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REAL-VALUED FUNCTIONS ON CERTAIN SEMI-METRIC SPACES

Harold R. Bennett

In [1], H. Blumberg showed that if \( f \) is a real-valued function on Euclidean \( n \)-space \( E_n \), then \( E_n \) contains a dense subspace \( Y \) (depending on \( f \)) such that \( f \) restricted to \( Y \) is continuous. In this paper it is shown that if \( f \) is a real-valued function on a regular semi-metrizable Baire space \( X \), then \( X \) has a dense subspace \( Y \) such that \( f \) restricted to \( Y \) is continuous. Other questions and extensions of Blumberg's theorem are in [2], [6] and [7].

In proving the indicated result, the concepts of First Category sets and Second Category sets are crucial. The following theorem (found in [5], page 82) is implicitly used: If \( \{X_\alpha\} \) is a family of sets open relative to the union \( S = \bigcup X_\alpha \) and if each \( X_\alpha \) is of the First Category, then \( S \) is also of the First Category.

All undefined terms and notations are as in [4].

1. Preliminaries

In the following definitions let \( f \) be a real-valued function on a topological space \( X \) and let \( x \in X \).

**Definition (1.1):** The function \( f \) is said to approach \( x \) First Categorically (written \( f^1 \to x \)) if there is an \( \epsilon > 0 \) and a neighborhood \( N(x, \epsilon) \) of \( x \) such that \( M(x, \epsilon) = \{z \in N(x, \epsilon) : |f(z) - f(x)| < \epsilon \} \) is a First Category set in \( X \).

**Definition (1.2):** The function \( f \) is said to approach \( x \) Second Categorically (written \( f^2 \to x \)) if given \( \epsilon > 0 \) there exists a neighborhood \( N(x, \epsilon) \) of \( x \) such that \( M(x, \epsilon) = \{z \in N(x, \epsilon) : |f(z) - f(x)| < \epsilon \} \) is a Second Category set in \( X \). The function \( f \) is said to approach \( x \) Second Categorically via \( R \) (written \( f^2 \to x \text{ via } R \)) if given \( \epsilon > 0 \), there is a neighborhood \( N(x, \epsilon) \) such that \( M(x, \epsilon) \cap R \) is a Second Category set in \( X \).
DEFINITION (1.3): An open set $U$ is a partial neighborhood of a point $x$ if either $x$ is in $U$ or $x$ is a limit point of $U$.

It follows from Definition 1.2 that $f \to x$ if given $\varepsilon > 0$ there is a partial $U$ of $x$ such that for any open subset $V$ of $U \{z \in V : |f(z) - f(x)| < \varepsilon\}$ is a Second Category subset of $U$.

DEFINITION (1.4): A function $f$ is said to approach $x$ densely (written $f \to x$ densely) if given $\varepsilon > 0$ there is a neighborhood $N(x, \varepsilon)$ of $x$ such that $M(x, \varepsilon) = \{z \in N(x, \varepsilon) : |f(z) - f(x)| < \varepsilon\}$ is dense in $N(x, \varepsilon)$. If $x$ is a limit point of $R$, then $f$ is said to approach $x$ densely via $R$ (written $f \to x$ densely via $R$) if $M(x, \varepsilon) \cap R$ is dense in $N(x, \varepsilon) \cap R$.

The following is a useful characterization of Definition 1.4.

THEOREM (1.5): Let $f$ be a real-valued function on a topological space $X$. If $x \in X$, then $f \to x$ densely if and only if for each partial neighborhood $U$ of $x$, $f(x)$ is a limit point of $f(U)$.

PROOF: Suppose $f \to x$ densely and $U$ is any partial neighborhood of $x$. Let $\varepsilon > 0$ be given, then $x$ has a neighborhood $N(x, \varepsilon)$ such that $M(x, \varepsilon)$ is dense in $M(x, \varepsilon)$. Thus there exists $z \in M(x, \varepsilon) \cap U$ such that $|f(z) - f(x)| < \varepsilon$.

Hence $f(x)$ is a limit point of $f(U)$.

To show the converse, suppose $f$ does not approach $x$ densely. Then there is an $\varepsilon > 0$ such that for each neighborhood $N$ of $x$, the set $\{z \in N : |f(x) - f(z)| < \varepsilon\}$ is not dense in $N$. Thus, there is a non-empty open set $U_N$ contained in $N$ such that for all $y \in U_N$, $|f(y) - f(x)| \geq \varepsilon$. Then $U = \bigcup \{U_N : N a neighborhood of x\}$ is a partial neighborhood of $x$ such that $f(x)$ is not a limit point of $f(U)$.

Let $Z^+$ denote the set of natural numbers.

THEOREM (1.6): Let $f$ be a real-valued function on a topological space $X$. Then $F_1 = \{x \in X : f \downarrow x\}$ and $F_2 = \{x \in X : f$ does not densely approach $x\}$ are sets of the First Category in $X$.

PROOF: If $x \in F_1$, then there is an $\varepsilon(x) > 0$ and a neighborhood $N(x, \varepsilon(x))$ of $x$ such that $M(x, \varepsilon(x))$ is a First Category set. There is no generality lost if it is assumed that $\varepsilon(x) = 1/m(x)$ for some $m(x) \in Z^+$. For each $k \in Z^+$ let $C(k) = \{x \in F_1 : m(x) = k\}$ and let $D(k) = \{d(k, i) : i \in Z^+\}$ be a countable
dense subset of $f(C(k))$. Let $D = \bigcup \{D(k) : k \in \mathbb{Z}^+\}$. If $d(m, i) \in D$ let

$$R(m, i) = \{x \in C(m) : d(m, i) \leq f(x) < d(m, i) + 1/m\}$$

and if $x \in R(m, i)$, let

$$RM(x, i) = \{z \in M(x, 1/m) : d(m, i) \leq f(z) < d(m, i) + 1/m\}.$$ 

Similarly define

$$L(m, i) = \{x \in C(m) : d(m, i) - 1/m < f(x) \leq d(m, i)\}$$

and if $x \in L(m, i)$, let

$$LM(x, i) = \{z \in M(x, 1/m) : d(m, i) - 1/m < f(z) \leq d(m, i)\}.$$ 

If $x$ and $y$ are in $R(m, i)$, then

$$RM(x, i) \cap N(y, 1/m) \subseteq RM(y, i).$$

For if $z \in RM(x, i) \cap N(y, 1/m)$, then $|f(z) - f(y)| < 1/m$ and

$$d(m, i) \leq f(z) < d(m, i) + 1/m$$

and hence, $z \in RM(y, i)$. Thus

$$T(m, i) = \bigcup \{RM(x, i) : x \in R(m, i)\}$$

and $S(m, i) = \bigcup \{LM(x, i) : x \in L(m, i)\}$ are First Category sets. Since

$$F_1 \subseteq \bigcup \{T(m, i) : m, i \in \mathbb{Z}^+\} \cup \bigcup \{S(m, i) : m, i \in \mathbb{Z}^+\},$$

it follows that $F_1$ is a First Category set. The theorem mentioned in the introduction was used in the proof of this theorem.

If $x \in F_2$, then there exists $\varepsilon(x) = \varepsilon > 0$ such that for each neighborhood $N(x, \varepsilon)$ of $x$, $M(x, \varepsilon)$ is not dense in $N(x, \varepsilon)$. Let $\{r_1, r_2, \cdots\}$ be the set of rational numbers and if $r_i < r_j$, let

$$F(i, j) = \{x \in F_2 : f(x) - \varepsilon(x) < r_i < f(x) < r_j < f(x) + \varepsilon(x)\}.$$ 

It follows that $F(i, j)$ is nowhere dense for suppose $0$ is an open set such that $F(i, j)^- \supset 0$. If $p$ and $q$ are in $F(i, j) \cap 0$, then $|f(p) - f(q)| < \varepsilon(p)$. 
Thus \( \{ q \in 0 : |f(p) - f(q)| < \varepsilon(p) \} \) is dense in 0. This contradiction shows that \( F(i, j) \) is nowhere dense and it follows that

\[
F_2 = \bigcup \{ F(i, j) : i, j \in \mathbb{Z}^+, r_i < r_j \}
\]

is a First Category set.

2. Semi-metrizable Baire spaces

In the following let all spaces be \( T_1 \) spaces.

**Definition (2.1):** A topological space is a Baire Space if the countable intersection of open dense sets is a dense set.

**Theorem (2.2):** If \( X \) is a Baire space and \( f \) is a real-valued function on \( X \), then there is a dense set \( D \) (depending on \( f \)) such that if \( x \in D \), then \( f \to x \) densely via \( D \).

**Proof:** Let \( F_1 = \{ x \in X : f \to x \} \). By Theorem 1.6, \( F_1 \) is a First Category set. Let \( R_1 = X - F_1 \). If \( f_2 \to x \), then \( f_2 \to x \) via \( R_1 \). Let \( F_2 = \{ x \in R_1 : f \) does not approach x densely via \( R_1 \} \). Again by Theorem 1.6, \( F_2 \) is a First Category set. Thus \( D = X - (F_1 \cup F_2) \) is a residual set and, since \( X \) is a Baire space, \( D \) is dense in \( X \). If \( f_2 \to x \) via \( R_1 \), then \( f_2 \to x \) via \( D \) and if \( x \in D \) then \( f \to x \) densely via \( R_1 \). Let \( x \in D \). If \( \varepsilon > 0 \) is given and \( U \cap D \) is any partial neighborhood of \( x \) in \( D \) (\( U \) is a partial neighborhood of \( x \) in \( X \)), then, since \( f \to x \) densely via \( R_1 \) there exists \( q \in U \cap R_1 \) such that \( |f(q) - f(x)| < \varepsilon/2 \). Since \( U \) is a neighborhood of \( q \), \( \{ z \in U : |f(z) - f(q)| < \varepsilon/2 \} \cap D \) is a nonempty Second Category set. Let \( y \) be any one of its elements. Then

\[
|f(y) - f(x)| \leq |f(y) - f(q)| + |f(q) - f(x)| < \varepsilon/2 + \varepsilon/2 < \varepsilon.
\]

Thus \( f(x) \) is a limit point of \( f(U \cap D) \) and, by Theorem 1.5, \( f \to x \) densely via \( D \).

**Definition (2.3):** A topological space is a semi-metric space if there is a function \( d \) with domain \( X \times X \) and range a subset of the non-negative real numbers such that

(i) \( d(x, y) = d(y, x) \geq 0 \),
(ii) \( d(x, y) = 0 \) if and only if \( x = y \), and
(iii) \( x \) is a limit point of a set \( M \) if and only if

\[
\inf \{d(x, y) : y \in M\} = d(x, M) = 0 \quad \text{(See [3])}.
\]

In [3], by letting \( g(n, x) = \text{int} \{y \in X : d(x, y) < 1/n\} \), R. W. Heath has shown the following equivalent condition for a space to be a semi-metric space.

**Theorem (2.4):** Let \( X \) be a regular space and \( G = \{g(n, x) : n \in \mathbb{Z}^+, x \in X\} \) a collection of open subset of \( X \). If \( G \) satisfies

(i) for each \( x \in X \), \( \{g(m, x) : m \in \mathbb{Z}^+\} \) is a non-increasing local base at \( x \), and
(ii) if \( y \in X \) and, for each \( n \in \mathbb{Z}^+ \), \( y \in g(n, x_n) \), then the point sequence \( x_1, x_2, \cdots \) converges to \( y \).

Then \( X \) is a semi-metric space.

This theorem is a useful tool in the following theorem.

**Theorem (2.5):** If \( f \) is a real valued function on a regular semi-metrizable Baire space \( X \), then there is a dense subset \( Y \) of \( X \) such that \( f \) restricted to \( Y \) is continuous.

**Proof:** Since \( X \) is semi-metrizable there exists a collection

\[
G = \{g(m, x) : m \in \mathbb{Z}^+, x \in X\}
\]

of open subsets of \( X \) satisfying parts (i) and (ii) of Theorem 2.5. Let \( D \) be a dense set in \( X \) such that if \( x \in D \), then \( f \to x \) densely via \( D \). The existence of \( D \) is guaranteed by Theorem 2.2. Construct a discrete subset \( B(1) = \{x(1, \alpha) : \alpha \in \Lambda(1)\} \) of \( X \) and a pairwise disjoint subcollection \( G(1) = \{g(n(1, \alpha), x(1, \alpha)) : \alpha \in \Lambda(1)\} \) of \( G \) such that

(i) \( \left( \bigcup \{g \in G(1)\} \right)^- = X \), and
(ii) for each \( \alpha \in \Lambda(1) \), \( g(n(1, \alpha), x(1, \alpha)) \) contains a dense subset \( h(1, \alpha) \subseteq D \) such that if \( z \in h(1, \alpha) \), then \( |f(z) - f(x(1, \alpha))| < 1 \).

To obtain \( B(1) \) and \( G(1) \) let \( \eta \) be a well ordering of \( D \) and let \( \varepsilon = 1 \) be given. Let \( x(1, 1) \) be the first element of \( \eta \) and let \( n(1, 1) \) be the first element of \( \mathbb{Z}^+ \) such that

\[
h(1, 1) = \{z \in g(n(1, 1), x(1, 1)) \cap D : |f(z) - f(x(1, 1))| < 1\}
\]

is dense in \( g(n(1, 1), x(1, 1)) \). Suppose that \( x(1, \beta) \) has been chosen for each
$\beta < \delta$ such that

$$g(n(1, \beta), x(1, \beta)) \cap g(n(1, \alpha), x(1, \alpha)) = \emptyset$$

if $\alpha < \delta$, $\beta < \delta$ and $\alpha \neq \beta$. Let $x(1, \delta)$ be the first element of $\eta$ such that $x(1, \delta) \not\in (\bigcup \{g(n(1, \beta), x(1, \beta)) : \beta < \delta\})^-$. Let $n(1, \delta)$ be the first element of $\mathbb{Z}^+$ such that

$$g(n(1, \delta), x(1, \delta)) \cap (\bigcup \{g(n(1, \beta), x(1, \beta) : \beta < \delta\})^- = \emptyset$$

and $h(1, \delta) = \{z \in g(n(1, \delta), x(1, \delta)) \cap \mathcal{D} : |f(z) - f(x(1, \delta))| < 1\}$ is dense in $g(n(1, \delta), x(1, \delta))$.

Let $A(1)$ be the set of all $\alpha$ which have been chosen in the process described above. Let $B(1) = \{x(1, \alpha) : \alpha \in A(1)\}$ and let

$$G(1) = \{g(n(1, \alpha), x(1, \alpha) : \alpha \in A(1)\}.$$

Let $H(1) = \bigcup \{h(1, \alpha) : \alpha \in A(1)\}$. It follows that if $x \in H(1)$, then $f \rightarrow x$ densely via $H(1)$. For if $x \in H(1)$, then there exists $\alpha \in A(1)$ such that $x \in h(1, \alpha)$. Thus $|f(x) - f(x(1, \alpha))| = 1 - \delta$ for some $\delta > 0$. But if $x \in H(1)$, then $x \in \mathcal{D}$. Thus given $\delta > 0$, there is a neighborhood $N(x, \delta)$ of $x$ such that

$$\{z \in N(x, \delta) \cap \mathcal{D} : |f(z) - f(x)| < \delta\}$$

is dense in $n(x, \delta)$. If

$$z \in N(x, \delta) \cap \mathcal{D} \cap g(n(1, \alpha), x(1, \alpha)),$$

then

$$|f(z) - f(x(1, \alpha))| \leq |f(z) - f(x)| + |f(x) - f(x(1, \alpha))| < \delta + 1 - \delta = 1.$$

Thus $z \in h(1, \alpha) \subseteq H(1)$.

Suppose $B(1), \cdots, B(k), G(1), \cdots, G(k), H(1), \cdots, H(k)$ have been chosen such that for $1 \leq i \leq k$

(i) $B(1) \subseteq \cdots \subseteq B(k)$,

(ii) if $g \in G(i)$, then $g$ is a member of the local base for some element of $B(i)$,

(iii) $(\bigcup \{g \in G(i)\})^- = X$,

(iv) if $g \in G(i+1)$, then there is a $g' \in G(i)$ such that $g' \supseteq g$,

(v) the elements of $G(i)$ are pairwise disjoint,
(vi) $D \supseteq H(1) \supseteq \cdots \supseteq H(k)$,

(vii) $H(i) = \bigcup \{h(i, \alpha) : \alpha \in A(i)\}$ where $h(i, \alpha) \subseteq H(i-1)$ and $h(i, \alpha)$ is a dense subset of $g(n(i, \alpha), x(i, \alpha))$ such that if $z \in h(i, \alpha)$, then $|f(z) - f(x(i, \alpha))| < 1/i$ and

(viii) if $x \in H(i)$, then $f \to x$ densely via $H(i)$.

To obtain $B(k+1)$, $G(k+1)$, and $H(k+1)$, let $g(n(k, \alpha), x(k, \alpha)) \in G(k)$. Let $x(k, \alpha) \in B(k+1)$ and let $n(k+1, \alpha)$ be the first element of $\mathbb{Z}^+$ such that $g(n(k+1, \alpha), x(k+1, \alpha))$ is dense in $g(n(k+1, \alpha), x(k, \alpha))$. Select from $B(k+1, \alpha)' = \{x(k+1, \beta) : \beta \in A(k+1, \alpha)\}$ and select from $G$ a pairwise disjoint collection such that

(i) if $g \in G(k+1, \alpha)'$, then $g \subseteq U$,

(ii) $(\bigcup \{g \in G(k+1, \alpha)\})' = U'$, and

(iii) for each $\beta \in A(k+1, \alpha)$, $g(n(k+1, \beta), x(k+1, \beta))$ contains a dense subset $h(k+1, \beta) \subset H(k)$ such that if $z \in h(k+1, \beta)$, then $|f(z) - f(x(k+1, \beta))| < 1/k+1$,

Then let

$B(k+1) = \bigcup \{B(k+1, \alpha) : \alpha \in A(k)\}$,

$G(k+1) = \bigcup \{G(k+1, \alpha) : \alpha \in A(k)\}$,

$H(k+1, \alpha) = \bigcup \{h(k+1, \beta) : \beta \in A(k+1, \alpha)\}$,

and

(iii) for each $\beta \in A(k+1, \alpha)$,

$g(n(k+1, \beta), x(k+1, \beta))$ contains a dense subset $h(k+1, \beta) \subset H(k)$ such that if $z \in h(k+1, \beta)$, then $|f(z) - f(x(k+1, \beta))| < 1/k+1$,

and

(iv) $B(k+1, \alpha)' \subseteq H(k)$.

Let $B(k+1, \alpha) = B(k+1, \alpha)' \cup \{x(k, \alpha)\}$ and let

$G(k+1, \alpha) = G(k+1, \alpha)' \cup \{g(n(k+1, \alpha), x(k, \alpha))\}$.
and
\[ H(k+1) = \bigcup \{ H(k+1, \alpha) : \alpha \in A(k) \}. \]

It clearly follows that the induction hypothesis is satisfied.

Let \( Y = \bigcup \{ B(n) : n \in \mathbb{Z}^+ \} \) and for each \( n \in \mathbb{Z}^+ \), let \( K(n) = \bigcup \{ g \in G(n) \} \). It follows that \( K = \bigcap \{ K(n) : n \in \mathbb{Z}^+ \} \) is dense since each \( K(n) \) is an open dense subset of \( X \). Notice that \( Y \) is a dense subset of \( K \) if for \( z \in K \), then, for each \( i \in \mathbb{Z}^+ \), there is an \( x(i, \alpha_i) \) such that \( z \in g(n(i, \alpha_i), x(i, \alpha_i)) \) and, since \( X \) is a semi-metric space, the point sequence \( x(1, \alpha_1), x(2, \alpha_2), \ldots \) converges to \( z \). Thus \( Y \) is dense in \( X \).

Let \( x \in Y \) and let \( \epsilon > 0 \) be given. Since \( x \in Y \), there exists \( i \in \mathbb{Z}^+ \) such that \( x \in B(j) \) for each \( j \geq i \) and there exists \( k \in \mathbb{Z}^+ \) such that \( 1/k < \epsilon \). Let \( m = \max \{ i, k \} \). Since \( x \in B(m) \), \( g(n, x) \in G(m) \) for some \( n \in \mathbb{Z}^+ \) and if \( z \in g(n, x) \cap Y \), then \( |f(z) - f(x)| < 1/m < \epsilon \). Thus \( f \) restricted to \( Y \) is a continuous function.

**Definition (2.6):** A semi-metric space \( X \) is said to be weakly complete provided there is a distance function \( d \) such that the topology of \( X \) is invariant with respect to \( d \) and if \( \{ M_i : i \in \mathbb{Z}^+ \} \) is a monotonic decreasing sequence of non-empty closed sets such that, for each \( n \in \mathbb{Z}^+ \), there exists a \( 1/n \)-neighborhood of a point \( P_n \in M_n \) which contains \( M_n \), then \( \bigcap \{ M_n : n \in \mathbb{Z}^+ \} \) is non-void.

Standard arguments show that a regular weakly complete semi-metric space is a Baire space. Thus the following is established.

**Corollary (2.7):** If \( f \) is a real-valued function in a regular, weakly complete semi-metric space \( X \), then \( X \) has a dense subset \( Y \) such that \( f \) restricted to \( Y \) is continuous.

**References**


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