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A WU FORMULA FOR EULER MOD 2 SPACES

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Since its introduction in [5], Euler mod 2 spaces, $E(2)$ space from now on, have attracted some interest, especially in regards to calculating their Stiefel-Whitney homology classes, see for example [1], [3]. In this paper we shall study w_{n-1} of an n-dimensional $E(2)$ space. We shall eventually obtain a formula which establishes the topological invariance for this class, this leads to a Wu type formula involving Sq^1 for this class in analogy with the situation in a smooth manifold. Then some examples are given to show that this formula is the best possible which one can obtain. One of the examples shows that w_{n-1} is not a homotopy invariant. Finally a general formula is given but not proved, which gives a method for defining the $S-W$ classes in the original complex without passing to the first derived. One final word is in order and that is that although everything is stated and proved in the context of $E(2)$ spaces the only hypothesis we really need is that the links of $n-1$ and $n-2$ dimensional simplexes have even Euler characteristic.

1. $w_{n-1}(K)$

Let K be an n-dimensional $E(2)$ space, that is K is a finite n-dimensional simplicial complex with the property that for any simplex, the number of simplexes in its link is even or equivalently the Euler characteristic of its link is even. The p-dimensional skeleton of a first derived of K, viewed as a p-chain with Z_2 coefficients, is a cycle whose homology class is called the p-dimensional Steifel-Whitney homology class of K and is written as $w_p(K)$. We observe that $w_n(K)$ is just the totality of all the n-simplexes in K and is clearly a topological invariant of K.

To each n-simplex in K arbitrarily choose an orientation. For each $(n-1)$-simplex s, the orientation induces on the set of n-simplexes which

* Supported by the Alexander von Humboldt Stiftung.

have s as a face, an equivalence relation defined as follows: two simplexes are equivalent if their orientations induce the same orientation on s. We denote by $A(s)$ and $B(s)$ the number of elements in each class and remark that they have the same parity which is called the parity of s with respect to the orientation. We write $P(s) = 0(1)$ if the parity is even (odd). In the rest of this section we assume that *everything* is taken mod 2 unless it is stated to the contrary.

THEOREM (1.1): *$\sum_S(P(s)+1)s$ is a cycle, whose homology class is $w_{n-1}(K)$.*

PROOF: We shall construct an n-chain, C, in K', whose boundary consists of all $n-1$-simplexes whose carrier is an n-simplex in K and those $n-1$-simplexes whose carrier is an $(n-1)$-simplex with odd parity. Now a typical n-simplex in K' can be expressed as $(s_0, s_1, \ldots, s_n)$ where s_j is a j-simplex in K. Order the vertices in s_n, so that $s_j = (v_0, \ldots, v_j)$, for every j. This well defined ordering on the vertices of s_n induces an orientation on s_n; we define C to consist of those simplexes $(s_0, s_1, \ldots, s_n)$, whose induced orientation on s_n agrees with the preassigned one on s_n.

Now an $(n-1)$-simplex in K' whose carrier is an n-simplex in K can be represented as $(s_0, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)$, where $0 \leqq j \leqq n-1$. Now there exactly two j-dimensional simplexes which can inserted in the above representation, however exactly one of the two will produce a simplex which is in C.

An $(n-1)$-simplex whose carrier is an $(n-1)$-simplex in K, looks like $(s_0, \ldots, s_{n-1})$. Now the coefficient of this simplex in ∂C, is equal to $P(s_{n-1})$, thereby proving the theorem.

COROLLARY (1.2): *$w_{n-1}(K)$ is 0 if and only if there exists an orientation for each n-simplex so that $P(s) = 1$ for every $(n-1)$-simplex.*

PROOF: Arbitrarily choose an orientation for each n-simplex and call the parity function P'. If $w_{n-1}(K)$ is 0, then there exists n-simplexes $t^1, \ldots, t^k$, so that $\sum_S(P'(s)+1) = \partial(t^1 + \ldots + t^k)$. Change the orientation on those n-simplexes and call the new parity function P. It is clear that P satisfies $P(s) = 1$ for each s.

The other implication is obvious.

Since K is an $E(2)$ space, the link of every $n-1$ simplex s, consists of $2k(s)$ vertices ($k(s)$ could be 0). Let Z_{n-1} be the chain consisting of those s so that $k(s)$ is even, that is $\sum_S(k(s)+1)s$.

LEMMA (1.3): *Z_{n-1} is a cycle.*

PROOF: Let r be an $(n-2)$-simplex and let C_r be its link. Now C_r is an 1-dimensional simplicial complex with $\chi C_r \equiv 0 \bmod 2$. Now the vertices in C_r correspond, via joining to r to $(n-1)$-simplexes in K and the 1-simplexes to n-simplexes in K. We call the order of a vertex, the number of 1-simplexes of which it is a face and denote it as $0(v)$, where v is the vertex. It is clear that $0(v) = 2k(v^*r)$. The lemma will be proved when we show that there is an even number of vertices v, with $0(v) \equiv 0 \pmod 4$. However this is an immediate consequence of the following:

LEMMA (1.4): *Let C be a 1-dimensional simplicial complex so that $0(v)$ is always even. Then the number of vertices v, with $0(v) \equiv 0 \pmod 4$ is equivalent to the χC modulo 2.*

PROOF:

Case a). If C has the property that $0(v)$ is either 0 or 2, then C is a disjoint union of circles and points and the proof is evident.

Case b). If v is a vertex with $0(v) > 2$, then add a new vertex v' to C disconnect 2 1-simplexes which meet v, and join them instead to v'. In this new complex C', the hypotheses of the lemma are still satisfied and both terms are changed by one. Keep on doing this until case a is reached. Hence the lemma is demonstrated.

We denote the homology class of Z_{n-1} as X_{n-1}. Now the carrier of Z_{n-1}, viewed as a point set of K is the closure of the union of the following two sets:

$$\text{*)} \qquad \{x \in K : H_n(|K|, |K|-x; Z)\} \cong Z^{2k}, k > 0.$$

$$\text{**)} \qquad \{x \in K : H_{n-1}(|K|, |K|-x; Z)\} \neq 0.$$

The first set are those points which lie on $(n-1)$-simplexes which have $k(s)$ even but not 0; the second set are those where $k(s)$ is 0. Hence X_{n-1} is a topological invariant. Now let B_* denote the Bockstein homomorphism in Z_2 homology.

THEOREM (1.5): $w_{n-1}(K) = B_*(w_n) + X_{n-1}$, *hence w_{n-1} is a topological invariant.*

PROOF: Using previous notation $B_*(w_n)$ has as a representative $\sum_s (A(s)-B(s))/2s$. It suffices to show that

$$P(s)+1 \equiv (A(s)-B(s))/2 + k(s) + 1 (\text{mod } 2).$$

Since $k(s) = (A(s)+B(s))/2$ and $P(s) \equiv A(s) \pmod 2$, the result follows. The topological invariance is a result of the fact that both w_n and X_{n-1} are topological invariants.

2. Wu type formula

Although $E(2)$ spaces do not satisfy Poincare duality, and are essentially homology objects, a relationship between a Steenrod square and w_{n-1} exists under certain conditions which is analogous to the situation when the $E(2)$ space is a smooth triangulation of a manifold. In this situation we know that if $Sq^1: H^{n-1}(K) \to H^n(K)$ is 0, then the first Wu class of the manifold and hence the first Stiefel-Whitney cohomology class is 0. However the Whitney theorem [2], implies that w_{n-1} is 0. We shall show that this is also valid for $E(2)$ spaces under the assumption that $k(s)$ is odd for every $(n-1)$-simplex, s.

This fact will follow quite readily from the following lemma, whose proof is immediate by an application of 'Pontryagin duality' and 1.5. Let $\mu: Z_p \to Z_p2$ be the homomorphism which takes 1 to p, and $\beta: Z_p2 \to Z_p$ be defined by sending 1 to 1 where p is a prime.

LEMMA (2.1): *$\beta_*: H_n(K; Z_p2) \to H_n(K; Z_p)$ is onto if and only if $\mu^*: H^n(K; Z_p) \to H^n(K; Z_p2)$ is a monomorphism.*

When we specialize to the case when p is 2, since Sq^1 is the coboundary homomorphism in the exact sequence induced by the coefficient homomorphism $0 \to Z_2 \xrightarrow{\mu} Z_4 \xrightarrow{\beta} Z_2 \to 0$, we get the following theorem

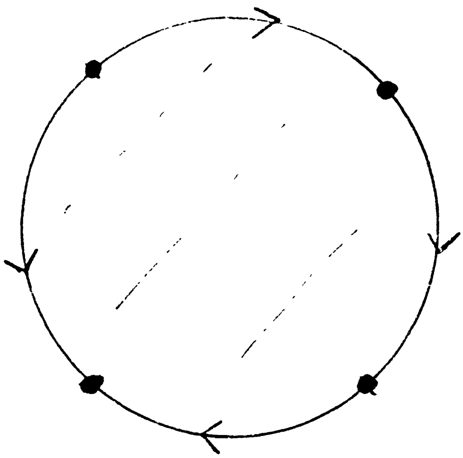
THEOREM (2.2): *Let K be an n-dimensional $E(2)$ space with $k(s)$ odd for every $(n-1)$-simplex, then $Sq^1: H^{n-1}(K) \to H^n(K) = 0$ implies that $w_{n-1}(K)$ is 0.*

PROOF: $Sq^1: H^{n-1}(K) \to H^n(K)$ equal to 0 implies that μ^* is one to one, hence by 2.1 we get that β_* is onto. Since X_{n-1} is 0 and B_* is 0, by applying 1.5 we arrive at the desired conclusion.

It should be noticed that under the additional hypothesis that $H_n(K; Z_2) \cong Z_2$, then the converse is also valid.

3. Examples

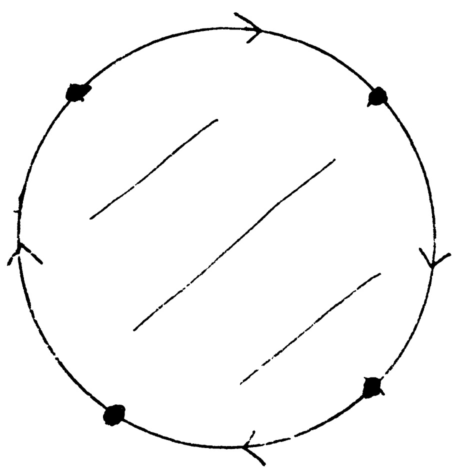
In the following examples, the spaces considered are what Ted Turner



Example 1

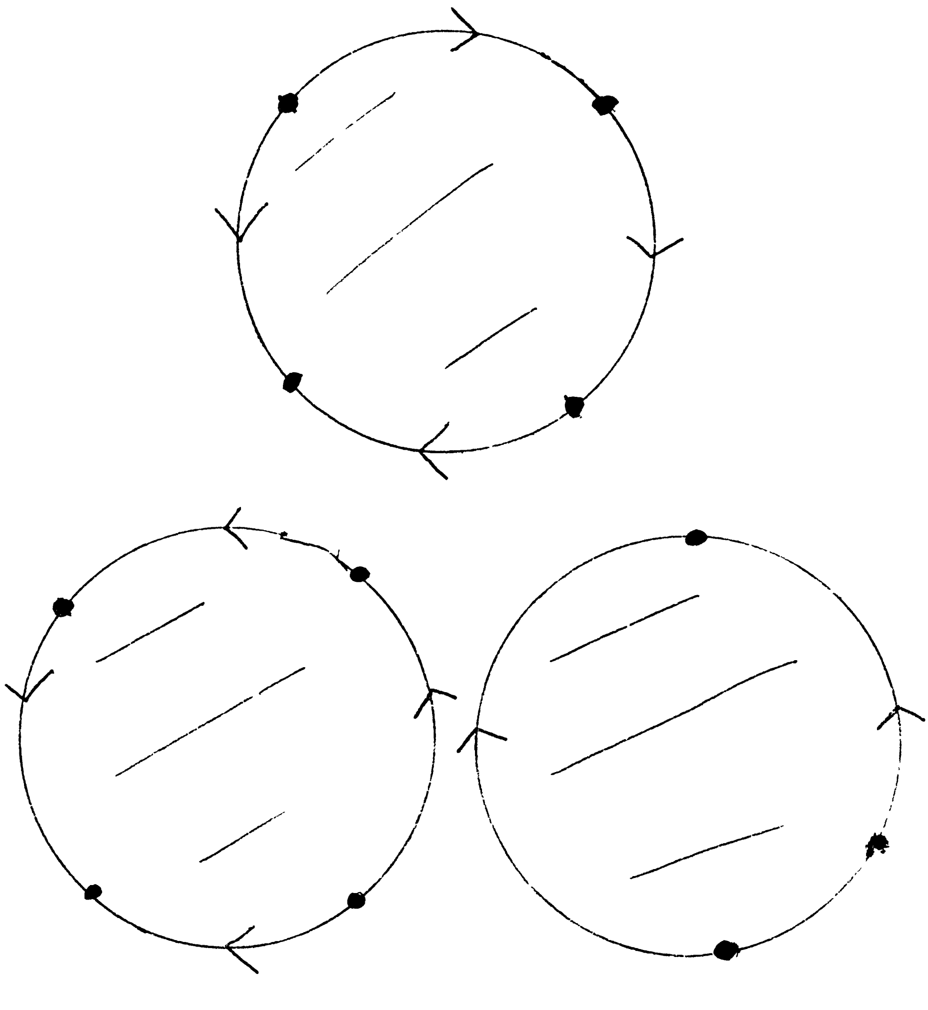
and the author call Quasi-regular complexes rather than simplicial complexes, the main result in [1] allows us to use the first derived of these to compute w_p. The 'obvious' first derived of these are simplicial complexes and one may do the calculations in these if it is desired.

EXAMPLE 1: Here K consists of a single 0, 1 and 2-cell. Since K is homotopy equivalent to RP^2, we have a Z_2 term in every dimension in Z_2 homology. It is easy to check that X_1 is also the generator in H_1, hence w_1 is 0. This example shows that the Stiefel-Whitney homology classes are not homotopy invariants for $E(2)$ spaces.



Example 2

EXAMPLE 2: Here K again consists of one cell in every dimension. Here we have that Sq^1 is 0, although w_1 is not the 0 element.



Example 3

EXAMPLE 3: K consists of 1 0-cell, 1 1-cell, and 3 2-cells. In this example $k(s)$ is odd for each 1 simplex. It is straightforeward to check that w_1 is 0, although Sq^1 is different from 0.

4. A different formula

In this section a formula will be given but not proven, which is a generalization of that given for w_{n-1}. It is interesting to note the not so accidental connection between this formula and the definition given by Steenrod in his original work on cohomological operations [6]. In fact except for a reversal of order the notion of *regularity* given here is identical to his.

DEFINITION: *Let K be a finite simplicial complex with the vertices ordered. Let t be a simplex in K which respect to the ordering is* $(v_0, \ldots, v_m)$. *A face s of t is said to be regular if* $s = (w_0, \ldots, w_p)$ *and the following condition is satisfied*

*) *If s is 0-dimensional then* $w_0 = v_0$

**) *If dimension of s is even then* $w_0 = v_0$, $w_1 = v_i$ *and* $w_2 = v_{i+1}$, $w_3 = v_j$ *and* $w_4 = v_{j+1}, \ldots,$ *and* $w_p = v_m$ *when p is odd.*

Now let $\partial_p t$ be the mod 2 union of all *regular* faces of t of dimension p. The following theorem is a product of joint work with Ted Turner.

THEOREM: *Let K be an E(2) space, then* $w_p(K) = \sum_{\dim t \geqq p} \partial_p t.$

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