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ON THE HOMOTOPY GROUPS OF SOME EQUIVARIANT AUTOMORPHISM GROUPS OF SPHERES

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We investigate the homotopy groups of the group of equivariant self-diffeomorphisms of a sphere S^{2dn-1}. The action involved is always the restriction to the unit sphere S^{2dn-1} of the representation $\rho_n \oplus \rho_n$ of $\Lambda_n = U(n)$ or $Sp(n)$ $(n \geqq 3)$. Here, ρ_n is the standard representation of real dimension $dn = 2n$ or $4n$, respectively. The equivariant automorphism group of (Λ_n, S^{2dn-1}) will be called Diff (Λ_n, S^{2dn-1}), or $D_n(\Lambda)$.

As in [3], we use the fact that non-trivial bundles over S^{k+1} with structure group $D_n(\Lambda)$ give rise to non-zero elements of $\pi_k D_n(\Lambda)$, and vice versa. The total space T of a bundle with structure group $D_n(\Lambda)$ and fibre S^{2dn-1} is a Λ_n-manifold with two orbit types and orbit space a manifold with boundary, and can therefore be handled using classification theorems by W. C. Hsiang, W. Y. Hsiang [5], and K. Jänich [6]. The data involved in classifying the Λ_n-manifold T are the orbit space T/Λ_n, which is the total space of a D^{d+1}-bundle over the same base (since $S^{2dn-1}/\Lambda_n \cong D^{d+1}$), and a reduction of the structure group of some bundle over the manifold boundary of T/Λ_n. In [3] we constructed non-trivial S^{2dn-1}-bundles having trivial D^{d+1}-bundle, but the above-mentioned reduction of a structure group was exotic. In the present note we construct non-trivial S^{2dn-1}-bundles starting with an exotic D^{d+1}-bundle, but the reduction of the structure group is trivial in some sense. More precisely, we have to use a non-linear D^{d+1}-bundle whose boundary is a trivial S^d-bundle. Our construction then gives an equivariantly non-linear S^{2dn-1}-bundle, as in [3].

$D_n(\Lambda)$ contains the subgroup of linear equivariant automorphisms of (Λ_n, S^{2dn-1}) which is isomorphic to Λ_2 [4]. That our construction yields non-linear bundles, means that non-zero elements of $\pi_k(\mathrm{Diff}(D^{d+1}; S^d))$ give rise to non-zero elements of $\pi_k D_n(\Lambda)$ which are not in the image of $\pi_k \Lambda_2$ (Theorem, Section 2). (Diff$(D^{d+1}; S^d)$ is the group of self-diffeomorphisms of D^{d+1} fixing the boundary S^d.) Unfortunately, not very much is known on the homotopy groups of $\mathrm{Diff}(D^{d+1}; S^d)$, for $d = 2, 4$. In [3] Theorem 4.8, we proved that $D_n(U)$ does not have the homotopy type of a finite CW complex. The corresponding result for Sp could not be proved by the methods of [3]. In this note we are able to prove it at least modulo a conjecture of D. Burghelea (Corollary, Section 3).

The analogue questions in the orthogonal case were answered in [3] by showing that the inclusion $O_2 \subset D_n(O)$ is a homotopy equivalence.

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The linear automorphisms of S^{2dn-1}, equivariant with respect to the action of Λ_n, form a group isomorphic to Λ_2 [4]. So Λ_2 acts on the orbit space S^{2dn-1}/Λ_n. S^{2dn-1} is a special Λ_n-manifold over D^{d+1} (e.g. [4]), so the orbit space S^{2dn-1}/Λ_n can be given a natural smooth structure diffeomorphic to D^{d+1}.

PROPOSITION 1 [3]: *The action of the group Λ_2 of equivariant linear automorphisms of $S^{2dn-1} (n \geqq 3)$ induced on the orbit space $S^{2dn-1}/\Lambda_n \cong D^{d+1}$ is smoothly equivalent to the orthogonal action of Λ_2 on D^{d+1} given by a homomorphism $\tau: \Lambda_2 \to O(d+1)$ with* $\ker \tau = \mathrm{center}\,(\Lambda_2)$.

This is Proposition 3.1 of [3]. There we proved only the topological equivalence of the action to the orthogonal one given by τ, and left the remainder of the proof to the reader. As this note heavily relies on the above proposition, we think it is adequate to present the remaining arguments.

First we claim that Λ_2 acts smoothly on the orbit space $S^{2dn-1}/\Lambda_n \cong D^{d+1}$. To see this we have to remember the definition of the smooth structure on the orbit space of a special Λ_n-manifold. In S^{2dn-1}, we replace the submanifold of singular orbits of the action of Λ_n by the total space of its normal sphere bundle [7]. Then Λ_n and Λ_2 still act smoothly, and the action of Λ_n has only one orbit type, so is a bundle over the orbit space. The base space B in such a bundle inherits a smooth structure from the total space. As B is diffeomorphic to D^{d+1} by hypothesis, the bundle is trivial, and its total space is diffeomorphic to $(\Lambda_n/\Lambda_{n-2}) \times D^{d+1}$. The action of Λ_2 is a smooth map

$$\Lambda_2 \times (\Lambda_n/\Lambda_{n-2}) \times D^{d+1} \to (\Lambda_n/\Lambda_{n-2}) \times D^{d+1}.$$

The induced action on the orbit space D^{d+1} of the Λ_n-action is therefore the induced map

$$\Lambda_2 \times D^{d+1} \to D^{d+1}$$

which is clearly differentiable.

In Section 3 of [3] we proved that this action has one fixed point, and in all non-fixed points the isotropy group is conjugate to $\Lambda_1 \times \Lambda_1$. As the action is linear in a neighborhood of the fixed point (by the slice theorem), there is an equivariant isomorphism φ of a small disk D_ε^{d+1} into D^{d+1} where Λ_2 acts on D_ε^{d+1} via the representation called τ in [3]. As the orbits of Λ_2 in $D^{d+1} - \varphi(\text{int } D_\varepsilon^{d+1})$ are $S^d \cong \Lambda_2/\Lambda_1 \times \Lambda_1$, and as the orbit space $(D^{d+1} - \varphi(\text{int } D_\varepsilon^{d+1}))/\Lambda_2$ is a compact connected 1-manifold with boundary, i.e. diffeomorphic to a real interval, it follows that the Λ_2-action on D^{d+1} is smoothly equivalent to the representation given by τ. This completes the proof of Proposition 1.

THEOREM: *Let* $j: \Lambda_2 \to D_n(\Lambda) = \text{Diff}(\Lambda_n, S^{2dn-1})$ *be the inclusion. Then there is a monomorphism*

$$g: \pi_k \,\text{Diff}(D^{d+1}; S^d) \to \pi_k D_n(\Lambda)/j_* \pi_k \Lambda_2$$

for every $k \geqq 0$ *and* $n \geqq 3$.

PROOF: The complement of an equivariant tubular neighborhood of the singular orbit bundle of S^{2dn-1} is equivariantly diffeomorphic to $(\Lambda_n/\Lambda_{n-2}) \times D_{1-\varepsilon}^{d+1}$. A diffeomorphism of D^{d+1} which is the identity on $N := D^{d+1} - \text{int } D_{1-2\varepsilon}^{d+1}$ therefore induces an equivariant diffeomorphism of S^{2dn-1} which is the identity in a neighborhood of the singular orbit bundle. This defines a homomorphism

$$h: \text{Diff}(D^{d+1}; N) \to D_n(\Lambda).$$

As $\pi_k \,\text{Diff}(D^{d+1}; N) \to \pi_k \,\text{Diff}(D^{d+1}; S^d)$ is bijective for all k ([2a] p. 120), by composition we have a homomorphism

$$g: \pi_k \,\text{Diff}(D^{d+1}; S^d) \to \pi_k D_n(\Lambda)/j_* \pi_k \Lambda_2.$$

To prove that g is monomorphic it suffices to show that the corresponding homomorphism

$$h_* : \pi_k \,\text{Diff}(D^{d+1}; N) \to \pi_k D_n(\Lambda)$$

has the property: If $x \in \pi_k \operatorname{Diff}(D^{d+1}; N)$, $x \neq 0$, then $h_*(x) \notin j_* \pi_k \Lambda_2$.

Bundles over S^{k+1} with structure group G are in 1-1 correspondence with the elements of $\pi_k G/\pi_0$, according to a well-known classification theorem [9]. Let $x \in \pi_k \operatorname{Diff}(D^{d+1}; N)$. Then x represents a D^{d+1}-bundle over S^{k+1}, and $h_*(x)$ represents a Λ_n-equivariant S^{2dn-1}-bundle over S^{k+1}. The orbit space of the latter is the D^{d+1}-bundle represented by x, which is non-linear if x is non-zero. But in that case, $h_*(x)$ cannot lie in $j_* \pi_k \Lambda_2$ because then the D^{d+1}-bundle would be linear, by Proposition 1. This proves the Theorem.

We give another description of the non-linear equivariant bundles constructed above, in the context of special Λ_n-manifolds [6] (compare Section 1). A non-zero element of $\pi_k \operatorname{Diff}(D^{d+1}; S^d)$ yields a non-trivial bundle over S^{k+1} with fibre D^{d+1}; moreover this bundle is non-linear. Let $\pi: E \to S^{k+1}$ be its projection. Clearly, the boundary ∂E is a trivial S^d-bundle over S^{k+1}. Choose a trivialization such that

$$\begin{array}{ccc} S^{k+1} \times S^d & \xrightarrow{\cong} & \partial E \\ & \searrow^{p_1} \quad \swarrow & \\ & S^{k+1} & \end{array}$$

is commutative. We construct, according to [6], a special Λ_n-manifold T with orbit types (Λ_{n-2}) and (Λ_{n-1}) over E using $E \times \Lambda_2$ as principal bundle of the principal orbit bundle. The other ingredient for our construction of T is the reduction of the structure group of the bundle

$$E \times \Lambda_2 \to E$$

over the boundary $\partial E \cong S^{k+1} \times S^d$ to $\Lambda_1 \times \Lambda_1$ (see [4] Section 3 and [3] Section 2). Such a reduction is a cross-section of the bundle $\partial E \times (\Lambda_2/\Lambda_1 \times \Lambda_1) \to \partial E$. Identifying $\Lambda_2/\Lambda_1 \times \Lambda_1$ with S^d and ∂E with $S^{k+1} \times S^d$, the reduction we use is the map

$$\begin{array}{ccc} S^{k+1} \times S^d & \to & S^{k+1} \times S^d \times S^d \\ \| & & \| \\ \partial E & \longrightarrow & \partial E \times (\Lambda_2/\Lambda_1 \times \Lambda_1) \end{array}$$

whose third component is given by the second projection $p_2: S^{k+1} \times S^d \to S^d$. This yields a Λ_n-manifold T with orbit space E, and the composition of the orbit map $T \to E$ with $\pi: E \to S^{k+1}$ is a bundle projection $T \to S^{k+1}$ The fibre of this bundle is S^{2dn-1}, the Λ_n-manifold over D^{d+1} (the fibre of π) that corresponds to the reduction of the structure group given by the identity map (point) $\times S^d = S^d \to S^d$ ([4] Prop. 3.2). So we have constructed a Λ_n-equivariant S^{2dn-1}-bundle over S^{k+1} with orbit space E, a non-linear D^{d+1}-bundle over S^{k+1}.

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COROLLARY TO THE THEOREM: *If* $\pi_2 \operatorname{Diff}(D^5; S^4) \neq 0$, *then* $D_n(Sp) = \operatorname{Diff}(Sp(n), S^{8n-1})$ *does not have the homotopy type of a finite CW complex.*

PROOF: By the Theorem, $\pi_2 D_n(Sp) \neq 0$, which is impossible if $D_n(Sp)$ has the homotopy type of a finite CW complex [1].

We cannot answer the question of whether $\pi_2 \operatorname{Diff}(D^5; S^4) \neq 0$, but only reduce it to a conjecture of D. Burghelea.

CONJECTURE 2 [2]: If $i \leqq 2n-2$, then $\pi_i \mathrm{Top}_n \to \pi_i \mathrm{Top}$ is onto.

PROPOSITION 2: *If Burghelea's Conjecture* (2) *is correct then* $\pi_2 \operatorname{Diff}(D^5; S^4)$ *is non-zero.*

CONSEQUENCE: If Burghelea's Conjecture (2) is correct then $D_n(Sp)$ does not have the homotopy type of a finite CW complex.

PROOF OF PROPOSITION (2): The homotopy sequence of the fibration $\mathrm{Top} \to \mathrm{Top}/O$ splits, i.e.

$$0 \to \pi_i O \to \pi_i \mathrm{Top} \to \pi_i(\mathrm{Top}/O) \to 0$$

is exact. This follows e.g. from Theorem 2.1.2) of [2]. As $\pi_8 O = Z_2$, we have $\pi_8 \mathrm{Top} \neq 0$. Applying Burghelea's Conjecture (2) for $i = 8$, $n = 5$, we obtain $\pi_8 \mathrm{Top}_5 \neq 0$. From $\pi_8 O_5 = 0$ [8] it follows that $\pi_8 \mathrm{Top}_5 \to \pi_8(\mathrm{Top}_5/O_5)$ is injective, therefore $\pi_8(\mathrm{Top}_5/O_5) \neq 0$. But

$$\pi_8(\mathrm{Top}_5/O_5) \cong \pi_2 \Omega^6(\mathrm{Top}_5/O_5) \cong \pi_2 \operatorname{Diff}(D^5; S^4)$$

by [2] Theorem 1.3, and we conclude that $\pi_2 \operatorname{Diff}(D^5; S^4) \neq 0$.

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