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D. R. LEWIS

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AN ISOMORPHIC CHARACTERIZATION OF THE SCHMIDT CLASS

D. R. Lewis [1]

This note gives an isomorphic characterization of the Hilbert-Schmidt norm within the class of unitarily invariant crossnorms on the space R of finite rank operators on l_2. Specifically if the completion of R under α has an unconditional basis (or more generally local unconditional structure) than α is equivalent to the Hilbert-Schmidt norm.

For convenience only real Banach spaces are considered. Notation and terminology is standard, with the following possible exceptions.

For E finite dimensional the unconditional basis constant of E is denoted by $\chi(E)$, and the parameter η is defined by $\eta(E) = \inf \|u\| \, \|v\| \chi(F)$, where the infimum is taken over all spaces F and pairs of operators $u : E \to F, v : F \to E$ satisfying $vu =$ identity. The definition of η is extended to infinite dimensional spaces by setting $\eta(G) = \inf_{\mathscr{D}} \sup_{E \in \mathscr{D}} \eta(E)$, with the infimum taken over all confinal collections of finite dimensional subspaces of G. G has *local unconditional structure* if $\eta(G) < \infty$.

R (respectively, $R(n)$) is the space of finite rank operators on l_2 (resp., l_2^n). A norm α on R (or $R(n)$) is called a *unitarily invariant crossnorm* if, for each $u \in R$, $\|u\| \leqq \alpha(u) \leqq i_1(u)$ and $\alpha(guh) = \alpha(u)$ for all isometries g and h. This coincides with Schatten's definition [7]. The completion of R under α is written $R(\alpha)$, and $R(\alpha, n)$ is $R(n)$ with norm α. Finally, π_p and i_p denote the p-absolutely summing and p-integral norms, respectively ([6]).

THEOREM: *If α is a unitarily invariant crossnorm on l_2 and if $R(\alpha)$ has local unconditional structure, then α is equivalent to the Hilbert-Schmidt norm.*

PROOF: Fix n and let P (respectively, Q) be the natural inclusion of $R(\alpha, n)$ (resp., $R(\alpha, n)'$) into $R(\pi_2, n)$. G denotes the group of isometries of l_2^n and dg is the normalized Haar measure on G. Let $\omega \in R(n)$ with $\omega \neq 0$, and write

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$(\omega^*\omega)^{\frac{1}{2}} = \sum_{i\leqq n}\lambda_i e_i \otimes e_i$, where (e_i) is some orthonormal basis for l_2^n and each $\lambda_i \geqq 0$. The dual of $R(\alpha, n)$ is naturally identified with $R(\alpha', n)$, where α' is the associate crossnorm of α([7]) and the action is the trace of the composition.

Define a probability measure μ on the closed unit ball of $R(\alpha, n)$ by setting

$$\mu(f) = \int_G \int_G f(\alpha(\omega)^{-1} g\omega h^*)dgdh.$$

Then for $u \in R(\alpha, n)$

$$\begin{aligned}\alpha(\omega)\mu(|\langle u, \cdot\rangle|) &= \iint |\text{trace } (ug(\omega^*\omega)^{\frac{1}{2}}h^*)|dgdh \\ &= \iint |\sum_{i\leqq n} \lambda_i(h(e_i), ug(e_i))|dgdh,\end{aligned}$$

the first since ω may be translated to $(\omega^*\omega)^{\frac{1}{2}}$ by an isometry. Now in the last integral replace g by gg_ε, where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ is an n-tuple of signs and $g_\varepsilon(e_i) = \varepsilon_i e_i$. Averaging over all such n-tuples ε and applying Khinchin's inequality yields

$$\begin{aligned}\alpha(w)\mu(|\langle u, \cdot\rangle|) &\geqq 3^{-\frac{1}{2}} \iint (\sum_{i\leqq n} \lambda_i^2 |(h(e_i), ug(e_i))|^2)^{\frac{1}{2}} dgdh \\ &\geqq 3^{-\frac{1}{2}} \left[\sum_{i\leqq n} \lambda_i^2 \left(\iint |h(e_i), ug(e_i))|dgdh\right)^2\right]^{\frac{1}{2}} \\ &\geqq 4^{-1}\pi_2(\omega) \iint |(h(e), ug(e))|dgdh\end{aligned}$$

for any vector $e \in l_2^n$ with norm one. Let dm be the normalized $(n-1)$-dimensional rotational invariant measure on the unit sphere of l_2^n and c_n the constant satisfying

$$\|x\| = c_n \int |(x, z)|m(dz), \quad x \in l_2^n.$$

This equality shows that for $\|e\| = 1$

$$\iint |(h(e), ug(e))|dgdh = \int c_n^{-1} \|u(x)\| m(dx)$$

$$= c_n^{-1} \int \Big(\sum_{i \leq n} |(x, u^*(e_i))|^2\Big)^{\frac{1}{2}} m(dx)$$

$$\geq c_n^{-1} \left[\sum_{i \leq n} \left(\int |(x, u^*(e_i))| m(dx) \right)^2 \right]^{\frac{1}{2}}$$

$$= c_n^{-2} \Big(\sum_{i \leq n} \|u^*(e_i)\|^2\Big)^{\frac{1}{2}},$$

and the last is nothing but $c_n^{-2}\pi_2(u)$. Combining inequalities now gives

$$\pi_2(u) \leqq 4c_n^2 \alpha(\omega)\pi_2(\omega)^{-1}\mu(|\langle u, \cdot\rangle|)$$

so that

$$\pi_1(Q) \leqq 4c_n^2 \alpha(\omega)\pi_2(\omega)^{-1}.$$

To estimate $\pi_1(P)$ let γ be the probability measure on the closed unit ball of $R(\alpha, n)'$ defined by

$$\gamma(f) = \int_G \int_G f(g(e) \otimes h(e)) dg dh,$$

where e is some fixed vector of norm one. The previous inequality shows

$$\pi_2(u) \leqq c_n^2 \gamma(|\langle \cdot, u\rangle|)$$

for all $u \in R(n)$, and consequently $\pi_1(P) \leqq c_n^2$. By the proof of Theorem 3.5 of [2]

$$n^2 \leqq \eta(R(\alpha, n))\pi_1(P)\pi_1(Q) \leqq \eta(R(\alpha, n)) 4c_n^4 \alpha(\omega)\pi_2(\omega)^{-1}$$

and by [1] $c_n \leqq (\pi n/2)^{\frac{1}{2}}$, so

$$\pi_2(\omega) \leqq \pi^2 \eta(R(\alpha, n))\alpha(\omega).$$

This first inequality is also valid if α is replaced by its associate α', and hence by duality (using $\pi_2' = \pi_2$)

$$\alpha(\omega) \leqq \pi^2 \eta(R(\alpha, n))\pi_2(\omega).$$

The constants in the last two inequalities are both dominated by $\pi^2\eta(R(\alpha))$, so that α and π_2 are $\pi^4\eta(R(\alpha))^2$ – equivalent on R. This completes the proof.

REMARK 1: An absolutely summing operator on a subspace of an $L_1(\mu)$-space or a quotient of a $C(K)$-space must factor through an L_1-space [2]. Thus the same proof as above shows that α is equivalent to the Hilbert-Schmidt norm if $R(\alpha)$ is isomorphic to a subspace of an L_1-space or to a quotient of a $C(K)$-space. In addition, by applying a basis selection theorem of Kadec-Pelczynski [3] and the result of McCarthy [5] that $R(c_p) \not\subset L_p$, it can be shown that α is equivalent to π_2 if $R(\alpha) \subset L_p$ for some $p > 2$. It seems a reasonable conjecture that α is equivalent to π_2 whenever $R(\alpha)$ embeds in a space with an unconditional basis.

REMARK 2: Given $\lambda \geqq 1$ there is a unitarily invariant crossnorm α on R which is equivalent to π_2 and such that the unconditional basis constant of $R(\alpha)$ is at least λ. To see this fix n and let ϕ_n be the symmetric gauge function ([7], Chapter V) on the space of finitely non-zero scalar sequences defined by

$$\phi_n(x) = \max \{(\sum_{k \geqq 1} |x_k|^2)^{\frac{1}{2}}, \quad \max_{|\sigma| = n} \sum_{k \in \sigma} |x_k|\}.$$

The crossnorm α on R induced by ϕ_n is clearly $n^{\frac{1}{2}}$-equivalent to π_2 and by the inequalities of the proof $\eta(R(\alpha)) \geqq \pi^{-2} n^{\frac{1}{2}}$. Similarly it is possible to produce unitary crossnorms β for which $\eta(R(\beta, n))$ increases very slowly. For instance, the crossnorm β induced by the symmetric gauge function

$$\psi(x) = \max_\sigma |\sigma|^{-\frac{1}{2}} \sum_{k \in \sigma} |x|$$

is, asymptotically, $(\log n)^{\frac{1}{2}}$-equivalent to π_2 on l_2^n.

REMARK 3: Suppose $R(\alpha, n)$ has a basis with the property that each permutation of the basis vectors naturally defines an isomorphism of norm at most λ. By [4] $\eta(R(\alpha, n)) \leqq 3\lambda$, so α is $9\pi^4\lambda^2$-equivalent to π_2 on l_2^n.

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Department of Mathematics
University of Florida
Gainesville, Florida 32601 U.S.A.