is faithful and primitive in KG.) There are only finitely many non-isomorphic such W, and therefore only finitely many non-isomorphic irreducible KG-modules faithful for G. Hence (b) implies (c). Trivially (c) implies (d).

The last part of the Theorem now also follows. For if e is a faithful primitive idempotent in KG, then eKG is an irreducible KG-module faithful for G; as we have just shown, every such module arises in this way. If e and f are idempotents in KG and $eKG \cong fKG$, then if θ: $eKG \to fKG$ is an isomorphism, we have $\theta(e) = f\theta(e) = \theta(e)f$; applying θ^{-1} we obtain $e = ef$. Similarly $f = fe$, so $e = f$.

To prove that (d) implies (a), we shall assume that KG contains no faithful primitive idempotent, and exhibit $2^{\aleph_0}$ non-isomorphic irreducible KG-modules faithful for G. Let

$$\Omega(G) = L_0 \leqq L_1 \leqq L_2 \leqq \ldots \leqq G$$

be a chain of finite subgroups with union G.

For $n = 0, 1, 2, \ldots$ let T_n denote the set of all n-tuples with each entry either 0 or 1. By induction we will construct for each integer n a finite subgroup H_n of G and for each $\varphi \in T_n$ a faithful primitive idempotent e_φ in KH_n. Firstly, let $H_0 = L_0 = \Omega(G)$. By Lemma 2.1, KH_0 contains a faithful primitive idempotent e.

Now suppose inductively that we have constructed H_n and $\{e_\varphi : \varphi \in T_n\}$. By Lemma 2.2 each e_φ is faithful for G, so by hypothesis is not primitive in KG. Hence we may choose a finite subgroup H_{n+1} of G containing L_{n+1} and such that for each $\varphi \in T_n$, e_φ decomposes in KH_{n+1}; say

$$e_\varphi KH_{n+1} = e_{(\varphi,0)}KH_{n+1} \oplus e_{(\varphi,1)}KH_{n+1} \oplus \cdots,$$

where $e_{(\varphi,0)}$ and $e_{(\varphi,1)}$ are primitive idempotents in KH_{n+1}. By Lemma 2.2, since $e_\varphi KH_{n+1} = e_\varphi KH_n|^{H_{n+1}}$, $e_{(\varphi,0)}$ and $e_{(\varphi,1)}$ are faithful for H_{n+1}. Thus we have chosen $e_{\varphi'}$ for each $\varphi' \in T_{n+1}$. This completes the inductive construction. Note that

$$\bigcup_{i=0}^{\infty} H_i = \bigcup_{i=0}^{\infty} L_i = G.$$

Let $\varphi = (a_1, a_2, a_3, \ldots)$ be an infinite sequence of 0's and 1's. Write $e_0(\varphi) = e$ and $e_n(\varphi) = e_{(a_1,\ldots,a_n)}$ $(n = 1, 2, 3, \ldots)$. By Lemma 3.2 there is a maximal ideal $M = M(\varphi)$ of KG with $1 - e_n(\varphi) \in M(\varphi)$ and $e_n(\varphi) \notin M(\varphi)$ for all n, and

$$C_G(KG/M(\varphi)) = \bigcup_{n=0}^{\infty} C_G(e_n(\varphi)) = 1.$$

Thus $V(\varphi) = KG/M(\varphi)$ is an irreducible KG-module faithful for G.

If $\varphi \neq \psi$ then $V(\varphi)$ and $V(\psi)$ are not KG-isomorphic. For if φ and ψ differ first in the n-th place, then $e_n(\varphi)e_n(\psi) = 0$; hence

$$e_n(\psi) = e_n(\psi)(1 - e_n(\varphi)) \in M(\varphi),$$

so $e_n(\psi)$ annihilates $V(\varphi)$. But $1 - e_n(\psi) \in M(\psi)$, so $e_n(\psi)$ acts as the identity on $V(\psi)$. This completes the proof of the Theorem.

In Lemma 2.12 of [1], S. D. Berman proves a result related to part of Theorem 3.3, for the special case of abelian p-groups. Note that a field K with prime field k is "of the first kind for p", in Berman's terminology, if and only if $|k(C_{p^\infty}) \cap K : k| < \infty$.

The following corollary to Theorem 3.3 generalizes Lemma 2.5 of [2].

THEOREM (3.4): *Let K be a field, k its prime field, and G an abelian almost locally cyclic group with Min such that $\operatorname{char} k \notin \pi(G)$. If $|k(G) \cap K : k| = \infty$, then KG contains no primitive idempotents. Suppose that $|k(G) \cap K : k| < \infty$. If C is any finite subgroup of G such that G/C is locally cyclic, then KG contains a non-zero finite number of primitive idempotents e with $C_G(e) = C$, and there is a one-to-one onto correspondence between such idempotents and isomorphism classes of irreducible KG-modules V with $C_G(V) = C$.*

PROOF: Let C be any finite subgroup of G. We may write

$$KG = \mathfrak{c}G \oplus \nu KG,$$

where $\mathfrak{c}G$ is the ideal of KG generated by the augmentation ideal $\mathfrak{c}$ of KC, and ν is the idempotent

$$\frac{1}{|C|}\sum_{x \in C} x.$$

It is easily deduced that the canonical group ring projection

$$KG \to K[G/C] \quad (\cong KG/\mathfrak{c}G \cong \nu KG)$$

determines a one-to-one map from the set of primitive idempotents e in KG with $C_G(e) = C$ onto the set of faithful primitive idempotents in $K[G/C]$. (Both these sets might be empty.)

Suppose KG contains a primitive idempotent e; we will show that $|k(G) \cap K : k| < \infty$. Let $C = C_G(e)$. By the above the image of e in $K[G/C]$ is a primitive idempotent faithful for G/C. Thus G/C is locally cyclic, and by Theorem 3.3 $|k(G/C) \cap K : k| < \infty$.

Since every image of G/C is an image of G, we have $k(G/C) \leqq k(G)$. Now let

$$F = k(\textstyle\prod O_p(G))$$

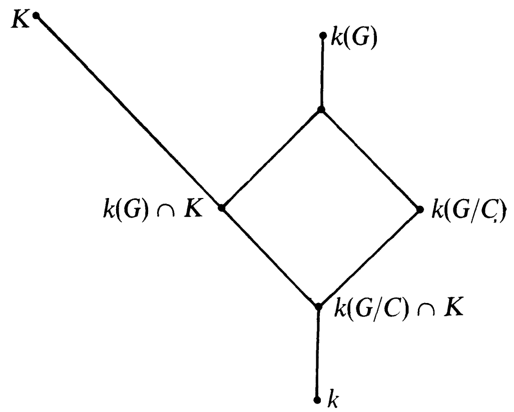
where the product is taken over those primes p such that $O_p(G)$ is finite. Then $|F : k| < \infty$ since G satisfies Min. Moreover $k(G) = F \cdot k(G/C)$. For $k(G)$ is determined by the exponents of the primary components of G, and since C is finite, if $\exp O_p(G) = \infty$ then $\exp O_p(G/C) = \infty$. Hence by Lemma 2.5,

$$|k(G) : k(G/C)| = |F \cdot k(G/C) : k(G/C)| \leqq |F : k| < \infty.$$

Now $k(G/C)$ is a union of finite normal extensions of k, so also of $k(G/C) \cap K$; Lemma 2.9(a) together with a local argument shows that $k(G/C)$ and K are linearly disjoint over $k(G/C) \cap K$. In particular, any subset of $k(G) \cap K$ which is linearly independent over $k(G/C) \cap K$ is a subset of $k(G)$ which is linearly independent over $k(G/C)$. Hence

$$|k(G) \cap K : k(G/C) \cap K| \leqq |k(G) : k(G/C)| < \infty.$$

We now have



$$|k(G) \cap K : k| = |k(G) \cap K : k(G/C) \cap K||k(G/C) \cap K : k| < \infty.$$

Now suppose that $|k(G) \cap K : k| < \infty$, and that C is a finite subgroup of G such that G/C is locally cyclic. Since $k(G/C) \leqq k(G)$ we also have $|k(G/C) \cap K : k| < \infty$. In view of the one-to-one correspondence mentioned in the first paragraph of this proof, an application of Theorem 3.3 to $K[G/C]$ yields the remaining statements of Theorem 3.4.

4. The socular series of KG

If V is a module recall that the *socle* $So(V)$ of V is the sum of all irreducible submodules of V. We define the *ascending socular series* of V by

$$So_0(V) = 0$$

$$So_1(V) = So(V)$$

$$\frac{So_{n+1}(V)}{So_n(V)} = So\left(\frac{V}{So_n(V)}\right), \qquad n = 1, 2, 3, \ldots.$$

In particular if A is a commutative ring, we obtain an ascending socular series of A considered as an A-module.

LEMMA (4.1): *Let G be a locally finite group and K a field with char $K \notin \pi(G)$. Then the socle of KG (considered as left or right KG-module) contains and is generated by all primitive idempotents in KG.*

PROOF: We consider the right module case; the proof for the left module case is analogous. If e is a primitive idempotent in KG then eKG is irreducible, for otherwise as G is locally finite there exists a finite subgroup H of G with $e \in KH$ such that eKH is reducible, a contradiction as KH is completely reducible and e is primitive in KH. Hence $e \in eKG \leqq So(KG_{KG})$.

Let N be a minimal right ideal of KG. Since G is locally finite there exists a finite subgroup H of G with $KH \cap N \neq 0$. As KH is completely reducible, $KH \cap N$ contains an idempotent e. Then $N = eKG$, so e is primitive in KG. Hence $So(KG_{KG})$ is generated as a right ideal by the primitive idempotents of KG.

THEOREM (4.2): *Let K be a field with prime field k, and G a periodic abelian group such that $\operatorname{char} k \notin \pi(G)$. If $|k(G) \cap K : k| = \infty$, then the socle of KG is zero. If $|k(G) \cap K : k| < \infty$, then the socle of KG is the intersection T of the maximal ideals M of KG such that $C_G(KG/M)$ is infinite.*

PROOF: If $|k(G) \cap K : k| = \infty$, then by Lemma 3.1 and Theorem 3.4, KG contains no primitive idempotents. Hence $So(KG) = 0$ by Lemma 4.1. Now assume that $|k(G) \cap K : k| < \infty$.

Suppose that N is a minimal ideal of KG, M is a maximal ideal, and $N \nleqq M$. Then $KG = N \oplus M$, so $C_G(KG/M) = C_G(N)$. Let $0 \neq \alpha \in N$; then $C_G(N)$ is contained in $C_G(\alpha)$, which is finite since it acts as a group of permutations on $\operatorname{supp} \alpha$. Hence $C_G(KG/M)$ is finite. It follows that $So(KG) \leqq T$.

To show that $T \leqq So(KG)$, suppose $0 \neq \alpha \in T$. Let $H = \langle \operatorname{supp} \alpha \rangle$, and write

$$\alpha = \alpha e_1 + \ldots + \alpha e_m$$

where the e_i are orthogonal primitive idempotents in KH, and $\alpha e_i \neq 0$ for each i. Since $e_i KH$ is irreducible, $\alpha e_i KH = e_i KH$, so there exists $\beta_i \in KH$ such that $e_i = \alpha e_i \beta_i$; thus $e_i \in T$. Hence it is sufficient to show that if H is a finite subgroup of G, e is a primitive idempotent in KH, and $e \in T$, then $e \in So(KG)$, i.e. if $e \notin So(KG)$ then $e \notin T$.

If $C_G(KG/M)$ is infinite for all maximal ideals M of KG, then T is the Jacobson radical of KG. But KG is semisimple (see Theorem 18.7 of [5]), so $T = 0 \leqq So(KG)$ as required. Hence we may assume that there exists a maximal ideal M of KG with $C = C_G(KG/M)$ finite. Then G/C embeds in the multiplicative subgroup of the field KG/M, so is locally cyclic whence countable. Thus G is also countable. Hence there exists a chain

$$H = H_0 \leqq H_1 \leqq \dots \leqq G$$

of finite subgroups with union G.

Assume first that G does not satisfy Min. Then by Lemmas 3.1 and 4.1 $So(KG) = 0$, so the condition that $e \notin So(KG)$ is vacuous; in effect we must show that T is also zero. We shall construct by induction a subchain $H_{n_0} \leqq H_{n_1} \leqq \dots$ of $H_0 \leqq H_1 \leqq \dots$ and for each i a primitive idempotent e_i in KH_{n_i} such that $e_i e_{i+1} = e_{i+1}$. Firstly, let $n_0 = 0$ and $e_0 = e$. Suppose we have already found n_i and e_i. Since G does not satisfy Min and $C_G(e_i)$ is finite, $\Omega(G)$ is not contained in $C_G(e_i)$, so there exists a non-trivial finite subgroup L_i of G with $C_G(e_i) \cap L_i = 1$. Choose n_{i+1} such that $H_{n_{i+1}} \geqq L_i H_{n_i}$. Let

$$v_i = \frac{1}{|L_i|} \sum_{x \in L_i} x$$

be the trivial primitive idempotent in KL_i, and choose a primitive idempotent e_{i+1} in $KH_{n_{i+1}}$ such that $(e_i v_i)e_{i+1} = e_{i+1}$; then also $e_i e_{i+1} = e_{i+1}$. Now $L_i \leqq C_G(e_{i+1})$, so $C_G(e_i) \lneqq C_G(e_{i+1})$. By Lemma 3.2 there exists a maximal ideal M of KG such that $e = e_0 \notin M$, and

$$C_G(KG/M) = \bigcup_{i=0}^{\infty} C_G(e_i),$$

which by construction is infinite. Thus $e \notin T$ as required. Hence we may assume that G satisfies Min.

If f is a primitive idempotent in KH_n for some $n \geqq 0$, consider the set of all sequences $(f_n, f_{n+1}, \dots)$ such that

(i) f_i is a primitive idempotent in KH_i for all $i \geqq n$;

(ii) $f_n = f$;

(iii) $f_i f_{i+1} = f_{i+1}$ for all $i \geqq n$.

If $m \geqq 0$ we shall say that f is *m-stationary* if for all such sequences $(f_n, f_{n+1}, \dots)$ and all $i \geqq 0$ we have $f_{n+m} = f_{n+m+i}$. Note that if

$$f = f_1' + \dots + f_t'$$

where the f_j' are orthogonal primitive idempotents in KH_{n+1}, then f is m-stationary (for $m \geqq 1$) if and only if each f_j' is $(m-1)$-stationary. Moreover f is 0-stationary if and only if it is primitive in KG. Hence if f is m-stationary and we write f as a sum of orthogonal primitive idempotents in KH_{n+m}, then each such idempotent will be 0-stationary; thus by Lemma 4.1 we have $f \in So(KG)$.

Now let e be a primitive idempotent in KH with $e \notin So(KG)$. Then $e = e_0$ is not m-stationary for any m. Hence among the finitely many orthogonal primitive idempotents in KH_1 whose sum is e_0, there must exist one, say e_1, which is not m-stationary for any m. Similarly we may choose a primitive idempotent e_2 in KH_2 which satisfies $e_1e_2 = e_2$ and is not m-stationary for any m, and so on. In this way we obtain a sequence $e_0 = e, e_1, e_2, \dots$ such that e_i is a primitive idempotent in KH_i, and $e_ie_{i+1} = e_{i+1}$.

Consider the chain of subgroups $C_G(e_0) \leqq C_G(e_1) \leqq \dots$, and suppose that $C = \bigcup_{i=0}^{\infty} C_G(e_i)$ is finite; then $C = C_G(e_n)$ for some n. For $i \geqq n$, e_iKH_i is an irreducible module faithful for H_i/C, so H_i/C is cyclic; hence G/C is locally cyclic. Also $|k(G/C) \cap K : k| \leqq |k(G) \cap K : k| < \infty$, so by Theorem 2.10 G/C contains a K-inductive subgroup. Thus we may choose $s \geqq n$ so that H_s/C is K-inductive in G/C. But e_s is a primitive idempotent in KH_s with $C_G(e_s) = C$, so e_s is primitive in KG, i.e. 0-stationary, a contradiction. It follows that $\bigcup_{i=0}^{\infty} C_G(e_i)$ is infinite, whence by Lemma 3.2 there is a maximal ideal M of KG such that $e = e_0 \notin M$ and $C_G(KG/M) = \bigcup_{i=0}^{\infty} C_G(e_i)$ is infinite. Hence $e \notin T$. This completes the proof of the theorem.

As an example we may take G to be a Prüfer group and K any field satisfying the hypotheses of Theorem 4.2. Then the augmentation ideal $\mathfrak{g}$ of KG is the only maximal ideal M such that $C_G(KG/M)$ is infinite. Hence $So(KG) = \mathfrak{g}$, a result obtained by W. Müller in [4] in the case where K is a subfield of the field of complex numbers. But $KG/\mathfrak{g}$ is the trivial irreducible KG-module, so $So_2(KG) = KG$. The next theorem generalizes this observation.

THEOREM (4.3): *Let K be a field with prime field k, and G an abelian almost locally cyclic group with Min such that char $k \notin \pi(G)$ and $|k(G) \cap K : k| < \infty$. Let m be the number of factors in a decomposition of the maximal divisible subgroup of G as a direct product of Prüfer groups. Then the ascending socular series of KG reaches KG after exactly $m+1$ steps, i.e. $So_m(KG) \neq KG = So_{m+1}(KG)$.*

PROOF: We may write

$$G = F \times \prod_{i=1}^{m} P_i,$$

where F is finite and for $i = 1, \dots, m$ P_i is a Prüfer p_i-group, where the

p_i are distinct primes. We proceed by induction on m. If $m = 0$ then G is finite, so KG is completely reducible and $So(KG) = KG$.

Suppose $m \geqq 1$. Let $\varphi_i\colon KG \to K[G/P_i]$ be the canonical projection of group rings, and define a KG-homomorphism θ by the commutativity of the diagrams

$$\begin{tikzcd} KG \arrow[r, "\theta"] \arrow[dr, "\varphi_i"'] & \bigoplus_{i=1}^{m} K[G/P_i] \arrow[d] \\ & K[G/P_i]. \end{tikzcd}$$

Then

$$\ker\theta = \bigcap_{i=1}^{m} \ker\varphi_i = \bigcap_{i=1}^{m} \mathfrak{p}_i G,$$

where $\mathfrak{p}_i G$ is the ideal of KG generated by the augmentation ideal $\mathfrak{p}_i$ of KP_i.

Since $KG/\mathfrak{p}_i G \cong K[G/P_i]$ and $K[G/P_i]$ is semisimple, it follows that $\mathfrak{p}_i G$ is the intersection of the maximal ideals M of KG containing it. But if $M \geqq \mathfrak{p}_i G$ then $C_G(KG/M)$ contains P_i so is infinite. Thus $\ker\theta$ is the intersection of certain maximal ideals M with $C_G(KG/M)$ infinite, so by Theorem 4.2 $\ker\theta \geqq So(KG)$. On the other hand if M is any maximal ideal of KG with $C_G(KG/M)$ infinite, then $C_G(KG/M)$ contains P_i for some i, whence $\ker\theta \leqq \mathfrak{p}_i G \leqq M$. Thus by Theorem 4.2 again we have $\ker\theta \leqq So(KG)$. Therefore $\ker\theta = So(KG)$.

Hence θ induces a KG-monomorphism

$$\frac{KG}{So(KG)} \to B = \bigoplus_{i=1}^{m} K[G/P_i].$$

By induction, the ascending socular series of $K[G/P_i]$ (as $K[G/P_i]$-module) reaches $K[G/P_i]$ after exactly m steps. Thus the ascending socular series of B (as KG-module) reaches B after m steps, i.e. $So_m(B_{KG}) = B$. Hence

$$\frac{So_{m+1}(KG)}{So(KG)} = So_m\left(\left(\frac{KG}{So(KG)}\right)_{KG}\right) = \frac{KG}{So(KG)},$$

whence $So_{m+1}(KG) = KG$. If $So_m(KG) = KG$ then we would have

$$So_{m-1}\left(\frac{KG}{So(KG)}\right) = \frac{KG}{So(KG)},$$

a contradiction as $K[G/P_i]$ is a quotient of $KG/So(KG)$ but

$$So_{m-1}(K[G/P_i]) \neq K[G/P_i].$$

This completes the proof of the theorem.

Despite Theorem 4.3 the group rings we have been studying do not seem to satisfy any form of the Jordan-Hölder Theorem. In fact, if K and G satisfy the hypotheses of Theorem 4.3 and G is infinite, we may enumerate the primitive idempotents of KG, say as $e_1, e_2, e_3, \ldots$. Then KG has a descending composition series

$$KG = V_0 > V_1 > V_2 > \ldots$$

of type ω, where for $n \geqq 1$

$$V_n = (1 - \sum_{i=1}^{n} e_i)KG.$$

(Since $\bigcap_{n=0}^{\infty} V_n$ contains no primitive idempotents it is disjoint from $So(KG)$ by Lemma 4.1, whence zero by Theorem 4.3.) For each $n \geqq 0$ the factor V_n/V_{n+1} is isomorphic to $e_{n+1}KG$, so $C_G(V_n/V_{n+1})$ is finite. Hence for example the trivial irreducible KG-module does not occur as a factor in the composition series.

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