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CONSTRUCTING THE MAXIMAL MONOIDS IN THE SEMIGROUPS $\mathscr{N}_n$, $\mathscr{S}_n$, AND Ω_n

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Abstract

This paper gives a constructive procedure for finding the maximal monoids in the semigroup of nonnegative matrices $\mathscr{N}_n$, in the semigroup of stochastic matrices $\mathscr{S}_n$, and in the semigroup of doubly stochastic matrices Ω_n.

Introduction

In providing a setting for this paper, one notes that much recent work has been concerned with algebraic systems within $\mathscr{N}_n$, the semigroup of nonnegative matrices. As a sampling of these algebraic studies, we first cite the work of Flor [2] in which some algebraic structure for the maximal groups in $\mathscr{N}_n$, as well as some algebraic structure for the semigroup of stochastic matrices $\mathscr{S}_n$ has been developed. Further Farahat [1] determined the structure of the maximal groups of doubly stochastic matrices Ω_n. Schwarz [7], [8] also studied the structure of maximal groups in $\mathscr{S}_n$ and Ω_n.

Research on a second type of algebraic problem in $\mathscr{N}_n$ has been conducted by Plemmons [4], [5] in which some of the Green relations are characterized for the semigroups $\mathscr{N}_n$, $\mathscr{S}_n$, and Ω_n. A third algebraic problem in $\mathscr{N}_n$, studied by Richman and Schneider [6], is concerned with factoring nonnegative matrices.

The work contained in this paper is a development of the structure of the maximal monoids in $\mathscr{N}_n$, $\mathscr{S}_n$, and Ω_n. We give a constructive characterization of these monoids and, as consequence, obtain all previous characterizations of maximal groups in $\mathscr{N}_n$, $\mathscr{S}_n$, and Ω_n.

Section 1

Much of the work concerned with the algebraic structure of nonnegative matrices uses, as a tool, a theorem of Flor [2] which specifies the form of nonnegative idempotent matrices. Although this theorem is basic, it seems that Flor provides the only proof. As Flor's theorem is also paramount for our work, we initiate this study with a new proof of this basic result.

THEOREM 1: *An nxn matrix $A \geqq 0$ is idempotent if and only if there is a permutation matrix P so that*

$$PAP^t = \begin{pmatrix} J & JB & 0 \\ 0 & 0 & 0 \\ CJ & CJB & 0 \end{pmatrix}$$

with main diagonal blocks square, C and B arbitrary and

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_r \end{pmatrix}$$

with each $J_k > 0$ and rank one, where $r = rank\ A$.

PROOF: Suppose first that $A \geqq 0$ is idempotent with no zero rows or columns. Suppose A is reducible. Then there is a permutation matrix Q so that

$$A_1 = QAQ^t = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

with X square. As A_1 is idempotent,

$$YX + ZY = Y.$$

Postmultiplying each matrix in the equation by X yields

$$YX + ZYX = YX,$$

i.e.

$$ZYX = 0.$$

As Z has no zero columns $YX = 0$. As X has no zero rows, $Y = 0$ and so

$$A_1 = \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix}.$$

By induction then, there is a permutation matrix P so that

$$PAP^t = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_r \end{pmatrix}$$

where each J_k is irreducible. Thus, by the Perron-Frobenius Theorem each $J_k > 0$ and rank one. Further, $r = \operatorname{rank} A$.

Now suppose $A \geqq 0$ is idempotent. Let P_1 be a permutation matrix so that

$$A_1 = P_1 A P_1^t = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$$

has no zero columns and A_{11} is square. As A_1 is idempotent, $A_{21}A_{11} = A_{21}$ and hence A_{11} has no zero columns. Pick a permutation matrix P_2 so that

$$P_2 A_{11} P_2^t = \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}$$

where $(B_{11} B_{12})$ has no zero rows. As above, B_{11} has no zero rows. Thus, pick a permutation matrix P_3 so that

$$P_3 B_{11} P_3^t = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_r \end{pmatrix} = J.$$

Hence a permutation matrix P exists so that

$$PAP^t = \begin{pmatrix} J & X & 0 \\ 0 & 0 & 0 \\ Y & Z & 0 \end{pmatrix}$$

which is idempotent. Thus,

$$\begin{aligned} JX &= X \\ YJ &= Y \\ YX &= Z \end{aligned}$$

and hence

$$PAP^t = \begin{pmatrix} J & JX & 0 \\ 0 & 0 & 0 \\ YJ & YJX & 0 \end{pmatrix}$$

from which the direct implication follows.

As the converse implication is only a direct calculation the theorem follows.

For the remainder of the work, we will utilize the symbol I to denote an idempotent in $\mathcal{N}_n$. We denote the i^{th} row of I by I_i. Further, we let

$M(I) = \{A \in \mathcal{N}_n | AI = IA = A\}$,

$G(I) = \{\text{units in } M(I)\}$,

$C(I)$ be the cone generated by the rows of I, and

End $C(I) = \{\text{cone endomorphisms of } C(I)\}$.

Section 2

The initial result of this paper, concerning the construction of $M(I)$, establishes an isomorphism between the monoids $M(I)$ and End $C(I)$.

LEMMA 1: *Let $\psi \in$ End $C(I)$. Let A be the matrix whose i^{th} row is $(I_i)\psi$. Then $AI = A$.*

PROOF: Suppose $(I_i)\psi = a_1I_1 + \ldots + a_nI_n$ where $a_k \geqq 0$ for $k = 1, 2, \ldots, n$. Then the i^{th} row of AI is

$$(a_1I_1 + \ldots + a_nI_n)I = a_1I_1I + \ldots + a_nI_nI = a_1I_1 + \ldots + a_nI_n.$$

Thus, $AI = A$.

THEOREM 2: *$M(I)$ is isomorphic to* End $C(I)$.

PROOF: For each $\psi \in \operatorname{End} C(I)$, denote by A_ψ the matrix of $\mathscr{N}_n$ whose i^{th} row is $(I_i)\psi$. Pick $\psi, \sigma \in \operatorname{End} C(I)$. Then the i^{th} row of $A_\psi A_\sigma$ is, by direct calculation and Lemma 1,

$$\sum_k [A_\psi]_{ik}[A_\sigma]_k = \sum_k [A_\psi]_{ik}[(I_k)\sigma]$$
$$= (\sum_k [A_\psi]_{ik} I_k)\sigma = ([A_\psi]_i I)\sigma = (I_i\psi)\sigma = (I_i)\psi \circ \sigma$$

Thus, $A_\psi A_\sigma = A_{\psi\sigma}$.

Let $1 \in \operatorname{End} C(I)$ be the identity cone endomorphism. Then $A_1 = I$ and from the above, $A_\psi I = IA_\psi = A_\psi$ for all $\psi \in \operatorname{End} C(I)$. Thus $A_\psi \in M(I)$.

Define $\pi: \operatorname{End} C(I) \to M(I)$ by the equation $(\psi)\pi = A_\psi$. This map is clearly a monomorphism. Finally, pick $A \in M(I)$. Define $(z)\psi = zA$ for all $z \in C(I)$. Since $AI = A$, $\psi \in \operatorname{End} C(I)$ and since $IA = A$, the i^{th} row of A is $(I_i)\psi$. Hence $A_\psi = A$ and π is onto. Thus $M(I)$ is isomorphic to $\operatorname{End} C(I)$.

As an immediate corollary, we have the following result concerning $G(I)$.

COROLLARY 1: *$G(I)$ is isomorphic to Aut $C(I)$.*

PROOF: The group of units in $M(I)$ is $G(I)$ while the group of units in $\operatorname{End} C(I)$ is $\operatorname{Aut} C(I)$. Since a monoid isomorphism maps units onto units, the result follows.

Theorem 1 and Theorem 2 provide the tools used to give a constructive procedure for finding $M(I)$. This procedure requires the following notions.

Let $V_n = \{1xn$ vectors of nonnegative numbers$\}$. If

$$\mathscr{a} = \{\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_m\} \subseteq \mathrm{V}_n$$

is such that

$$\alpha_i a_i = \sum_{k \neq i} \alpha_k a_k, \qquad \text{for } \alpha_k \geqq 0,$$

implies that $\alpha_k = 0$, $k = 1, 2, \ldots, m$, then $\mathscr{a}$ is said to be an independent set. If $\mathscr{S} \subseteq V_n$ is closed under the cone operations of addition and multiplication by nonnegative scalars, and $\mathscr{S}$ contains an independent set $\mathscr{a} = \{a_1, a_2, \ldots, a_m\}$ such that

$$\mathscr{S} = \{\sum_{k=1}^{n} \alpha_k a_k | \alpha_k \geqq 0\}$$

then $\mathscr{a}$ is said to be a basis for $\mathscr{S}$. Two easily deducible remarks follow.

LEMMA 2: *If*

$$\mathscr{S} = \{\sum_{k=1}^{m} \alpha_k a_k | \alpha_k \geqq 0 \text{ and } a_k \in V_n\},$$

then $\mathscr{S}$ is a cone which has a basis in $\{a_1, a_2, \ldots, a_m\}$. In particular, if I is an idempotent in $\mathscr{N}_n$, then $C(I)$ has a basis contained among the rows of I.

LEMMA 3: *If $\mathscr{S}$ is a cone with a basis, each $\psi \in \operatorname{End} \mathscr{S}$ is completely determined by its action on the basis. In particular, each $\psi \in \operatorname{End} C(I)$ is completely determined by its action on the basis.*

As a matter of nomenclature we will denote the basis of $C(I)$ by $\{d_1, d_2, \ldots, d_r\}$ where d_k is any specified row of I intersecting J_k. Thus from the preceding lemma, $A_\psi \in M(I)$ is completely determined once $(d_1)\psi, (d_2)\psi, \ldots, d_r)\psi$ have been specified.

To give a constructive procedure for finding $M(I)$, we first note that $M(I)$ is a closed cone. Thus $M(I)$ can be constructed if its edges can be determined. This is the thrust of the next theorem.

THEOREM 3: *A_ψ is on an edge of $M(I)$ if and only if there exists i and j such that $(d_i)\psi = \lambda_j d_j$, $\lambda_j > 0$ and $(d_k)\psi = 0$ for $k \neq i$.*

PROOF: A_ψ is on an edge of $M(I)$ if and only if ψ is on an edge of $\operatorname{End} C(I)$. Thus, the theorem follows by elementary calculations.

As an example of the construction procedure of $M(I)$ from this theorem, consider

$$I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $M(I)$ has edges

$$\left\{ \lambda_1 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \lambda_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \right.$$
$$\left. \text{where } \lambda_1, \lambda_2, \lambda_3 \text{ and } \lambda_4 \text{ are nonnegative} \right\}.$$

By applying the above work, and the following lemma, we can also construct $G(I)$.

LEMMA 4: *$A_\psi \in G(I)$ if and only if there is a permutation π of $\{1, 2, \ldots, r\}$ and positive numbers $\lambda_1, \ldots, \lambda_r$ so that*

$$(d_i)\psi = \lambda_i d_{\pi(i)}.$$

PROOF: If

$$(d_i)\psi = \sum_{k=1}^{r} \lambda_k d_k, \quad \text{then} \quad d_i = \sum_{k=1}^{r} \lambda_k (d_k)\psi^{-1}.$$

Since ψ^{-1} is a cone automorphism, we conclude that for precisely one k, $\lambda_k(d_k)\psi^{-1} = d_i$, i.e. $(d_i)\psi = \lambda_k d_k$. Thus the result follows.

From this lemma if

$$I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$G(I) = \left\{ \begin{pmatrix} \lambda & \lambda & 0 \\ \lambda & \lambda & 0 \\ 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} \beta & 0 & 0 \\ \beta & 0 & 0 \\ 0 & \lambda & \lambda \end{pmatrix} \text{for all } \lambda > 0 \text{ and } \beta > 0 \right\}.$$

This result also gives Flor's Theorem.

THEOREM (Flor): *$G(I)$ is isomorphic to the group of $r \times r$ monomial nonnegative matrices.*

Having resolved the problem of constructing $M(I)$ in $\mathcal{N}_n$, it is now our direction to consider the semigroup $\mathcal{S}$ of stochastic matrices, and give a constructive procedure for finding the maximal monoids in this semigroup. For this, let

$$M_{\mathcal{S}}(I) = \{\text{stochastic } A | AI = IA = A \text{ and } I \in \mathcal{S}\}.$$

THEOREM 4: *Let I be an idempotent in $\mathcal{S}$. Let*

$$\text{End } C(I) = \{\psi \in \text{End } C(I) | (d_i)\psi = \sum_{k=1}^{r} \lambda_k d_k \textit{ implies that } \sum_{k=1}^{r} \lambda_k = 1\}.$$

Define $\pi\colon \mathrm{End}_{\mathscr{S}}\, C(I) \to M_{\mathscr{S}}(I)$ *by* $(\psi)\pi = A_\psi$ *where the* i^{th} *row of* A_ψ *is* $(I_i)\psi$. *Then* π *is an isomorphism from* End $C(I)$ *onto* $M_{\mathscr{S}}(I)$.

Proof: The restriction to $M_{\mathscr{S}}(I)$ of the isomorphism $\pi\colon M(I) \to \mathrm{End}\, C(I)$ given in Theorem 2 is the desired isomorphism.

To construct $M_{\mathscr{S}}(I)$, we note that this monoid is a compact convex set. Thus this set is constructable once its vertices are known. This is the intent of the next theorem.

Theorem 5: A_ψ *is a vertex of* $M_{\mathscr{S}}(I)$ *if and only if there is a transformation* π *of* $\{1, 2, \ldots, r\}$ *such that* $(d_i)\psi = d_{\pi(i)}$.

Proof: A_ψ is a vertex of $M_{\mathscr{S}}(I)$ if and only if ψ is a vertex of $\mathrm{End}_{\mathscr{S}}\, C(I)$.

It is an easy calculation to show that if $(d_i)\psi = d_{\pi(i)}$ for some mapping π from $\{1, 2, \ldots, r\}$ into itself, then ψ is a vertex. Thus, suppose

$$(d_i)\psi = \sum_{k=0}^{r} \lambda_k d_k \quad \text{where} \quad \sum_{k=1}^{r} \lambda_k = 1,$$

and suppose without loss of generality that λ_1 and λ_2 are positive. Define

$$(d_k)\psi_1 = \begin{cases} d_1 & \text{for } k = i \\ (d_k)\psi & \text{for } k \neq i \end{cases}$$

and

$$(d_k)\psi_2 = \begin{cases} (1-\lambda_1)^{-1} \sum\limits_{k=2}^{r} \lambda_k d_k & \text{for } k = i \\ (d_k)\psi & \text{for } k \neq i. \end{cases}$$

Then $\psi_1, \psi_2 \in \mathrm{End}_{\mathscr{S}}\, C(I)$ and $\psi = \lambda_1\psi_1 + (1-\lambda_1)\psi_2$ with ψ neither ψ_1 nor ψ_2. Thus, ψ is not a vertex of $\mathrm{End}_{\mathscr{S}}\, C(I)$ and the result follows.

As an example of how this theorem is used to construct $M_{\mathscr{S}}(I)$, consider

$$I = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the set of vertices of $M_{\mathscr{S}}(I)$ is

$$\left\{ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \right\}.$$

For the construction of $G_{\mathcal{S}}(I)$ we use the following.

LEMMA 5: *$A_\psi \in G(I)$ if and only if there is a permutation π of $\{1, 2, \ldots, r\}$ so that $(d_i)\psi = d_{\pi(i)}$.*

PROOF: As in Lemma 4.

As an example, if

$$I = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$G_{\mathcal{S}}(I) = \left\{ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \right\}.$$

This work also gives the following theorem of Schwarz. (See [2].)

THEOREM (Schwarz.): *$G_{\mathcal{S}}(I)$ is isomorphic to S_r, the symmetric group on r letters.*

Having resolved the problem of constructing $M_{\mathcal{S}}(I)$, our final work concerns a constructive procedure for finding the maximal monoids in the semigroup of doubly stochastic matrices, $M_\Omega(I)$. In this case it should be noted that

$$I = \begin{pmatrix} J_{1/m_1} & & 0 \\ & J_{1/m_2} & \\ 0 & & J_{1/m_r} \end{pmatrix}$$

where $J_{1/k}$ is the $k \times k$ rank one doubly stochastic matrix.

Our first result, in this case, determines the basic nature of the matrices $M_\Omega(I)$.

THEOREM 6: *Let I be an idempotent in Ω. Let*

$$\mathrm{End}_\Omega\, C(I) = \{\psi \in \mathrm{End}\, C(I) | (d_i)\psi = \sum_{k=1}^{r} \lambda_{ik} d_k \text{ where } \sum_{k=1}^{r} \lambda_{ik} = 1 \text{ and } \sum_{k=1}^{r} m_k \lambda_{ki} = m_i\}.$$

Define $\pi: \mathrm{End}_\Omega\, C(I) \to M_\Omega(I)$ by $(\psi)\pi = A_\psi$ where the i^{th} row of A_ψ is $(d_i)\psi$. Then π is an isomorphism from $\mathrm{End}_\Omega\, C(I)$ *onto $M_\Omega(I)$.*

PROOF: By previous results, we need only show A_ψ is doubly stochastic. It is clear that A_ψ is stochastic. Further, as

$$\sum_{k=1}^{r} m_k[(d_k)\psi]_i = \sum_{k=1}^{r} m_k[\sum_{j=1}^{r} \lambda_{kj} d_j]_i = \sum_{k=1}^{r} m_k[\lambda_{ki} 1/m_i] = 1,$$

the theorem follows.

A result needed for our development follows as a corollary.

COROLLARY 2: Let $D = \text{diag.}\ (m_1, m_2, \ldots, m_r)$ and $M_\psi = (\lambda_{ij})$. Then $A_\psi \in M_\Omega(I)$ if and only if DM_ψ has row and column sums $m_1, m_2, \ldots, m_r$ respectively.

To construct $M_\Omega(I)$, we note that this set is a closed convex set. Thus, this set can be constructed if its vertices can be determined. This provides the direction for our remaining work.

LEMMA 6: *Let* $\mathscr{M} = \{DM_\psi | \psi \in \text{End}_\Omega\, C(I)\}$. *The map* $\pi: \mathscr{M} \to \mathscr{M}_\Omega(I)$ *defined by* $(M_\psi)\pi = A_\psi$ *is an isomorphism between the convex sets* $(\mathscr{M}, +)$ *and* $(\mathscr{M}_\Omega(I), +)$.

As π provides a one to one correspondence between the vertices of $(\mathscr{M}, +)$ and those of $M(I)$, our intent is to find the vertices of $(\mathscr{M}, +)$. The construction of these vertices may be accomplished by applying the general procedure of Ryser and Jurkat [3].

To indicate how these vertices can be constructed, consider

$$I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, by applying the Ryser-Jurkat procedure to find the vertices in $(\mathscr{M}, +)$ and then π to determine the vertices in $M_\Omega(I)$ we find

$$\left\{ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & T & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right\} \text{to be the set of vertices of } M_\Omega(I).$$

From this work, and the following lemma, we can also construct $G_\Omega(I)$.

LEMMA 7: *$A_\psi \in G_\Omega(I)$ if and only if DM_ψ has precisely one nonzero entry in each row and column.*

PROOF: This condition is both necessary and sufficient for ψ to have an inverse.

It should be noted that any such $DM_\psi = R$ has the property if $r_{ij} > 0$ then $r_{ij} = m_i = m_j$. Thus, to construct $G_\Omega(I)$, it follows that $A_\psi \in G_\Omega(I)$ if and only if $(d_i)\psi = d_j$ for $m_i = m_j$. As an example, consider

$$I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$G(I) = \left\{ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

This work can also be used to give a theorem of Farahat, [1].

THEOREM (Farahat): *Partition $\{m_1, m_2, \ldots, m_r\}$ by equality into sets* $X_1, X_2, \ldots, X_l$. Then $G(I)$ is isomorphic to $S_{X_1} \otimes S_{X_2} \otimes \ldots \otimes S_{X_l}$ where S_X is the symmetric group on X.

REFERENCES

[1] H. K. FARAHAT: The semigroup of doubly stochastic matrices. *Proc. Glasgow Math. Assoc.* 7 (1966) 178–183.

[2] P. FLOR: On groups of nonnegative matrices. *Compositio Math. 21* (1969) 376–382.

[3] W. B. JURKAT and H. J. RYSER: Term ranks and permanents of nonnegative matrices. *Journal of Algebra*, *5* (1967) 342–357.

[4] J. S. MONTAGUE and R. J. PLEMMONS: Convex Matrix Equations. *Bull. Amer. Math. Soc. 78* (1972) 965–968.
[5] R. J. PLEMMONS: Regular Nonnegative Matrices. *Proc. Amer. Math. Soc. 39* (1973) 26–32.
[6] D. J. RICHMAN and HANS SCHNEIDER: Primes in the semigroup of nonnegative matrices. (to appear in LMA)
[7] S. SCHWARZ: On the structure of the semigroup of stochastic matrices. *Magyar Tud. Akad. Mat. Kutato Inst. Kozl. 9* (1964) 297–311.
[8] S. SCHWARZ: A note on the structure of the semigroup of doubly stochastic matrices. *Mat. Casopis Sloven. Akad. Vied. 17* (1967) 308–316.

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