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LOCAL CONTRACTIBILITY OF SPACES OF HOMEOMORPHISMS

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Abstract

In this paper we give a new and simple proof of local contractibility of the space of homeomorphisms of a finite polyhedron. We find that the local contraction preserves PL homeomorphisms so we obtain the new result that the space of PL homeomorphisms of a finite polyhedron is locally contractible.

1. Introduction

If X is a topological space, let $\mathscr{H}(X)$ denote the group of homeomorphisms of X with the compact-open topology. If X is a polyhedron then $\mathrm{PL}(X)$ is the subgroup of PL homeomorphisms.

THEOREM (1): *Let K be a finite polyhedron. Then for each neighbourhood $\mathscr{U}$ of 1 in $\mathscr{H}(K)$ there is a neighbourhood $\mathscr{V}$ of 1 and a map*

$$\varphi: \mathscr{V} \times [0, 1] \rightarrow \mathscr{U}$$

satisfying

(i) $\varphi(h, 0) = h$ *for each* $h \in \mathscr{V}$;
(ii) $\varphi(h, 1) = 1$ *for each* $h \in \mathscr{V}$;
(iii) $\varphi(1, t) = 1$ *for each* $t \in [0, 1]$;
(iv) $\varphi(\{h\} \times [0, t])$ *is a* PL *isotopy for each* $h \in \mathrm{PL}(K) \cap \mathscr{V}$ *and each* $t \in [0, 1)$.

COROLLARY: *For a finite polyhedron K, the spaces $\mathscr{H}(K)$ and $\mathrm{PL}(K)$ are locally contractible.*

The corollary, together with theorem 1.9 of Geoghegan [3, p. 466] and the theorem of Haver [4, p. 281] tell us that PL(K) is an absolute neighbourhood retract.

An alternative proof that $\mathscr{H}(K)$ is locally contractible appears in [5] and an alternative proof of local contractibility of PL(M) for a compact PL manifold M appears in [2].

We now give an indication of the proof of theorem (1). Suppose that $h: K \to K$ is a homeomorphism which is near the identity 1. Using the idea in Section 8 of [1] we modify h in a neighbourhood of each vertex of K to h_1 so that h_1 is very close to 1 in a small neighbourhood of the 0-skeleton. The process is then repeated to modify h_1 in a neighbourhood of the barycentre of each 1-simplex to h_2 so that h_2 is very close to 1 in a small neighbourhood of the 1-skeleton. In this way we obtain an isotopy from h to a homeomorphism which is very close to 1 on the whole of K. Repeating the whole process, we obtain a sequence of isotopies in which the initial homeomorphism of an isotopy is the end product of the previous isotopy and the final homeomorphism of an isotopy is much closer to 1 than the initial homeomorphism. These isotopies may then be stacked together to give the map φ.

Extensions and variations of theorem (1) are possible. For example, by choosing the auxiliary functions in Proposition (5) to be smooth, one obtains

THEOREM (2): *The group of diffeomorphisms of a compact differentiable manifold with the C^r-topology (any $r \geqq 0$) is locally contractible.*

One can also give an alternative proof of Siebenmann's theorem [5, p. 132], as well as obtain relative and non-compact versions.

2. Notation and preliminary results

All spaces of embeddings (and homeomorphisms) are assumed to have the compact-open topology. When we say that an embedding h' constructed from an embedding h is *canonical* we mean that the function $h \mapsto h'$ on the embedding spaces is continuous.

If X is a topological space, by the (*open*) *cone* over X, denoted cX, is meant the quotient space obtained from the disjoint union of $X \times [0, \infty)$ and the singleton $\{0\}$ by identifying each point of $X \times \{0\}$ with 0. The point 0 is the *vertex* of cX. If $A \subset X$ then $cA \subset cX$ in a natural way. If $\alpha \in [0, \infty)$, let $c_\alpha X$ (resp. $\bar{c}_\alpha X$) be the subspace of cX obtained from $X \times [0, \alpha)$ (resp. $X \times [0, \alpha]$). If $y \in cX$ and $\alpha \in [0, \infty)$, define αy as follows:

y is the image of some $(x, \beta) \in X \times [0, \infty)$. Let αy denote the image in cX of $(x, \alpha\beta)$. For $B \subset cX$ and $\alpha \in [0, \infty)$, let

$$\alpha B = \{\alpha y \in cX | y \in B\}.$$

If $y \in cX$ is the image of (x, β), write $|y| = \beta$. Let

$$s_\alpha B = \{y \in cB| \, |y| = \alpha\} = \bar{c}_\alpha B - c_\alpha B.$$

PROPOSITION (3): *Let X be a metrisable space. Then X is compact if and only if its cone cX is metrisable by a metric d satisfying:*
(i) *$d(\alpha y, \alpha z) = \alpha d(y, z)$ for any $y, z \in cX$ and $\alpha \in [0, \infty)$;*
(ii) *$d(\alpha y, \beta y) = |\alpha - \beta| \cdot |y|$ for any $y \in cX$ and $\alpha, \beta \in [0, \infty)$.*

PROOF: If X is compact then we may embed X in the unit sphere of the Hilbert space l_2; cX then embeds in l_2 as all rays from the origin through X. The metric on cX induced from the norm on l_2 satisfies (i) and (ii).

Conversely, if cX is metrisable then the vertex 0 has a countable neighbourhood basis, so $X \times \{0\}$ has a countable neighbourhood basis in $X \times [0, 1]$. Thus X is sequentially compact and hence compact.

In the sequel, whenever we require a metric on cX for a compact metrisable space X we will assume that it satisfies conditions (i) and (ii) of Proposition (3).

If K is a finite polyhedron and σ a simplex of K, let b^σ denote the barycentre of σ and K' the first barycentric subdivision of K. Let $lk(b^\sigma, K')$ denote the link of b^σ in K'. Since K is finite, we may choose any metric on K; so suppose K is metrised by a metric which is linear on each simplex and which assigns to each edge the length 1. Then the closed star of b^σ in K' may be naturally identified with $\bar{c}_{\frac{1}{2}} lk(b^\sigma, K')$.

We require the following result which is 1.7 of [5].

PROPOSITION (4): *Let $h: F \to F'$ be an open embedding of locally compact locally connected Hausdorff spaces. Let $C \subset F$ be compact. If $g: F \to F'$ is another open embedding sufficiently near h in the compact-open topology then $h(C) \subset g(F)$. If, further, $g = h$ outside C, then $h(F) = g(F)$.*

3. The basic construction

The following result is essentially Edwards' wrapping process, cf. [5, Proposition 4.9] and [1, Section 8], except that we stop short of the final wrapping step.

Proposition (5): *Let α, δ and r be positive real numbers with $r < 1$ and let A be a closed subset of the compact, locally connected, metrisable space X. Let U be an open neighbourhood of $\bar{c}_\alpha A - c_{\alpha r^8} A$ in cX. Then there is a neighbourhood W of A in X such that for all sufficiently small $\varepsilon > 0$ and all open embeddings $h: c_\alpha X \to cX$ within ε of the inclusion i and within $\delta\varepsilon$ of i on U, there is an isotopy $h_t: c_\alpha X \to cX$ satisfying:*

(i) $h_t = i$ *if* $h = i$;
(ii) $h_0 = h$;
(iii) h_t *is canonical*;
(iv) $h_t | c_\alpha X - c_{\alpha r^4} X = h | c_\alpha X - c_{\alpha r^4} X$;
(v) $h_1(x) = r^3 h(x/r^3)$ *if* $x \in \bar{c}_{\alpha r^7} X$;
(vi) h_t *is within* 6ε *of* i;
(vii) $h_1 | c_\alpha W - c_{\alpha r^7} W$ *is withih* $6\delta\varepsilon$ *of* i;
(viii) *if* X *is a polyhedron and* h *is* PL *then so is* h_t.

Proof: Let W be a neighbourhood of A in X small enough so that the closure of $c_\alpha W - c_{\alpha r^8} W$ in cX lies in U. Suppose ε is a small positive number and $h: c_\alpha X \to cX$ is an open embedding within ε of i and within $\delta\varepsilon$ of i on U.

Define $\bar{\kappa}, \bar{\lambda}: [0, \infty) \to [0, \infty)$ as follows: $\bar{\kappa}(y) = y/r^3$; $\bar{\lambda}$ is multiplication by r^3 on $[0, \alpha r^3]$, takes $[\alpha r^3, \alpha r^2]$ linearly onto $[\alpha r^6, \alpha r^2]$ and is the identity on $[\alpha r^2, \infty)$. Let $\bar{\lambda}_t, \bar{\mu}_t: [0, \infty) \to [0, \infty)$ be PL isotopies satisfying:
(a) $\bar{\lambda}_0 = 1$ and $\bar{\lambda}_1 = \bar{\lambda}$;
(b) $\bar{\lambda}_t = 1$ on $[\alpha r^2, \infty)$;
(c) $\bar{\mu}_t = (\bar{\lambda}_t \bar{\kappa})^{-1}$.
Define $\lambda_t, \mu_t: cX \to cX$ by $\lambda_t(0) = \mu_t(0) = 0$ and if $x \in cX - \{0\}$, let

$$\lambda_t(x) = \frac{\bar{\lambda}_t(|x|)x}{|x|}, \qquad \mu_t(x) = \frac{\bar{\mu}_t(|x|)x}{|x|}.$$

Define the required isotopy $h_t: c_\alpha X \to cX$ by

$$h_t(x) = \begin{cases} h\mu_t h^{-1}\lambda_t h(x/r^3) & \text{if } |x| \leqq \alpha r^4 \\ h(x) & \text{if } |x| \geqq \alpha r^4, \end{cases}$$

see figure 1.

Provided ε is small enough, the first line of the definition of $h_t(x)$ is meaningful by Proposition (4). If $|x| = \alpha r^4$ then $x/r^3 \in s_{\alpha r} X$ so if ε is small enough, $h(x/r^3) \in c_\alpha X - c_{\alpha r^2} X$ on which $\lambda_t = 1$; thus

$$h^{-1}\lambda_t h(x/r^3) = x/r^3,$$

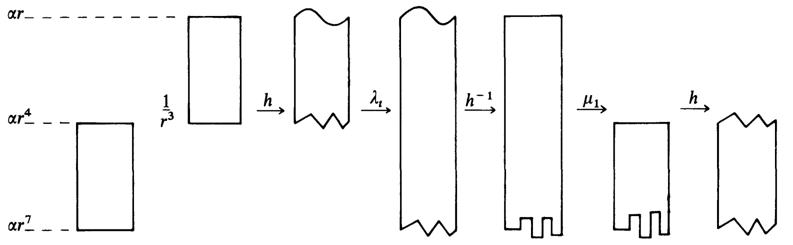

Figure 1. Definition of h_1 on $c_{\alpha r}4X - c_{\alpha r}7X$

and application of μ_t here is multiplication by r^3, so

$$h\mu_t h^{-1}\lambda_t h(x/r^3) = h(x)$$

when $|x| = \alpha r^4$. Thus h_t is well-defined.

The family h_t is clearly an isotopy satisfying (i) to (v) and (viii). Proof of satisfaction of (vi) is similar to that of (vii). We will verify the latter.

Suppose $x \in c_\alpha W - c_{\alpha r^7} W$. If $|x| \geqq \alpha r^4$ then $h_t(x) = h(x)$ so $h_t(x)$ is within $\delta\varepsilon < 6\delta\varepsilon$ of x. If $|x| \leqq \alpha r^4$ then $x/r^3 \in c_{\alpha r} W - c_{\alpha r^4} W$, so

$$d(h(x/r^3), x/r^3) < \delta\varepsilon.$$

Since λ_1 expands by at most a factor of 4, this implies

$$d(\lambda_1 h(x/r^3), \lambda_1(x/r^3)) < 4\delta\varepsilon.$$

Thus

$$d(h^{-1}\lambda_1 h(x/r^3), \lambda_1(x/r^3)) < 5\delta\varepsilon,$$

provided ε is small enough, so, since μ_1 does not increase distances, we have

$$d(\mu_1 h^{-1}\lambda_1 h(x/r^3), \mu_1\lambda_1(x/r^3)) < 5\delta\varepsilon,$$

and hence

$$d(h_1(x), x) < 6\delta\varepsilon,$$

as required by (vii).

REMARKS: There is nothing special about the constant 6 appearing in (vi) and (vii) of Proposition (5) and in Proposition (6) below except that it is independent of the embedding h. One could reduce the size of this constant by adjusting the auxiliary functions λ_t and μ_t, although it would have to exceed 3, this being the number of applications of h (or its inverse) in the definition of h_t. As was pointed out by the referee, the scale αr, αr^2, $\alpha r^3, \dots$ is not necessary: in fact Edwards and Kirby, and Siebenmann use a linear scale. However the above scale appears most suited to our application as it better respects the cone structure we impose on our polyhedra.

PROPOSITION (6): *Let α and δ be positive real numbers and let A be a closed subset of the compact, locally connected, metrisable space X. Let U be an open neighbourhood of $s_\alpha A$ in cX. Then there is a neighbourhood V of $c_\alpha A$ in cX such that for each sufficiently small $\varepsilon > 0$ and each open embedding $h: c_\alpha X \to cX$ within ε of the inclusion i and within $\delta\varepsilon$ of i on U, there is an isotopy $h_t: c_\alpha X \to cX$ of open embeddings satisfying*:

(i) $h_t = i$ *if* $h = i$;
(ii) $h_0 = h$;
(iii) h_t *is canonical*;
(iv) h_t *agrees with* t *near* $s_\alpha X$;
(v) h_t *is within* 6ε *of* i;
(vi) $h_1|V$ *is within* $6\delta\varepsilon$ *of* i;
(vii) *if* X *is a polyhedron and* h *is* PL *then so is* h_t.

PROOF: Choose $r < 1$ so that $\bar{c}_\alpha A - c_{\alpha r^8} A \subset U$, and let n be a positive integer so that $r^{3n} \leqq 6\delta$. Let W be the neighbourhood of A given by Proposition (5) and let

$$V = c_\alpha W \cup c_{\alpha r^{3n+4}} X.$$

For $\varepsilon > 0$ sufficiently small, if $h: c_\alpha X \to cX$ is an open embedding within ε of i and within $\delta\varepsilon$ of i on U, we will construct a canonical isotopy $h_t: c_\alpha X \to cX$ of open embeddings parametrised by $[0, n]$ so as to satisfy conditions (i) to (vii) above (but with h_n in place of h_1 in (vi)). By reparametrising the isotopy we obtain the desired result. Given $x \in c_\alpha X$, $t \in [0, n]$, say $t \in [k, k+1]$, let

$$h_t(x) = \begin{cases} h_k(x) & \text{if } |x| \geqq \alpha r^{3k+4} \\ r^{3k} h_{t-k}(x/r^{3k}) & \text{if } |x| \leqq \alpha r^{3k+4}. \end{cases}$$

The isotopy h_{t-k} in the second line of this definition is that given by Proposition (5). We have used condition (v) of Proposition (5) to enable us

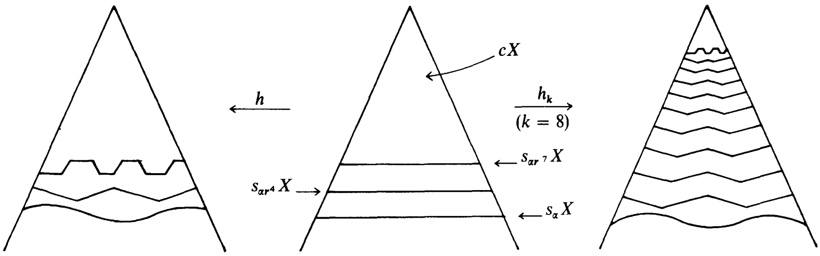

Figure 2. The repetitive nature of h_k

to isotope h to an embedding which repeats itself as we move toward the vertex of the cone. See Figure 2.

Satisfaction of most of the conditions of Proposition (6) follows from the corresponding condition in Proposition (5). Condition (vi) of Proposition (6) follows from condition (vii) of Proposition (5) for points outside $c_{\alpha r^{3n+4}}X$ and for $x \in c_{\alpha r^{3n+4}}X$, we have

$$h_n(x) = r^{3n}h(x/r^{3n}),$$

so since

$$d(h(x/r^{3n}), x/r^{3n}) < \varepsilon,$$

$$d(h_n(x), x) < r^{3n}\varepsilon \leqq 6\delta\varepsilon,$$

by choice of n.

4. Proof of the main results

Proof of Theorem (1): Suppose $\dim K = n$. For a homeomorphism $h: K \to K$ sufficiently close to 1, construct a canonical isotopy $h_t: K \to K$, $t \in [-1, n]$, with $h_{-1} = h$ and h_n twice as close to 1 as was h, by induction up the skeleton of K as follows. For each vertex v of K, recall from the end of Section 2 that the closed star of v in K' is identified with $\bar{c}_{\frac{1}{2}}lk(v, K')$. Applying Proposition (6) to h on this cone gives us an isotopy $^v h_t$, $t \in [-1, 0]$, from h to h_0 which is very close to 1 in a neighbourhood of v. By constructing this isotopy simultaneously about each vertex of K we obtain the isotopy h_t, $t \in [-1, 0]$, from $h = h_{-1}$ to h_0 which is very close to 1 in a neighbouthood of the 0-skeleton of K.

Inductively, if h_t has been constructed for $t \in [-1, k]$ so that h_k is very close to 1 in a neighbourhood of the k-skeleton of K, and σ is a (k+1)-simplex of K, then apply Proposition (6) to h_k on the cone $\bar{c}_{\frac{1}{2}} lk(b^\sigma, K')$, and simultaneously on the corresponding cones for the other $(k+1)$-simplices of K, to obtain an isotopy h_t, $t \in [k, k+1]$, so that h_{k+1} is very close to 1 in a neighbourhood of the $(k+1)$-skeleton.

By 'very close' above, one might mean the following: if h is within ε of 1, we want h_k to be within $\varepsilon/2.6^{n-k}$ of 1 in a neighbourhood of the k-skeleton. Further details are left to the reader.

If we set $k = n$ in our definition of 'very close', we see that the isotopy h_t takes $h_{-1} = h$ to h_n which is within $\varepsilon/2$ of 1 on a neighbourhood of the n-skeleton of K, i.e. on all of K. By reparametrising, we obtain an isotopy h_t, $t \in [0, \frac{1}{2}]$, from $h_0 = h$ so that $h_{\frac{1}{2}}$ is twice as close to 1 as was h. Repeating the process over and over, we obtain a canonical isotopy h_t, $t \in [0, 1)$, from $h_0 = h$ so that $h_{1-1/2^k}$ is twice as close to 1 as is $h_{1-1/2^{k-1}}$. We can then set $h_1 = 1$ to obtain the required canonical isotopy $\varphi(h, t)$.

PROOF OF THEOREM (2): For this we need smooth versions of Propositions (5) and (6). These are easily attained by making λ_t and μ_t in the proof of Proposition (5) smooth. Condition (v) of Proposition (5) actually holds in a neighbourhood of $\bar{c}_{\alpha r^7} X$ so when we piece together the isotopies in the proof of Proposition (6), this will be done smoothly.

Relative versions

If, for example, K is a finite polyhedron and A, B are closed subsets of K so that B is a neighbourhood of A then any homeomorphism of K sufficiently close to 1 which is already 1 on B can be deformed to 1 leaving A fixed. This construction is carried out in the usual way, i.e. choose a subdivision of K so fine that no simplex meets both A and $K-B$. Proceed as in the proof of theorem (1). Since the homeomorphism is already 1 on the closed star of A, the deformation will leave this set, and hence also A, fixed. In the case where A is a subpolyhedron of K we can dispense with the set B. Similarly, in the case where A is a subpolyhedron of K, any homeomorphism of the pair (K, A) sufficiently close to 1 can be deformed through homeomorphisms of (K, A) to 1. One can formulate analogues in the smooth case.

Siebenmann's Deformation theorem

(cf. [5], p. 132); We can adapt the above proof to give an alternative proof of Siebenmann's theorem. For example in the case where (Siebenmann's notation) X is a finite polyhedron, subdivide sufficiently so that no simplex meets both A and $Cl(X-A')$ and that the closed star of C lies in U, where C consists of the union of all (closed) simplices meeting B. Barycentrically subdivide X and apply the ideas of the inductive part of the proof of theorem (1) to $st'(C)$ to bring the embedding twice as close to 1 on $st'(C)$. Again subdivide X and repeat the process on $st''(C)$. Continuing in this way we obtain an isotopy of embeddings converging to an embedding which is the inclusion on C and agrees with the old embedding outside $st(C)$. Unfortunately this limiting embedding need not be PL even if the original embedding is PL. Thus although we obtain Siebenmann's theorem 2.3, we do not obtain PL or smooth analogues of this result.

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