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SOME LINEAR TOPOLOGICAL PROPERTIES OF THE SPACES $C_p$
OF OPERATORS ON HILBERT SPACE

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1. Introduction

The purpose of this paper is to investigate some linear topological properties of the Banach spaces $C_p$, $1 \leq p < \infty$, consisting of all compact operators $x$ on the separable infinite dimensional Hilbert space $l_2$ for which $\|x\|_p = (\text{trace} (x^* x)^{p/2})^{1/p} < \infty$. Our main interest is in comparing the properties of these spaces with some known results concerning the structure of the function spaces $L_p = L_p(0, 1)$.

In section 2, we make some preliminary observations (which follow directly from known results) concerning projections in $C_p$. We discuss in particular the subspace $T_p$ of $C_p$ of those operators whose matrix representation with respect to a fixed orthonormal basis is triangular (the role of $T_p$ in $C_p$ is quite analogous to that of the Hardy space $H_p$ in $L_p$). We mention an analogy between the Haar basis in $L_p$ and an unconditional Schauder decomposition of $C_p$ and single out a subspace $S_p$ of $C_p$ which plays a central role in the study of $C_p$. The space $S_p$ is the direct sum

$$S_p = (C_p^1 \oplus C_p^2 \oplus \cdots \oplus C_p^n \oplus \cdots)_p$$

where $C_p^n$ denotes the space of all operators $x$ on the $n$-dimensional Hilbert space $l_2^n$ with $\|x\| = (\text{trace} (x^* x)^{p/2})^{1/p}$.

In section 3 we study basic sequences in $C_p$. For $2 \leq p < \infty$ the situation is simple and very similar to that in $L_p$ (as presented in [10]). Every normalized basic sequence has a subsequence equivalent either to the unit vector basis in $l_2$ or $l_p$. It turns out that the same result is valid (but the proof somewhat less simple) also for $1 \leq p < 2$. This is in marked contrast to the situation in $L_p$, $1 \leq p < 2$, where the structure of basic

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sequences is known to be far more involved (and interesting).

In section 4 we study the structure of ‘small’ subspaces of $C_p$. The situation is analogous to that discussed by Johnson and Odell [9] in the case of $L_p$. For $\infty > p > 2$ we show that a subspace of $C_p$ which does not contain a subspace isomorphic to $l_2$ is isomorphic to a subspace of $S_p$. For $1 \leq p < 2$ this is no longer true. The ‘right’ result for $1 < p < 2$ turns out to be the following. A subspace $X$ of $C_p$ embeds into $S_p$ if and only if every normalized basic sequence in $X$ has a subsequence which is $K$-equivalent to the unit vector basis of $l_p$ (where the constant $K$ depends only on $X$). This result is more complete than the analogue obtained in [9] for $L_p$.

In section 5 we study subspaces of $C_p$ which contain a copy of $S_p$. These spaces are characterized by their local structure.

In section 6 we classify up to isomorphism all the ‘obvious’ complemented subspaces of $C_p$, by showing the non-existence of certain embeddings. In this connection we also prove that for $1 \leq p < \infty$, $p \neq 2$, $L_p$ cannot be isomorphically embedded in $C_p$.

The main point in the study we present here is in the comparison between the properties of $C_p$ and the other two familiar spaces associated with the index $p$, i.e., the sequence space $l_p$ and the function space $L_p$. The space $C_p$ can be viewed as the natural matrix space associated to $p$ and as in the case of $L_p$ its structure is governed by an interplay between $l_2$ and $l_p$. From the results proved here combined with known results (several of those will be quoted in this and in the next section) there emerges what we think is a quite interesting picture. These results should motivate on the one hand a deeper study of the structure of $C_p$ and on the other hand a careful study of matrix spaces associated to other symmetric sequence spaces (e.g. Orlicz spaces $l_M$). The results proved in the present paper depend on properties which are well known to characterize $l_p$ among sequence spaces (perfectly homogeneous bases for example) and thus their analogues for $l_M$ are false. What we have in mind when we speak of the study of more general matrix spaces is the study in the context of matrices of questions which are usually considered in the sequence space or function space settings. Here is one such example. Can two matrix spaces associated e.g. with Orlicz spaces $l_{M_1}$ and $l_{M_2}$ be isomorphic without being identical? It is known that for sequence spaces this happens quite often while known partial results suggest that for function spaces this cannot happen. For a background to the comments made in this paragraph and for a general reference to the terminology of Banach space theory we refer to [13].

The basic properties of $C_p$ as a Banach space are presented in [1], [4] and [15]. Let us recall that $C_2$ is the space of Hilbert Schmidt
operators. As a Banach space it is isometric to $l_2$ and so its structure is simple and well known. Therefore the case $p = 2$ will often be excluded from the discussion below. The space $C_1$ is the space of nuclear operators on $l_2$ and is also called the space of the trace class operators. We shall use the notation $C_\infty$ to denote the space of all compact operators on $l_2$ with the usual operator norm. Most of our attention will be given to the spaces $C_p$ with $1 < p < \infty$ and $p \neq 2$. It is well known that if $1 < p \leq \infty$ and $p^{-1} + q^{-1} = 1$ then $C_p^*$ is isometric to $C_q$, the pairing between these spaces is given by $\langle x, y \rangle = \text{trace}(y^*x)$. The space $C_p$ is reflexive iff $1 < p < \infty$.

Here are some known results concerning the structure of $C_p$ which we shall need in the future.

(i) The spaces $C_p$ are uniformly convex for $1 < p < \infty$ and their modulus of convexity is up to a bounded factor the same as that for $L_p$ (cf. [15] and [16]). More precisely there are positive constants $\alpha_p$ and $\beta_p$ so that

\begin{equation}
\alpha_p \varepsilon^p \leq \delta_{C_p}(\varepsilon) \leq \beta_p \varepsilon^p \quad \text{if} \quad \infty > p > 2, \quad 0 < \varepsilon \leq 1
\end{equation}

\begin{equation}
\alpha_p \varepsilon^2 \leq \delta_{C_p}(\varepsilon) \leq \beta_p \varepsilon^2 \quad \text{if} \quad 1 < p < 2, \quad 0 < \varepsilon \leq 1.
\end{equation}

Closely related to (1.2) and (1.3) is the behaviour of terms of the form ‘average over all choices of signs of $\|\sum_{j=1}^n \pm x_j\|$, $\{x_j\}_{j=1}^n \in C_p^*$. The following inequalities concerning these expressions are proved in [15] and [16]: For $1 \leq p < \infty$ there is a constant $K_p$ such that for every integer $n$ and every choice of $\{x_j\}_{j=1}^n$ in $C_p$

\begin{equation}
K_p(\sum_{j=1}^n \|x_j\|^p)^{\frac{1}{p}} \leq \left(\int_0^1 \| \sum_{j=1}^n r_j(t)x_j\|^p dt\right)^{1/p} \leq \left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p},
\end{equation}

\begin{equation}
1 \leq p \leq 2
\end{equation}

\begin{equation}
\left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p} \leq \left(\int_0^1 \| \sum_{j=1}^n r_j(t)x_j\|^p dt\right)^{1/p} \leq K_p(\sum_{j=1}^n \|x_j\|^2)^{\frac{1}{2}},
\end{equation}

\begin{equation}
2 \leq p < \infty
\end{equation}

where the $r_j(t)$ denote the Rademacher functions.

(ii) The spaces $C_p$, $p \neq 2$, have no local unconditional structure (cf. [6]). We do not need here the definition of this notion but only the following consequence of the result of [6]. There is a sequence $\lambda(p, n)$ with $\lim_{n \to \infty} \lambda(p, n) = \infty$ for $p \neq 2$ so that if $X$ is Banach space with an unconditional basis (with unconditional constant equal to 1) if $T : C_p^* \to X$
is an isomorphism into and if $P$ is a projection from $X$ onto $TC^n_p$, then $\|T\| \cdot \|T^{-1}\| \cdot \|P\| \geq \lambda(p, n)$.

(iii) The space $C_p$ is not isomorphic to a subspace of $L_p$ if $1 \leq p < \infty$, $p \neq 2$. This fact is due to McCarthy [15] (for $p > 2$ the proof given by McCarthy is complete. For $1 \leq p < 2$ his basic idea works but some details have to be changed. Another proof for the case $1 \leq p < 2$ which gives also more precise information is given in [6]).

2. Preliminary observations

In this section we make some essentially known preliminary observations concerning some projections in $C_p$ and their ranges.

The elements of $C_p$ are by definition operators on $l_2$. We shall often work with the matrix representation of these elements with respect to a fixed orthonormal basis $\{e_i\}_{i=1}^\infty$ of $l_2$. The matrix $x(i, j)$ representing the element $x \in C_p$ is defined by $x(i, j) = (xe_i, e_j) 1 \leq i, j < \infty$. We shall often use the elements $u_{i,j} \in C_p$ defined by

$$u_{i,j}(k, l) = \delta_i^k \delta_j^l \quad 1 \leq i, j, k, l < \infty,$$

i.e., $u_{i,j}$ is the operator whose matrix has only one non-zero entry and this is 1 in the $(i, j)$th place. We mention in passing that in a suitable ordering the $\{u_{i,j}\}_{i,j=1}^\infty$ form a Schauder basis of $C_p$ (cf. [3], [11]).

The first projection we consider is the triangular projection $P_T$ defined by

$$P_T x(i, j) = \begin{cases} x(i, j) & i \geq j \\ 0 & j > i \end{cases}$$

This projection is known (cf. [14] and for a much simpler proof [5, pp. 118-120]) to be bounded in $C_p$ if $1 < p < \infty$ and not bounded in $C_1$ and $C_\infty$ (actually for $p = 1$ or $p = \infty$ (2.2) is defined only for $x$ belonging to the linear span of the $u_{i,j}$ and since $P_T$ is not bounded there it cannot be extended to the whole space). The range $P_T C_p$ of the projection $P_T$ is denoted by $T_p$. More precisely (in order to take into account also $p = 1, \infty$) we denote by $T_p$ the subspace of $C_p$ consisting of those $x$ for which $x(i, j) = 0$ for $j > i$.

Another important projection, actually a whole class of projections, in $C_p$ is obtained as follows. Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be two collections ($n$ is finite or $\infty$) of subsets of the integers such that $A_k \cap A_l = B_k \cap B_l = \emptyset$ if $k \neq l$. Corresponding to these families of subsets of the integers we define a projection as follows:
It is trivial to check that for every $1 \leq p \leq \infty$ and every choice of $\{A_k\}$ and $\{B_k\}$ the projection $P(\{A_k\}, \{B_k\})$ has norm 1. It is also evident that

$$\|P(\{A_k\}, \{B_k\})x\| = (\sum_k \|x_k\|_p^{1/p})$$

where $x_k(i, j) = x(i, j)$ if $(i, j) \in A_k \times B_k$ and $= 0$ otherwise. (If $p = \infty$ the sum in the right hand side will be replaced by $\sup_k \|x_k\|$.) We shall use the same convention in the future without mentioning it specifically. Whenever we use $l_p$ in the context of $p = \infty$ we shall mean the space $c_0$. Also direct sums in the $l_p$ sense will mean in case $p = \infty$ direct sums in the sense of $c_0$.) In particular if $A_k = B_k = \{k\}$ for $k = 1, 2, \cdots$ then $P(\{A_k\}, \{B_k\})$ is equal to the projection of a matrix onto its diagonal, and its range is isometric to $l_p$.

**Proposition 1:** The space $C_p$ is isomorphic to its subspace $T_p$ if and only if $1 < p < \infty$.

**Proof:** Assume that $1 < p < \infty$. Since $P_T$ is bounded we have

$$C_p = P_T C_p \oplus (I - P_T)C_p = T_p \oplus (I - P_T)C_p.$$ 

It is clear that $(I - P_T)C_p$ is isometric to $T_p$, and hence ($\cong$ denotes isomorphism)

$$C_p \cong T_p \oplus T_p$$

Let now $\{A_k\}_{k=1}^\infty$ be a sequence of disjoint infinite subsets of the integers. By (2.4), $P(\{A_k\}, \{A_k\})T_p$ is isometric to $(T_p \oplus T_p \oplus \cdots)_p$, i.e.,

$$T_p \cong (T_p \oplus T_p \oplus \cdots)_p \oplus X.$$ 

It follows from (2.6) that $T_p \cong T_p \oplus T_p$ and thus by (2.5) $C_p \cong T_p$.

If $p = 1$ or $p = \infty$ the space $C_p$ is not isomorphic to $T_p$ since $C_p$ does not have an unconditional Schauder decomposition into finite dimensional spaces (cf. [11]) while $T_p$ has such a decomposition. In fact for every $1 \leq p \leq \infty$ we have $T_p = \sum_{j=1}^p E_j$ where $E_j = \text{span} \{ u_{i,j}, 1 \leq i \leq j \}$ and the unconditional constant of this decomposition is evidently equal to 1.
Let us make some remarks concerning the notion of finite-dimensional Schauder decompositions which entered into the proof of Proposition 1. In [11] it was shown that $C_p$ (or more generally a symmetric matrix space in the terminology of [11]) has an unconditional finite-dimensional Schauder decomposition if and only if the triangular projection is bounded. The 'if' part follows also from the argument of Proposition 1. The interesting part is the 'only if' part. Its proof in [11] actually shows somewhat more which we would like to point out here (in view of the analogy with the Haar basis in $L_p$ cf. [12]).

**Definition:**
(i) A Schauder decomposition $X = \sum_n \oplus E_n$ is said to be equivalent to a Schauder decomposition $Y = \sum \oplus F_n$ if there is an isomorphism $T$ from $X$ onto $Y$ so that $TE_n = F_n$ for all $n$.

(ii) If $X = \sum \oplus E_n$ is a Schauder decomposition of $X$, if $n_1 = 1 < n_2 < n_3 < \cdots$ is an increasing sequence of integers and

$$F_j \subset E_{n_j} \oplus E_{n_j+1} \oplus \cdots \oplus E_{n_{j+1}}$$

then the $\{F_j\}_{j=1}^\infty$ are called a block decomposition of $\{E_n\}_{n=1}^\infty$. (The $\{F_j\}_{j=1}^\infty$ form a Schauder decomposition of their closed linear span.)

(iii) A finite dimensional Schauder decomposition $X = \sum \oplus E_n$ is said to be reproducible if whenever $X \subset Y$ and $Y$ has a finite dimensional Schauder decomposition $\sum \oplus F_n$ then the $\{E_n\}_{n=1}^\infty$ are equivalent to a block decomposition of the $\{F_n\}_{n=1}^\infty$.

We introduced here this definition of a reproducible decomposition since the matrix spaces provide natural examples for such decompositions which are not already bases.

For every integer $n$ let $P_n$ be the projection on $C_p$ defined by

(2.7) $P_n x(i, j) = \begin{cases} x(i, j) & \text{if } 1 \leq i, j \leq n \\ 0 & \text{otherwise.} \end{cases}$

Observe that $P_n$ is a special case of the family of projections defined in (2.3) (take $A = B = \{1, 2, \cdots, n\}$) and that $P_n C_p$ is isometric to the space we denoted in section 1 by $C_n^*$.

**Proposition 2:**

(i) For $1 \leq p \leq \infty$ the decomposition $C_p = \sum_{n=1}^\infty \oplus (P_n - P_{n-1})C_p$ is reproducible.

(ii) For $1 \leq p \leq \infty$ the decomposition $T_p = \sum_{n=1}^\infty \oplus (P_n - P_{n-1})T_p$ is reproducible.

This proposition is actually valid in the more general context of
symmetric matrix spaces. The proof of (i) is essentially given in [11] and we shall not repeat it here. The proof of (ii) is similar and even simpler. Observe that the decomposition in (i) is unconditional only if \(1 < p < \infty\). The decomposition in (ii) is exactly the decomposition which appeared in the proof of Proposition 1 and it is unconditional for every \(p\).

The ranges of the projections (2.3) will be classified in section 6. We deal here only with one special case dealing with the space \(S_p\) which plays a central role in section 4.

**PROPOSITION 3**: Let \(1 \leq p \leq \infty\) and let \(\{n_k\}_{k=1}^{\infty}\) and \(\{m_k\}_{k=1}^{\infty}\) be sequences of integers so that \(\sup_k (\min (n_k, m_k)) = \infty\). Let \(\{A_k\}\) and \(\{B_k\}\) be families of disjoint subsets of the integers so that \(|A_k| = n_k\) and \(|B_k| = m_k\) for every \(k\). Then \(P(\{A_k\}, \{B_k\})C_p\) is isomorphic to \(S_p = (\sum \oplus C^n_p)_{\infty}\).

**PROOF**: Let \(X = P(\{A_k\}, \{B_k\})C_p\). Let \(h \to k_h\) be a one to one map of the integers into themselves so that \(h \leq \min (n_{k_h}, m_{k_h})\). Let \(A'_h\) and \(B'_h\) be subsets of \(A_{k_h}\) and \(B_{k_h}\) respectively so that \(|A'_h| = |B'_h| = h\). Then \(P(\{A'_k\}, \{B'_k\})X\) is a complemented subspace of \(X\) which is isometric to \(S_p\). Hence \(X \approx S_p \oplus Y\) for some space \(Y\). A similar remark shows that \(S_p \approx X \oplus W\) and \(S_p \approx (S_p \oplus S_p \oplus \cdots) \oplus Z\) for some spaces \(W\) and \(Z\). A simple application of the decomposition method shows that \(S_p \approx X\).

Observe in particular that Proposition 3 implies that \(S_p\) is isomorphic to \((S_p \oplus S_p \oplus \cdots)_{\infty}\).

Another projection which will be of great use in the sequel is the projection \(E_n, n = 1 \cdots\) defined by

\[
E_n x(i, j) = \begin{cases} x(i, j) & \text{min } (i, j) \leq n \\ 0 & \text{otherwise} \end{cases}
\]

(2.8)

The projection \(E_n\) is the sum of two projections of the type (2.3) and thus in particular \(\|E_n\| \leq 2\). Its use in classifying subspaces of \(C_p\) is demonstrated in

**PROPOSITION 4**:

(i) A subspace \(X\) of \(C_p, 1 \leq p \leq \infty, p \neq 2\) is isomorphic to \(l_2\) if and only if the restriction \(E_n|X\) of \(E_n\) to \(X\) is an isomorphism for some \(n\). Consequently every subspace of \(C_p\) which is isomorphic to \(l_2\) is complemented in \(C_p\).

(ii) If \(X\) is subspace of \(C_p\) such that \(E_n|X\) fails to be an isomorphism for every \(n\), then for every \(\varepsilon > 0\) there is a subspace \(Z\) of \(X\) such that \(d(Z, l_p) \leq 1 + \varepsilon\) and there is a projection of norm \(\leq 1 + \varepsilon\) from \(C_p\) onto \(Z\).
In particular every subspace $X$ of $C_p$ has a subspace which is isomorphic either to $l_2$ or to $l_p$.

This proposition was observed by several mathematicians independently. A proof of it is given in [2] (and in [8] for $p = 1, \infty$) and will not be given here.

3. Basic sequences

This section is devoted to the proof of the following theorem.

**Theorem 1:** Let $\{x_n\}_{n=1}^\infty$ be a normalized basic sequence in $C_p$, $1 \leq p < \infty$. Then there is a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}$ which is equivalent to the unit vector basis in $l_2$ or in $l_p$. Moreover the subsequence may be chosen so that the span of $\{x_{n_k}\}$ is complemented in $C_p$.

**Proof:** The following three cases $2 \leq p < \infty$, $1 < p \leq 2$, and $p = 1$ will be discussed separately.

Consider first the reflexive case i.e., $1 < p < \infty$. The sequence $\{x_n\}$ tends weakly to 0 and thus by a standard perturbation argument (and passing to a subsequence if necessary) we may assume that there is a sequence of integers $m_1 < m_2 < \cdots$ so that

$$x_n \in (P_{m_n+1} - P_{m_n})C_p \quad n = 1, 2, \cdots$$

and thus in particular (since $X = \sum \oplus (P_{m+1} - P_m)X$ is an unconditional decomposition) that the sequence $\{x_n\}$ is an unconditional basic sequence.

Assume now that $2 \leq p < \infty$. If there is an integer $m$, a subsequence $\{n_k\}$ of the integers and a $\delta > 0$ so that

$$\|E_m x_{n_k}\| > \delta \quad k = 1, 2, \cdots$$

holds then $\{x_{n_k}\}$ is equivalent to the unit vector basis in $l_2$. Indeed, since the projections $E_m$ and $P_l$ commute, the sequence $\{E_m x_{n_k}\}_{k=1}^\infty$ is an unconditional semi-normalized basic sequence in $E_m C_p$ which is isomorphic to a Hilbert space. Hence there is a $\rho > 0$ such that for every choice of $\{\lambda_k\}$

$$\|\sum_k \lambda_k x_{n_k}\| \geq 2^{-1} \|\sum_k \lambda_k E_m x_{n_k}\| \geq \rho(\sum_k |\lambda_k|^2)^{1/2}.$$ 

On the other hand by (1.5) and the fact that $\{x_{n_k}\}$ is unconditional we get that
for some $R$. The span of $\{x_{n_k}\}$ is complemented in $C_p$ by the general fact that every isomorph of $l_2$ is complemented in $C_p$ (see Proposition 4).

If (3.2) does not hold then $\lim_{n \to \infty} \|E_m x_n\| = 0$ for every $m$. A standard argument shows now that for every given sequence $\varepsilon_k$ of positive numbers there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, vectors $\{y_k\}$ in $C_p$ and a sequence $t_k$ of integers so that for $k = 1, 2, \cdots$

$$
(3.4) \quad \|y_k - x_{n_k}\| \leq \varepsilon_k, \quad y_k = P_{t_k} y_k, \quad E_{t_k-1} y_k = 0, \quad \|y_k\| = 1.
$$

Hence for every choice of $\{\lambda_k\}$, $\|\sum_k \lambda_k y_k\| = (\sum_k |\lambda_k|^p)^{1/p}$ (this is a special case of (2.4)) and there is a projection of norm 1 from $C_p$ onto span $\{y_k\}$.

By the usual perturbation argument it follows that if the $\{\varepsilon_k\}$ are small enough the sequence $\{x_{n_k}\}$ is equivalent to the unit vector basis in $l_p$ and its span is complemented in $C_p$.

We pass now to the case $1 < p < 2$. If there is a $\delta > 0$ such that for every $m$ there is an $n$ with

$$
(3.5) \quad \|(I - E_m) x_n\| > \delta
$$

then there is a subsequence of $\{x_n\}$ which is equivalent to the unit vector basis in $l_p$. Indeed, if (3.5) holds we can in view of (3.1) find increasing sequences of integers $\{n_k\}$ and $\{t_k\}$ so that $\|Q_k x_{n_k}\| > \delta$ for every $k$ and $Q_k x_n = 0$ if $k \neq l$ where $Q_k = (I - E_{t_k-1}) P_{t_k}$. The projection $\sum_k Q_k$ is a projection of norm 1 in $C_p$ (it is a projection of the type (2.3)) and hence for every choice of $\{\lambda_k\}$ we have

$$
(3.6) \quad \|\sum_k \lambda_k x_{n_k}\| \geq \|(\sum_k Q_k)(\sum_k \lambda_k x_{n_k})\| = \|\sum_k \lambda_k Q_k x_{n_k}\| \geq \delta (\sum_k |\lambda_k|^p)^{1/p}.
$$

On the other hand since $\{x_{n_k}\}$ is an unconditional normalized basic sequence it follows from (1.4) that

$$
(3.7) \quad \|\sum_k \lambda_k x_{n_k}\| \leq R (\sum_k |\lambda_k|^p)^{1/p}
$$

for some $R$. The relations (3.6) and (3.7) show that $\{x_{n_k}\}$ is equivalent to the unit vector basis in $l_p$. The operator $P$ defined by

$$
(3.8) \quad Px = \sum_{k=1}^{\infty} \text{trace} (Q_k x \cdot y_k) x_{n_k}
$$

where $y_k = |Q_k x_{n_k}|^{p-2}(Q_k x_{n_k})^*/||Q_k x_{n_k}||^p$, is, as can be easily checked,
a bounded linear projection from \( C_p \) onto \( \overline{\text{span}} \{x_n\} \).

We turn now to the case where (3.5) fails, i.e.,

\[
\lim_{m \to \infty} \|(I - E_m)x_n\| = 0 \quad \text{uniformly in } n
\]

and show that in this case there is a subsequence of \( \{x_n\} \) which is equivalent to the unit vector basis in \( l_2 \) (as before, \( 1 < p < 2 \)). In this case the proof is a little less trivial and requires the following lemma.

**Lemma 1:** Let \( 1 < p < \infty \), let \( m \) be an integer and let \( \{y_n\}_{n=1}^{\infty} \) be a normalized basic sequence in \( E_mC_p \). Then there is a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) so that for all \( \{\lambda_k\} \)

\[
4^{-1}(\sum_k |\lambda_k|^2)^{\frac{1}{2}} \leq \|\sum_k \lambda_k y_{n_k}\| \leq 4(\sum_k |\lambda_k|^2)^{\frac{1}{2}}.
\]

The point in the lemma is the fact that in spite of the fact that \( d(E_mC_p, l_2) \) tends to infinity with \( m \) the constants in (3.10) are independent of \( m \).

**Proof of the Lemma:** There is no loss of generality to assume that \( y_n \in (P_{n+1} - P_n)C_p \) for some increasing sequence of integers. We shall assume in addition that \( y_n(i, j) = 0 \) if \( i > m \) and prove that in this case (3.10) holds with 4 replaced by 2. This will prove (3.10) in the general case since each \( y_n \in E_mC_p \) has a natural decomposition of the form \( y_n = y'_n + y''_n \) with \( y'_n(i, j) = 0 \) if \( j > m \) and \( y''_n(i, j) = 0 \) if \( i > m \). From our assumptions it follows that \( y^*_n y_k = 0 \) if \( n \neq k \) and that \( y_n(e_i) = 0 \) if \( i > m \). For every \( n \) let \( u_n \) be a unitary operator in \( l_2 \) such that \( u_n(e_i) = e_i \) for \( 1 \leq i \leq m \) and \( u_n y_n(e_i) \in \text{span} \{e_j\}_{j=1}^{2m} \) for \( 1 \leq i \leq m \) (and thus for all \( i \)). By the compactness of the unit ball in \( C_{2m}^p \) there is a sequence of integers \( \{n_k\} \) so that \( u_{n_k} y_{n_k} \) converges in norm to some operator \( x_0 \). Assume for the moment that \( u_{n_k} y_{n_k} = x_0 \) for all \( k \). Then for all scalars \( \{\lambda_k\} \)

\[
(\sum \lambda_k^* y_{n_k})(\sum \lambda_k^* y_{n_k}) = \sum_k |\lambda_k|^2 x_0^* u_{n_k} u_{n_k}^* x_0 = \sum_k |\lambda_k|^2 x_0^* x_0
\]

and hence

\[
\|\sum_k \lambda_k y_{n_k}\| = (\sum_k |\lambda_k|^2)^{\frac{1}{2}}.
\]

If instead of \( u_{n_k} y_{n_k} = x_0 \) we have only \( \|u_{n_k} y_{n_k} - x_0\| \leq \varepsilon_k \) with \( \varepsilon_k \) sufficiently small we get instead of (3.11) the relation (3.10) with 2 say, instead of 4.

We return to the proof of Theorem 1. Let \( \{\varepsilon_k\}_{k=1}^{\infty} \) be any sequence of positive numbers. By (3.9) there is an integer \( t_1 \) such that
By Lemma 1 there is a subsequence \( \{n_k^1\}_{k=1}^{\infty} \) of the integers so that

\[
4^{-1}(\sum_{k} |\lambda_k|^2\|E_{t_k}x_{n_k^1}\|^2)^{\frac{1}{2}} \leq \|E_{t_1}(\sum_{k} \lambda_k x_{n_k^1})\| \leq 4(\sum_{k} |\lambda_k|^2\|E_{t_k}x_{n_k^1}\|^2)^{\frac{1}{2}}.
\]

By (3.1) and (3.9) we can choose an integer \( t_2 \) so that

\[
E_{t_2}x_{n_1^1} = x_{n_1^1}, \quad \|(I-E_{t_2})x_n\| < \epsilon_2, \quad n = 1, 2, \cdots.
\]

By Lemma 1 there is a subsequence \( \{n_k^2\} \) of \( \{n_k^1\} \) so that

\[
\|(E_{t_2} - E_{t_1})\sum_{k} \lambda_k x_{n_k^2}\| \leq 4(\sum_{k} |\lambda_k|^2\|(E_{t_2} - E_{t_1})x_{n_k^2}\|^2)^{\frac{1}{2}}.
\]

We continue inductively to construct an increasing sequence of integers \( \{t_k\}_{k=1}^{\infty} \) and sequences \( \{n_k^1\}_{k=1}^{\infty} \) so that \( \{n_k^{l+1}\}_{k=1}^{\infty} \) is for every \( l \) a subsequence of \( \{n_k^l\}_{k=1}^{\infty} \) and

\[
E_{t_{k+1}}x_{n_k^k} = x_{n_k^k}, \quad \|(I-E_{t_{k+1}})x_n\| < \epsilon_{k+1}, \quad n = 1, 2, \cdots
\]

\[
\|(E_{t_1} - E_{t_{l-1}})\sum_{k} \lambda_k x_{n_k^l}\| \leq 4(\sum_{k} |\lambda_k|^2\|(E_{t_1} - E_{t_{l-1}})x_{n_k^l}\|^2)^{\frac{1}{2}}.
\]

Let \( n_k = n_k^k \) be the diagonal sequence. We claim that \( \{x_{n_k}\} \) is equivalent to the unit vector basis in \( l_2 \). Indeed, we have (with \( E_{t_0} = 0 \)) by (3.16) and (3.17) that

\[
\|\sum_{k=1}^{\infty} \lambda_k x_{n_k}\| = \|\sum_{k=1}^{\infty} \sum_{l=1}^{k} \lambda_k(E_{t_l} - E_{t_{l-1}})x_{n_k}\|
\]

\[
= \|\sum_{k=1}^{k+1} \sum_{l=1}^{k+1} \lambda_k(E_{t_l} - E_{t_{l-1}})x_{n_k}\|
\]

\[
\leq \|\sum_{k=1}^{\infty} \lambda_k(E_{t_{k+1}} - E_{t_k})x_{n_k}\| + \sum_{l=1}^{\infty} \|\sum_{k=1}^{l} \lambda_k(E_{t_k} - E_{t_{l-1}})x_{n_k}\|
\]

\[
\leq 2 \|\lambda_1\| \epsilon_1 + 4(\sum_{k=1}^{\infty} |\lambda_k|^2)^{\frac{1}{2}} + 8 \sum_{l=2}^{\infty} (\sum_{k=1}^{\infty} |\lambda_k|^2 \epsilon_{l-1}^2)^{\frac{1}{2}}
\]

\[
\leq (\sum_{k=1}^{\infty} |\lambda_k|^2)^{\frac{1}{2}}(4 + 10 \sum_{l=1}^{\infty} \epsilon_l).
\]

An estimate of \( \|\sum_{k} \lambda_k x_{n_k}\| \) from below (to show that it is greater than
\( p(\sum_k |\lambda_k|^2)^{\frac{1}{p}} \) for some \( p > 0 \) can be obtained either by using the fact that \( \{x_{nk}\}_{k=1}^{\infty} \) is an unconditional basic sequence and (1.4) or by using the left hand side inequality of (3.13) which combined with computation in (3.18) gives \( \|\sum_k \lambda_k x_{nk}\| \geq (4^{-1} - 10 \sum_i |e_i|)(\sum_k |\lambda_k|^2)^{\frac{1}{p}} \).

It remains to consider the case \( p = 1 \). If the sequence \( \{x_n\} \) satisfies (3.1) then the proof works just as in the case \( 1 < p < 2 \). The only difference between the case \( 1 < p < 2 \) and \( p = 1 \) is that the sequence \( \{x_n\} \) need not be an unconditional basic sequence. However the use of unconditionality can be avoided in the two places it was used, since (3.7) is trivial for \( p = 1 \) and as we have just observed the use of the left hand inequality of (3.13) avoids the use of unconditionality at the end of the proof. A general normalized basic sequence \( \{x_n\}_{n=1}^{\infty} \) in \( C_1 \) has a subsequence of the form \( x_{nk} = y + z_{nk} \) where \( z_{nk} \to 0 \) in the \( w^* \) topology of \( C_1 \) and \( \|z_{nk}\| \geq \delta > 0 \) for all \( k \) and some \( \delta \). By the preceding observation we may assume also that \( \{z_{nk}\} \) is equivalent either to the unit vector basis in \( l_1 \) and then the same is true for \( \{x_{nk}\} \) or to the unit vector basis in \( l_2 \) and then (since \( x_{nk} \) is a basic sequence) \( y \) must be equal to 0. This concludes the proof of the theorem for all \( 1 \leq p < \infty \).

**Remark:** Instead of assuming in the statement of Theorem 1 that \( \{x_n\} \) is a normalized basic sequence we could assume of course that \( x_n \to 0 \) weakly \( (w^* \text{ if } p = 1) \) and that \( \|x_n\| \to 0 \).

### 4. Subspaces of \( S_p \)

The first theorem in this section deals with subspaces of \( C_p, 2 < p < \infty \) which embeds into \( S_p \). The theorem and its proof is an adaptation of the work of Johnson and Odell [9] to the setting of \( C_p \).

**Theorem 2:** A subspace \( X \) of \( C_p, 2 < p < \infty \), is isomorphic to a subspace of \( S_p \) if and only if \( X \) has no subspace isomorphic to \( l_2 \).

**Proof:** The `only if` part is obvious. In order to prove the `if` part we observe first that by our assumption

\[
(4.1) \quad \lim_{n \to \infty} ||E_m(I - P_n)||_X = 0 \quad m = 1, 2, \cdots.
\]

Indeed, if (4.1) fails it follows easily that there is a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( X \) which tends weakly to \( 0 \) but for which \( \inf_n ||E_m x_n|| > 0 \) for some \( m \). By the proof of Theorem 1 such a sequence has a subsequence which is equivalent to the unit vector basis in \( l_2 \) contradicting our assumption.
Let $\delta > 0$ be given. In view of (4.1) there is a sequence of integers 
\[ \{n_k\}_{k=1}^{\infty} \] so that
\[ \| (I - P_{n_{k+1}}) E_{n_k} x \| \leq 2^{-k} \delta, \quad k = 1, 2, \ldots. \] (4.2)

Put $R_k = P_{n_{k+1}} - P_{n_k}$ and $Q_k = R_k (I - E_{n_{k-1}})$ for $k = 0, 1, 2, \ldots$ (where $P_{n_0} = 0$, $E_{n_0} = E_{n-1} = 0$). Observe that if $|k - h| > 1$, then $Q_k$ and $Q_h$ map into disjoint rectangles, i.e., there are sets $\{A_k\}$ and $\{B_k\}$ of integers so that $Q_k x(i, j) = 0$ unless $(i, j) \in A_k \times B_k$ and $A_k \cap A_h = B_k \cap B_h = \emptyset$ if $|k - h| > 1$.

Let us consider first the odd indices. We have by (2.4) and the preceding remark for every $x \in C_P$

\[ \| \sum_{k=0}^{\infty} Q_{2k+1} x \| = (\sum_{k=0}^{\infty} \| Q_{2k+1} x \|^p)^{1/p} \] (4.3)

and by (4.2) for $x \in X$

\[ \| \sum_{k=0}^{\infty} (R_{2k+1} - Q_{2k+1}) x \| \leq \| x \| \sum_{k=0}^{\infty} 2^{-2k} \cdot \delta \leq 2 \delta \| x \|. \] (4.4)

Hence for $x \in X$

\[ \| \sum_k R_{2k+1} x \| \leq \| \sum_k Q_{2k+1} x \| + 2 \delta \| x \| \\
= (\sum_k \| Q_{2k+1} x \|^p)^{1/p} + 2 \delta \| x \| \leq (\sum_k \| R_{2k+1} x \|^p)^{1/p} \\
+ \| x \| (\sum_k (2^{-2k} \delta)^p)^{1/p} + 2 \delta \| x \| \leq (\sum_k \| R_{2k+1} x \|^p)^{1/p} + 4 \delta \| x \|. \] (4.5)

A similar computation shows that

\[ \| \sum_k R_{2k+1} x \| \geq (\sum_k \| R_{2k+1} x \|^p)^{1/p} - 4 \delta \| x \| \] (4.6)

and working with the even indices we get similarly

\[ (\sum_{k=0}^{\infty} \| R_{2k} x \|^p)^{1/p} - 6 \delta \| x \| \leq \| \sum_{k=0}^{\infty} R_{2k} x \| \leq (\sum_{k=0}^{\infty} \| R_{2k} x \|^p)^{1/p} + 6 \delta \| x \|. \] (4.7)

Combining (4.5) and (4.7) we get for $x \in X$

\[ \| x \| = \| \sum_k R_k x \| \leq \| \sum_k R_{2k} x \| + \| \sum_k R_{2k+1} x \| \\
\leq 10 \delta \| x \| + 2^{1/q} (\sum_k \| R_k x \|^p)^{1/p} \]
and hence
\[(4.8) \quad \|x\| \leq 2^{1/q}(1 - 10\delta)^{-1}(\sum_k \|R_k x\|^p)^{1/p}.\]

By (4.6), (4.7) and the fact that the decomposition \(C_p = \sum_k R_k C_p\) is unconditional (with an unconditional constant \(M_p\)) we get that
\[(4.9) \quad 2M_p\|x\| \geq \|\sum_k R_{2k} x\| + \|\sum_k R_{2k+1} x\| \geq (\sum_k \|R_k x\|^p)^{1/p} - 10\delta\|x\|.
\]

It follows from (4.8) and (4.9) that if \(\delta\) is chosen small enough then for \(x \in X\)
\[(4.10) \quad (\sum_k \|R_k x\|^p)^{1/p} / 3M_p \leq \|x\| \leq 2(\sum_k \|R_k x\|^p)^{1/p}.\]

Since \(R_k C_p \subset C_p^{nk+1}\) it follows from (4.10) that the map \(x \rightarrow (R_1 x, R_2 x, \cdots)\) defines an isomorphism from \(X\) into \(S_p\).

The direct extension of Theorem 2 to the case \(1 < p < 2\) fails to be true. There is a subspace \(X\) of \(C_p\), \(1 < p < 2\), which does not contain a copy of \(l_2\) but which does not embed into \(S_p\). This counter-example is identical to that used in [9] for \(L_p\) and we recall it briefly. Let \(\delta > 0\) and let \(X_{\delta}\) be the subspace of \((l_p \oplus l_2)_p\) which is spanned by the vectors \(g_{i,\delta} = e_i + \delta f_i\) \(i = 1, 2, \cdots\) where \(e_i\) and \(f_i\) denote the unit vectors in \(l_2\) and \(l_p\) respectively. The sequence \(\{g_{i,\delta}\}_{i=1}^\infty\) is a symmetric basic sequence which is equivalent to \(\{f_i\}_{i=1}^\infty\) but the equivalence constant is large if \(\delta\) is small (if \(T : l_p \rightarrow X_{\delta}\) is defined by \(T f_i = g_{i,\delta}\) then \(\|T\| \cdot \|T^{-1}\| \geq \lambda^{-1}\)).

It is trivial to verify that every normalized basic sequence in \(S_p\) has for every \(\varepsilon > 0\) a subsequence which is \((1 + \varepsilon)\) equivalent to the unit vector basis \(l_p\). It follows from this observation that the distance from \(X_{\delta}\) to any subspace of \(S_p\) is \(\geq \lambda^{-1}\), and thus \(X = (\sum_n \oplus X_{1/n})_p\) is not isomorphic to a subspace of \(S_p\). The space \(X\) is clearly isometric to a subspace of \((l_2 \oplus l_2 \oplus l_2 \oplus \cdots)_p\) which in turn is isometric to a subspace of \(C_p\) (it is the range of a projection of the type (2.3) if we take \(A_k = \{k\}\) and \(B_k\) a family of disjoint infinite subsets of the integers). It is also evident that \(X\) has no subspace isomorphic to \(l_2\). Let us also remark that by Theorem 1 (or as can be seen directly) every normalized basic sequence in \(X\) has a subsequence which is equivalent to the unit vector basis in \(l_p\).

Theorem 2 may be restated as follows: A subspace \(X\) of \(C_p\), \(2 \leq p < \infty\), is isomorphic to a subspace of \(S_p\) if and only if every normalized basic sequence in \(X\) has a subsequence which is equivalent to the unit vector basis in \(l_p\). It turns out that by adding a uniformity condition on the equivalence we get a result which is valid also for \(1 < p \leq 2\).
THEOREM 3: Let \( X \) be a subspace of \( C_p \), \( 1 < p < \infty \). Then \( X \) is isomorphic to a subspace of \( S_p \) if and only if there is a constant \( K \) so that every normalized basic sequence in \( X \) has a subsequence which is \( K \) equivalent to the unit vector basis in \( l_p \).

The 'only if' part of the theorem is obvious. As remarked above the 'if' part for \( 2 \leq p < \infty \) is contained already in Theorem 2 and the uniformity assumption (i.e. the existence of \( K \)) need not be assumed a-priori. On the other hand the example given above shows that for \( 1 < p < 2 \) the uniformity assumption is essential. The proof we give to the 'if' part of Theorem 3 will show also that the distance coefficient of \( X \) from a suitable subspace of \( S_p \) can be estimated from above by a function of \( K \). In [9] Johnson and Odell prove an analogue of Theorem 3 in the case of \( L_p \). However, they use there an extra assumption on \( X \) and their result is thus less complete. The proof we give here, which avoids any extra assumption, is completely different from the argument used in [9]. Let us also point out that in contrast to the situation in \( L_p \), \( 1 < p < 2 \), the assumption of the 'if' part of Theorem 3 is, in view of Theorem 1, quite close to the assumption that \( X \) has no subspace isomorphic to \( l_2 \).

For the proof of Theorem 3 we need two lemmas.

**Lemma 2:** Let \( n_0 = 1 < n_1 < n_2 < \cdots \) be a finite or infinite sequence of integers. Consider the set of pairs of integers

\[
D(n_1, n_2, \cdots) = \bigcup_{k \geq 1} \{(i, j); n_k \leq i < \infty, 1 \leq j \leq n_{k-1}\}
\]

and let \( Q(n_1, n_2, \cdots) \) be the projection on \( C_p \) \( 1 < p < \infty \) defined by

\[
Q(n_1, n_2, \cdots) x(i, j) = \begin{cases} x(i, j) & \text{if } (i, j) \in D(n_1, n_2, \cdots) \\ 0 & \text{otherwise} \end{cases}
\]

Then there is a number \( M \) (depending on \( p \) but not on the integers \( n_1, n_2, \cdots \)) so that \( \|Q(n_1, n_2, \cdots)\| \leq M \).

**Proof:** We assume that the sequence \( \{n_k\} \) is infinite (for finite sequences the verification is the same but with a little different notation). Put \( A_k = \{i; n_k-1 \leq i < n_k\} \) and \( B_k = \{i; n_k-1 < i \leq n_k\} \), \( k = 1, 2, \cdots \) and \( B_0 = \{1\} \). The lemma is a consequence from the boundedness of the triangular projection \( P_T \) and the following easily verified identity

\[
Q(n_1, n_2, \cdots) + P(A_k, \{B_{k-1}\}) + P_T P(A_k, \{B_k\}) = P_T.
\]
**Lemma 3:** Let $1 < p < 2$ and let $X$ be a subspace of $\mathcal{C}_p$ for which there is a constant $K$ as in the statement of Theorem 3. Assume that $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of elements in $T_p = P_T \mathcal{C}_p$ so that

\begin{equation}
(4.13) \quad u_n + v_n \in X, \quad \|u_n\| = 1, \quad n = 1, 2, \cdots
\end{equation}

\begin{equation}
(4.14) \quad u_n, v_n \to 0 \text{ weakly}
\end{equation}

\begin{equation}
(4.15) \quad \| \sum_n \lambda_n u_n \| \leq 4(\sum_n |\lambda_n|^2)^{\frac{1}{2}} \quad \text{for all } \{\lambda_n\}.
\end{equation}

Then for all but finitely many indices $n$ we have

\begin{equation}
(4.16) \quad \|v_n\| \geq 1/(2K + 2).
\end{equation}

**Proof:** By (4.14) we may assume that (after passing to a suitable subsequence) $\{v_n\}$ is an unconditional basic sequence with an unconditional constant as close to 1 as we wish (recall that $v_n \in T_p$, and use the decomposition of $T_p$ appearing in Proposition 2 (ii)). Hence in view of (1.4) we may assume without loss of generality that for all $\{\lambda_n\}$

\begin{equation}
(4.17) \quad \| \sum_n \lambda_n v_n \| \leq 2(\sum \|v_n\|^p)^{1/p}.
\end{equation}

By our assumption on $K$ and (4.14) we may assume also that for all $\{\lambda_n\}$

\begin{equation}
(4.18) \quad (\sum |\lambda_n|^p\|u_n + v_n\|^p)^{1/p}/(K + \frac{1}{3}) \leq \| \sum \lambda_n (u_n + v_n) \|.
\end{equation}

Finally, if (4.16) fails, there is no loss of generality to assume that in addition to all the above we have that $\|v_n\| \leq 1/(2K + 2)$ for all $n$ and thus also

\begin{equation}
(4.19) \quad \|u_n + v_n\| \geq 1 - \|v_n\| \geq (2K + 1)/(2K + 2) \quad n = 1, \cdots
\end{equation}

By (4.15), (4.17), (4.18) and (4.19) we get for all integers $m$

\begin{equation}
(4.20) \quad m^{1/p}(2K + 1)/(K + \frac{1}{3})(2K + 2) \leq (\sum_{n=1}^m \|u_n + v_n\|^p)^{1/p}/(K + \frac{1}{3}) \leq \| \sum_{n=1}^m u_n \| + \| \sum_{n=1}^m v_n \| \leq 4m^{\frac{1}{2}} + 2m^{1/p}/(2K + 2).
\end{equation}

Evidently (4.20) is false if $m$ is large enough and this contradiction proves the lemma.
We now turn to the proof of Theorem 3 itself, \((1 < p < 2)\). We assume, as we may in view of Proposition 1, that \(X \subseteq T_p\), i.e., that for all \(x \in X\), \(x(i,j) = 0\) if \(j > i\).

The main step of the proof is the construction of an infinite sequence \(n_0 = 1 < n_1 < n_2 < \cdots\) of integers so that for all \(x \in X\)

\[
\| (I - Q(n_1, n_2, \cdots))x \| \geq \|Q(n_1, n_2, \cdots)x\|/2M(2K + 2).
\]

The sequence will be constructed inductively so that for every \(k\) and every \(x \in X\)

\[
\| (I - Q(n_1, n_2, \cdots, n_k))x \| \geq \|Q(n_1, n_2, \cdots, n_k)x\|/(2 - k^{-1})M(2K + 2).
\]

The choice of \(n_1\) is similar to the induction step so we present here only the induction step. Let us assume that \(n_1, n_2, \cdots, n_k\) have already been chosen so that (4.22) holds. If no suitable \(n_{k+1}\) exists then for all \(n > n_k\) there is an \(x_n \in X\) with \(\|x_n\| = 1\) for which

\[
x_n = y_n + z_n, \quad y_n = Q(n_1, n_2, \cdots, n_k, n)x_n, \quad \|z_n\| \leq \|y_n\|/(2 - (k + 1)^{-1})M(2K + 2).
\]

By passing to a subsequence we may assume that \(\{y_n\}\) and \(\{z_n\}\) converge weakly to \(y\) and \(z\) respectively. Clearly \(y + z \in X\) and it follows from (4.23) that

\[
y = Q(n_1, n_2, \cdots, n_k)y + Q(n_1, n_2, \cdots, n_k)z = 0.
\]

Hence, by (4.22)

\[
\|z\| \geq \|y\|/(2 - k^{-1})M(2K + 2).
\]

By comparing (4.23) with (4.25) we infer in particular that it is not true that \(\lim (\|y_n - y\| + \|z_n - z\|) = 0\) and thus

\[
\|y + z - y_n - z_n\| \geq \delta
\]

for some \(\delta > 0\) (recall that by (4.24) \(\lim_n \|Q(n_1, n_2, \cdots, n_k, n)z\| = 0\) and thus \(\|y - y_n\| + \|z - z_n\| \leq (2M + 2)||y + z - y_n - z_n\|\) for large \(n\).

Let \(\varepsilon > 0\) (its precise choice will be specified later) and choose an integer \(m > n_k\) so that

\[
\|P_m z - z\| < \varepsilon
\]
\((P_m\) is the projection defined by (2.2)). Put
\[
t_n = y - y_n, \quad s_n = z - z_n, \quad w_n = E_m s_n
\]

where \(E_m\) is the projection defined by (2.8). We claim that for infinitely many integers \(n\)
\[
\|(s_n - w_n)\| \geq \|t_n + w_n\|/(2K + 2).
\]

Indeed, since \((s_n - w_n) + (t_n + w_n) = y + z - y_n - z_n\), (4.29) is a consequence of (4.26) if \(\lim \|t_n + w_n\| = 0\). If \(\|t_n + w_n\|\) is bounded from below then (4.29) is a consequence of Lemma 3 when applied to \(u_n = (t_n + w_n)/\|t_n + w_n\|\) and \(v_n = (s_n - w_n)/\|t_n + w_n\|\). The sequence \(\{u_n\}\) and \(\{v_n\}\) certainly satisfy (4.13) and (4.14) while (4.15) can be obtained by passing to a suitable subsequence (use Lemma 1 and the fact that \(u_n = E_m u_n\) for all \(n\)).

Since \(\lim_n \|Q(n_1, n_2, \ldots, n_k, n)z\| = 0\) it follows from (4.23) and (4.28) that
\[
\lim_n \|Q(n_1, n_2, \ldots, n_k, n)w_n\| = 0, \quad Q(n_1, n_2, \ldots, n) \tau_n = \tau_n, \text{ all } n.
\]

By (4.29) and (4.30) and Lemma 2 we get that for infinitely many indices \(n\)
\[
\|s_n - w_n\|^p \geq \|t_n\|^p \cdot (2K + 2)^{-p} M^{-p} - \varepsilon^p.
\]

Observe next that on \(T_p\), \(P_m + I - E_m\) is a projection of norm 1 of the type (2.3) and thus for all \(n\)
\[
\|z_n\|^p \geq \|(P_m + I - E_m)z_n\|^p = \|P_m z_n\|^p + \|(I - E_m)z_n\|^p = \|P_m z - P_m s_n\|^p + \|(I - E_m)z - s_n + w_n\|^p.
\]

Now, \(\lim_n \|P_m s_n\| \to 0\) (since \(s_n \to 0\) weakly and \(\dim P_m C_p < \infty\)) and thus by (4.27) and (4.32) we get that for sufficiently large \(n\)
\[
\|z_n\|^p \geq \|z\|^p + \|s_n - w_n\|^p - 4\varepsilon^p.
\]

Also since \(t_n \to 0\) weakly it follows from (1.4) (or more precisely its special case called the Clarkson inequality) that for sufficiently large \(n\)
\[
\|y_n\|^p \leq \|y\|^p + \|t_n\|^p + \varepsilon^p.
\]

Combining (4.25), (4.31), (4.33) and (4.34) we get that there exist infinitely
many integers \( n \) for which

\[
(4.35) \quad \| z_n \|^p \geq \| z \|^p + \| s_n - w_n \|^p - 4\varepsilon^p \\
\geq \| y \|^p((2 - k^{-1})M(2K + 2))^{-p} + \| t_n \|^p(M(2K + 2))^{-p} - 5\varepsilon^p \\
\geq (\| y \|^p + \| t_n \|^p)((2 - k^{-1})M(2K + 2))^{-p} - 5\varepsilon^p \\
\geq \| y_n \|^p((2 - k^{-1})M(2K + 2))^{-p} - \varepsilon^p(5 + ((2 - k^{-1})M(2K + 2))^{-p}).
\]

But (4.35) contradicts (4.23) if \( \varepsilon \) is chosen small enough. This contradiction establishes the existence of a suitable \( n_{k+1} \). Thus we can choose the infinite sequence \( \{ n_k \} \) so that (4.22) holds for all \( k \) and therefore (4.21) holds as well.

It follows from (4.21) that the restriction of \( I - Q(n_1, n_2, \ldots) \) to \( X \) is an isomorphism. The proof of Theorem 3 is now completed by showing that \( Z = (I - Q(n_1, n_2, \cdots))T_p \) is isomorphic to a subspace of \( S_p \). The proof of this fact is similar to some arguments which were used in the proof of Theorem 2 and we repeat them here only briefly. Denote by \( \hat{Q}_k \) \( k = 1, 2, \cdots \) the projection on \( T_p \) defined by

\[
\hat{Q}_k x(i, j) = \begin{cases} 
 x(i, j) & \text{if } n_{k-1} \leq i < n_k, \ n_{k-2} < j \leq n_k \\
 0 & \text{otherwise}
\end{cases}
\]

where \( n_{-1} = 0 \). Then \( Z = \sum_k \hat{Q}_k Z \) and this decomposition is unconditional, with unconditionality constant equal to 1. Moreover for every \( z \in Z \)

\[
\| \sum_k \hat{Q}_2k z \|^p = \sum_k \| \hat{Q}_2k z \|^p, \quad \| \sum_k \hat{Q}_{2k+1} z \|^p = \sum_k \| \hat{Q}_{2k+1} z \|^p
\]

and this implies that \( Z \) is isomorphic to \( (\sum_k \hat{Q}_k Z)_p \) which in turn is isometric to a subspace of \( S_p \).

Note: The above isomorphism of \( Z \) into \( S_p \) has constant \( \leq 4 \). Using this fact and (4.22) we get an isomorphism of \( X \) into \( S_p \) with constant \( \leq \alpha \cdot M \cdot K \), where \( \alpha \) is an absolute constant, \( M \) depends only on \( p \), and \( K \) is the constant which appears in the statement of Theorem 3.

5. Subspaces of \( C_p \) which contain \( S_p \)

The purpose of this section is to prove that the answer to the question whether or not a given subspace \( X \) of \( C_p \) contains in it a copy of \( S_p \) depends only on the local structure of it.
THEOREM 4: Let $X$ be a subspace of $C_p$, $1 \leq p \leq \infty$. Then $S_p$ is isomorphic to a subspace of $X$ if and only if there is a constant $K$ so that for every $n$ there is an isomorphism $T$ from $C^n_p$ into $X$ with $\|T\| \cdot \|T^{-1}\| \leq K$.

PROOF: The 'only if' part is obvious and so we have to prove the 'if' part for $2 < p \leq \infty$ and $1 \leq p < 2$. Like in many other situations these two cases have to be considered separately.

Assume first that $2 < p \leq \infty$. We shall prove that for every given sequence of positive $\{\varepsilon_n\}$ there exist an increasing sequence of integers $m_n$ and operators $T_n : C^n_p \rightarrow X$ so that for $n = 1, 2, \cdots$

\begin{equation}
\|P_{m_n} T_n - T_n\| \leq \varepsilon_n
\end{equation}

\begin{equation}
\|E_{m_n} T_{n+1}\| \leq \varepsilon_n
\end{equation}

\begin{equation}
\|x\| \leq \|T_n x\| \leq K \|x\| \quad x \in C^n_p.
\end{equation}

Once (5.1), (5.2) and (5.3) are established the result follows easily. Indeed if the $\{\varepsilon_n\}$ are small enough we get from (5.1) and (5.2) that for any choice of $x_n \in C^n_p$

\begin{equation}
\frac{1}{2}(\|T_n x_n\|^p)^{1/p} \leq \|\sum T_n x_n\| \leq 2(\sum \|T_n x_n\|^p)^{1/p}
\end{equation}

and thus in view also of (5.3) the mapping $\hat{T} : S_p \rightarrow X$ defined by

\begin{equation}
\hat{T}(x_1, x_2, \cdots) = \sum_n T_n x_n
\end{equation}

is an isomorphism.

We construct the operators $\{T_n\}$ and the integers $\{m_n\}$ inductively in the following order $T_1, m_1, T_2, m_2, \cdots$. The operator $T_1$ is any operator satisfying (5.3). Once $T_n$ is constructed we choose $m_n$ so that (5.1) holds. The main point in the proof is to show given $m_n$ how to construct $T_{n+1}$ so that (5.2) holds as well as (5.3) (with $n$ replaced by $n+1$).

Let $N \geq 2n+2$ be such that

\begin{equation}
2K(N-1)^{1/p} < (N-1)^{\frac{3}{2}} \varepsilon_n/m_n(n+1)^2
\end{equation}

where $(N-1)^{1/p}$ is replaced by 1 if $p = \infty$. By Ramsey’s theorem [7], there is an $M = M(N)$ such that whenever the set of pairs of integers $(i, j)$, $1 \leq i, j \leq M$, $i < j$, is divided into two parts then there is a set $A \subset \{1, \cdots, M\}$ of cardinality $|A| = N$ so that all pairs $(i, j)$ with $i, j \in A$ (and $i < j$) belong to one of these parts. By our assumption there is an
operator $T : C_p^M \to X$ so that

$$\|x\| \leq \|Tx\| \leq K\|x\| \quad \text{for all } x \in C_p^M.$$  

Let $u_{i,j}$ denote the usual basis vectors of $C_p^M$. We show now that there is no subset $A$ of $\{1, \cdots, M\}$ with $|A| = N$ so that

$$\|E_{m_n} Tu_{i,j}\| \geq \varepsilon_n/(n+1)^2 \quad i, j \in A, \quad i < j.$$  

Indeed assume that $A = \{i_1, i_2, \cdots, i_N\}$ were such a set and consider the vectors $v_k = u_{i_k, i_{k+1}}$, $k = 1, 2, \cdots, N - 1$ in $C_p^M$. The $\{v_k\}_{k=1}^{N-1}$ are isometrically equivalent to the unit vector basis of $l_p^{N-1}$ and hence by (5.7) we have for every choice of signs

$$\| \sum_{k=1}^{N-1} \pm E_{m_n} Tv_k \| \leq 2K(N-1)^{1/p}.$$  

Since $d(E_{m_n} C_p, l_2) \leq m_n$ we get by the generalized parallelogram equality in $l_2$ (i.e., (1.4) and (1.5) which reduce to equalities with $K_p = 1$ if $p = 2$) that

$$\int_0^1 \| \sum_k r_k(t)E_{m_n} Tv_k \|^2 \, dt \geq \sum_k \|E_{m_n} Tv_k\|^2/m_n^2 \geq (N-1)\varepsilon_n^2/m_n^2(n+1)^4$$  

but this together with (5.9) contradicts the choice of $N$ in (5.6).

In view of what we just proved and Ramsey’s theorem it follows that there is a subset $A$ of $\{1, 2, \cdots, M\}$ with $|A| = N$ so that

$$\|E_{m_n} Tu_{i,j}\| \leq \varepsilon_n/(n+1)^2 \quad i, j \in A, \quad i < j.$$  

Since $N \geq 2(n+1)$ there are disjoint subsets $A_1$ and $A_2$ of $A$ so that $|A_1| = |A_2| = n+1$. Clearly (5.11) holds for all $(i, j) \in A_1 \times A_2$ and $Y = \text{span} \{u_{i,j}, (i, j) \in A_1 \times A_2\}$ is isometric to $C_p^{n+1}$. Clearly every $y \in Y$ has the form $y = \sum_{i,j} z_{i,j} u_{i,j}$ with $|z_{i,j}| \leq \|y\|$ for all $(i, j) \in A_1 \times A_2$. Hence by (5.11) we get that $\|E_{m_n} Ty\| \leq \varepsilon_n\|y\|$ for all $y \in Y$. In view of this fact and (5.7) we deduce that we can take as $T_n + 1$ the restriction of $T$ to $Y$. This concludes the proof for $2 < p \leq \infty$.

Assume now that $1 \leq p < 2$. In this case we shall prove that given any sequence $\{\varepsilon_n\}$ of positive numbers it is possible to choose an increasing sequence of integers $m_n$ and operators $T_n : C_p^n \to X$ so that for $n = 1, 2, \cdots$

$$\|P_{m_n} T_n - T_n\| \leq \varepsilon_n$$  

...
As in the proof of the case \( p > 2 \) we show first how to conclude the proof once (5.12)–(5.15) are satisfied. Put \( Q_1 = P_m \) and \( Q_n = P_m(I - E_{m_{n-1}}) \) for \( n \geq 2 \). It follows from (5.12) and (5.13) that if the \( \varepsilon_n \) are small enough then

\[
\|P_{m_n}T_{n+1}\| \leq \varepsilon_n
\]

(5.13)

\[
\|(I - E_{m_n})T_{n+1}x\| \geq \|x\|/2, \quad x \in C_p^{n+1}
\]

(5.14)

\[
\|x\| \leq \|T_nx\| \leq K\|x\|, \quad x \in C_p^n.
\]

(5.15)

As in the proof of the case \( p > 2 \) we show first how to conclude the proof once (5.12)–(5.15) are satisfied. Put \( Q_1 = P_m \) and \( Q_n = P_m(I - E_{m_{n-1}}) \) for \( n \geq 2 \). It follows from (5.12) and (5.13) that if the \( \varepsilon_n \) are small enough then

\[
\|\sum_n T_nx_n\| \leq M(\sum_n \|T_nx_n\|^p)^{1/p}
\]

for some constant \( M \) and all choices of \( x_n \in C_p^n \) (if \( 1 < p < 2 \) use the fact that the decomposition \( P_{m_{n+1}} - P_m \) is unconditional as well as (1.4). For \( p = 1 \) (5.16) is trivial). It follows from (5.12), (5.13) and (5.14) that if the \( \varepsilon_n \) are small enough we have for all \( x_n \in C_p^n \)

\[
\|(\sum_j Q_j)(\sum_n T_nx_n)\| \geq \|\sum_n Q_n T_nx_n\| - \sum_n \|(\sum_{j < n} Q_j)T_nx_n\| - \sum_n \|(\sum_{j > n} Q_j)T_nx_n\|
\]

\[
\geq (\sum_n \|Q_n T_nx_n\|^p)^{1/p} - 2\sum_n \varepsilon_n \|x_n\| \geq 4^{-1}(\sum_n \|x_n\|^p)^{1/p}.
\]

(5.15), (5.16) and (5.17) combine to show that (5.5) defines an isomorphism \( \tilde{T} \) from \( S_p \) into \( X \).

We now pass to the inductive construction of \( T_1, m_1, T_2, m_2, \ldots \). As in the case \( p > 2 \) the only fact which needs verification is that given \( m_n \) we can find a \( T_{n+1} \) so that (5.13), (5.14) and (5.15) (for \( n+1 \)) hold. Let \( N \) be an integer so that

\[
KN^{1/2} < N\varepsilon_n/3(n+1)^2
\]

(5.18)

\[
N^{1/p} > 4Km_nN^{1/2}.
\]

(5.19)

Since the space \( P_{m_n}C_p \) is finite dimensional there is a finite set of points which is an \( \varepsilon_n/3(n+1)^2 \) net in the ball of radius \( K \) and center 0 in \( P_{m_n}C_p \). By Ramsey’s theorem and by applying the assumption in the theorem to \( C_p^M \) for a suitable \( M \) (much larger than \( N \)) it follows that there is an operator \( T : C_p^{n+1} \to X \) and an element \( x_0 \in P_{m_n}C_p \) so that
\begin{align}
(5.20) \quad ||x|| \leq ||Tx|| \leq K||x||, \quad x \in C^{(n+1)N}_p \\
(5.21) \quad ||P_{mn}Tu_{i,j} - x_0|| \leq \epsilon_n/3(n+1)^2 \quad 1 \leq i, j \leq (n+1)N.
\end{align}

Since \( \sum_{j=1}^{N} u_{i,j} \| \leq N^{\frac{1}{2}} \) we get from (5.20) and (5.21) that

\begin{align}
(5.22) \quad N||x_0|| \leq \sum_{j=1}^{N} ||P_{mn}Tu_{i,j} - x_0|| + ||P_{mn}|| \cdot ||T|| \cdot \sum_{j=1}^{N} u_{i,j} || \\
\quad \leq N\epsilon_n/3(n+1)^2 + KN^{\frac{1}{2}}
\end{align}

and hence by (5.18), \( ||x_0|| \leq 2\epsilon_n/3(n+1)^2 \), and therefore (5.21) implies

\begin{align}
(5.23) \quad ||P_{mn}Tu_{i,j}|| \leq \epsilon_n/(n+1)^2 \quad 1 \leq i, j \leq (n+1)N.
\end{align}

We represent now the set \( \{1, 2, \cdots, (n+1)N\} \) as a union of \( N \) disjoint sets \( \{A_k\}_{k=1}^{N} \) with \( |A_k| = n+1 \) for every \( k \). Let \( Y_k = \text{span} \{u_{i,j}, (i,j) \in A_k \times A_k\} \). Each \( Y_k \) is isometric to \( C^{n+1}_p \) and the restriction of \( T \) to every \( Y_k \) satisfies (5.13) and (5.15) (with \( T_{n+1} = T_{|yk|} \)). We show that for at least one \( k \) also (5.14) holds and this will conclude the proof. Assume that (5.14) fails for every \( k \), i.e., there is a \( y_k \in Y_k \) with \( ||y_k|| = 1 \) and

\begin{align}
(5.24) \quad ||(I - E_{mn})Ty_k|| \leq 1 \quad k = 1, 2, \cdots, N.
\end{align}

Since the \( \{y_k\}_{k=1}^{N} \) are equivalent to the unit vector basis in \( l^p \), we get by using (1.4), (5.24), the fact that \( d(E_{mn}, C_p, l_2) \leq m_n \) that

\begin{align}
(5.25) \quad N^{1/p} = \left( \int_{0}^{1} \|| \sum_{k=1}^{N} r_k(t)Ty_k||^p dt \right)^{1/p} \leq \left( \int_{0}^{1} \|| \sum_{k=1}^{N} r_k(t)Ty_k||^p dt \right)^{1/p} \\
\quad \leq \left( \int_{0}^{1} \|| \sum_{k=1}^{N} r_k(t)E_{mn}Ty_k||^p dt \right)^{1/p} + \left( \int_{0}^{1} \|| \sum_{k=1}^{N} r_k(t)(I - E_{mn})Ty_k||^p dt \right)^{1/p} \\
\quad \leq \left( \int_{0}^{1} \|| \sum_{k=1}^{N} r_k(t)E_{mn}Ty_k||^2 dt \right)^{1/2} + \left( \sum_{k=1}^{N} \||(I - E_{mn})Ty_k||^p \right)^{1/p} \\
\quad \leq m_n \left( \sum_{k=1}^{N} ||E_{mn}Ty_k||^2 \right)^{1/2} + N^{1/p}/2 \leq m_n \cdot 2KN^{1/2} + N^{1/p}/2.
\end{align}

However (5.25) contradicts (5.19) and this concludes the proof.
6. Complemented subspaces of $C_p$

In this section we classify up to isomorphism the ranges of the projections $P(\{A_k\}, \{B_k\})$ and obtain some related result concerning the possibility of embedding certain of these spaces into each other. We first introduce some more notations. By $C_p^{n, \infty}$ we denote the subspace of $C_p$ consisting of all those $x$ for which $x(i, j) = 0$ if $j > n$. Clearly for every $n$ $C_p^{n, \infty}$ is isomorphic to $l_2$ but $d(C_p^{n, \infty}, l_2) \to \infty$ as $n \to \infty$ (unless $p = 2$). The space $(\sum_{n=1}^{\infty} \oplus C_p^{n, \infty})$ is denoted by $S_p^\infty$.

**Theorem 5:** Let $1 < p < \infty$, $p \neq 2$. The infinite-dimensional ranges of projections of the type $P(\{A_k\}, \{B_k\})$ (defined by (2.3)) are isomorphic to one of the following spaces:

\[ l_2, \quad l_p, \quad l_2 \oplus l_p, \quad (l_2 \oplus l_2 \oplus \cdots)_p, \quad S_p, \quad S_p \oplus l_2, \]

\[ S_p \oplus (l_2 \oplus l_2 \oplus \cdots)_p, \quad S_p^\infty, \quad C_p. \]

All the spaces in this list are of a different isomorphism type.

**Proof:** It is easy to see that all the spaces in the list are even isometric to $P(\{A_k\}, \{B_k\})C_p$ for a suitable choice of $\{A_k\}$ and $\{B_k\}$. A simple application of the decomposition method (similar to that used in the proof of Proposition 3 which is a special case of the present theorem) shows that for every choice of $\{A_k\}$ and $\{B_k\}$, the range of $P(\{A_k\}, \{B_k\})$ is isomorphic to one of the spaces in the list. We omit the easy details.

We pass to the proof of the second statement of Theorem 5 which is less trivial. It is a well known fact that the first four spaces in the list are of different isomorphism types. None of them is isomorphic to any of the last five spaces (the first four spaces have an unconditional basis while the last five fail even to have a local unconditional structure [6], cf. the end of section 1).

We now consider the last five spaces in the list. They are written in increasing order; each one is obviously isometric to a subspace of the space following it in the list. Among them the space $S_p$ is singled out by the fact that it does not contain a subspace isomorphic to $l_2$. To conclude the proof of Theorem 5 it is enough to prove the following three propositions (observe that non-isomorphism for $1 < p < 2$ implies non-isomorphism for $2 < p < \infty$ by duality):

**Proposition 5:** For $1 \leq p < \infty$, $p \neq 2$, $S_p \oplus (l_2 \oplus l_2 \oplus \cdots)_p$ is not isomorphic to a subspace of $S_p \oplus l_2$. 
PROPOSITION 6: For $1 \leq p < 2$, $S_p^\infty$ is not isomorphic to a subspace of $S_p \oplus (l_2 \oplus l_2 \oplus \cdots)_p$.

PROPOSITION 7: For $1 \leq p < 2$, $C_p$ is not isomorphic to a subspace of $S_p^\infty$.

The proof of Propositions 5 and 6 is routine. We give e.g. a brief outline of the proof of Proposition 6. Let $T$ be an operator from $S_p^\infty$ into $S_p \oplus (l_2 \oplus \cdots \oplus l_2 \oplus \cdots)_p$ and let $P$ be the natural projection of the latter space onto its summand $S_p$. Since $C_p^{\infty,\infty}$ is isomorphic to $l_2$ and any operator from $l_2$ to $l_p$ is compact (here $p < 2$ is used) it follows easily that the restriction of $PT$ to every summand $C_p^{\infty,\infty}$ of $S_p^\infty$ is compact. Using this observation we deduce easily that if $T$ is an isomorphism it is possible to find also an isomorphism from $S_p^\infty$ into $(l_2 \oplus l_2 \oplus \cdots)_p$ which is known to be impossible.

The proof of Proposition 7 is less routine and we present it in detail.

PROOF OF PROPOSITION 7: Assume that there is an isomorphism $T$ from $C_p$ into $S_p^\infty = (C_p^{\infty,\infty} \oplus C_p^{2,\infty} \oplus \cdots)_p$ and let $u_{i,j}$ be the usual basis vectors in $C_p$. Put

$$Tu_{i,j} = (x_{i,j}(1), x_{i,j}(2), \cdots)$$

with $x_{i,j}(n) \in C_p^{\infty,\infty}$ for all $i, j, n$. Put also $\alpha_{i,j}(n) = ||x_{i,j}(n)||$ and

$$\alpha_{i,j} = (\alpha_{i,j}(1), \alpha_{i,j}(2), \cdots) \in l_p \quad 1 \leq i, j < \infty.$$  

Clearly $||x_{i,j}|| = ||Tu_{i,j}|| \leq ||T||$ (the norm of $\alpha_{i,j}$ and all the norms of other $\alpha$ vectors in the sequel is taken in $l_p$). The bounded sequence $\{\alpha_{1,j}^j\}_{j=1}^\infty$ of vectors in $l_p$ has a subsequence $\{\alpha_{i,j_k}^j\}_{k=1}^\infty$ converging weakly (if $p = 1, w^*$) to some vector $\alpha_1 \in l_p$. We claim that

$$\lim_{k \to \infty} ||x_{1,j_k} - \alpha_1|| = 0.$$  

Indeed if (6.3) fails there is a $\rho > 0$ so that (after passing again to a subsequence if necessary) $\alpha_{1,j_k} = \alpha_1 + \beta_k$ where $||\beta_k|| \geq \rho$ and the $\{\beta_k\}$ have almost disjoint supports in $l_p$. Thus for every integer $m$

$$m^{\frac{1}{p}} ||T|| = \left( \sum_{k=1}^m ||u_{1,j_k}|| \right) \frac{1}{m} \sum_{k=1}^m ||Tu_{1,j_k}|| \geq \frac{1}{m} \sum_{k=1}^m ||\beta_k|| \geq \frac{m^{1/p}}{4}$$

but this is a contradiction for large $m$.

The same argument which we gave for $\alpha_{1,j}$ clearly works for $\{\alpha_{i,j}^j\}_{j=1}^\infty$.
for every $i$ and thus by the diagonal procedure we may find a subsequence of the integers (which in order to avoid more indices we continue to call $\{j_k\}_{k=1}^\infty$) so that
\begin{equation}
\lim_{k \to \infty} \|x_{i, j_k} - x_i\| = 0 \quad i = 1, 2, \cdots.
\end{equation}
Using once again the same argument this time for the vectors $\{x_i\}_{i=1}^\infty$ we get that there is a subsequence $\{i_l\}_{l=1}^\infty$ of the integers and a vector $x$ in $l_p$ so that
\begin{equation}
\lim_{l \to \infty} \|x_{i_l} - x\| = 0
\end{equation}
(we use here the fact that for every $k$ span $\{u_{i, j_k}\}_{i=1}^\infty$ is isometric to $l_2$).
By (6.5) and (6.6) it follows that there are increasing sequences of integers $\{i_h\}_{h=1}^\infty$ and $\{j_h\}_{h=1}^\infty$ so that
\begin{equation}
\|x_{i_h, j_h} - x\| \leq 2^{-h} \quad h = 1, 2, \cdots.
\end{equation}
Put $v_h = u_{i_h, j_h} = h = 1, 2, \cdots$. The vectors $\{v_h\}_{h=1}^\infty$ are isometrically equivalent to the unit vector basis in $l_p$ and by (6.7) we have
\begin{equation}
Tv_h = (y_h(1), y_h(2), \cdots) + z_h
\end{equation}
with $y_h(n) \in C_p^\infty$, $\|y_h(n)\| = a(n)$, $z_h \in S_p^\infty$, $\|z_h\| \leq 2^{-h}$. Since $d(C_p^\infty, l_2) \leq n$ we get for every $m$ and $n$
\begin{equation}
\left( \int_0^1 \| \sum_{h=1}^m r_h(t)y_h(n) \|^p dt \right)^{1/p} \leq \left( \int_0^1 \| \sum_{h=1}^m r_h(t)y_h(n) \|^2 dt \right)^{1/2} \leq n \left( \sum_{h=1}^m \|y_h(n)\|^2 \right)^{1/2} = n a(n)m^{1/2}.
\end{equation}
By using (1.4) we get another estimate for the same expression
\begin{equation}
\left( \int_0^1 \| \sum_{h=1}^m r_h(t)y_h(n) \|^p dt \right)^{1/p} \leq \left( \sum_{h=1}^m \|y_h(n)\|^p \right)^{1/p} = a(n)m^{1/p}.
\end{equation}
Combining (6.9) and (6.10) we get for any $m$ and $n$
\begin{equation}
m^{1/p} ||T^{-1}||^{-1} \leq ||T^{-1}||^{-1} \left( \int_0^1 \| \sum_{h=1}^m r_h(t)v_h \|^p dt \right)^{1/p} \leq \left( \sum_{h=1}^m ||z_h|| + \left( \int_0^1 \| \sum_{h=1}^m r_h(t)y_h(k) \|^p dt \right)^{1/p} \right.
\end{equation}
\begin{equation}
\leq 2 + \sum_{k=1}^n \left( \int_0^1 \| \sum_{h=1}^m r_h(t)y_h(k) \|^p dt \right)^{1/p} + \left( \int_0^1 \| \sum_{h=n+1}^\infty r_h(t)y_h(k) \|^p dt \right)^{1/p}
\end{equation}
\begin{equation}
\leq 2 + m^{1/2} \sum_{k=1}^n k a(k) + m^{1/p} \left( \sum_{k=n+1}^\infty a^p(k) \right)^{1/p}.
\end{equation}
However if \( n \) is chosen so that \( \|T^{-1}\|/(\sum_{k=n+1}^{\infty} 2^k)^{1/p} \) is smaller than 1 then (6.11) fails to hold for large enough \( m \). This contradiction concludes the proof of Proposition 7 and thus of Theorem 5.

**Remark:**
(i) Theorem 5 is true also in the cases \( p = 1 \) and \( p = \infty \), provided we add to the list given in the statement of theorem 5 two more spaces:

\[
\left( \sum_{k=1}^{\infty} \oplus l_2^p \right); \quad \left( \sum_{k=1}^{\infty} \oplus l_2^p \oplus l_2 \right)
\]

(Here, as before, for \( p = \infty \) the direct sum is taken in the sense of \( c_0 \), and \( l_\infty \) means \( c_0 \).)

When showing that all the spaces in the enlarged list are of different isomorphic types, after some trivial observations, we need only to compare two pairs of spaces: \( l_p \) with \( (\sum_{k=1}^{\infty} \oplus l_2^p) \), and \( l_2 \oplus l_p \) with \( l_2 \oplus (\sum_{k=1}^{\infty} \oplus l_2^p) \). Since \( l_p \) (\( p = 1, \infty \)) has unique unconditional basis (see [13], I.2.c), and the natural basis of \( (\sum_{k=1}^{\infty} \oplus l_2^p) \) is unconditional but is not equivalent to the unit-vector-basis of \( l_p \), we see that \( l_p \not\cong (\sum_{k=1}^{\infty} \oplus l_2^p) \).

To prove that \( l_2 \oplus l_p \not\cong l_2 \oplus (\sum_{k=1}^{\infty} \oplus l_2^p) \), it is enough to prove that \( (\sum_{k=1}^{\infty} \oplus l_2^p) \) is not isomorphic to a complemented subspace of \( l_2 \oplus l_p \). But the infinite-dimensional complemented subspaces of \( l_2 \oplus l_p \) are known to be the trivial ones: \( l_2 \), \( l_p \) and \( l_2 \oplus l_p \) (see[17]), so the proof is complete.

(ii) We do not know whether the list given in Theorem 5 exhausts all the possible isomorphism types of complemented subspaces of \( C_p \). A comparison with \( L_p \) does not seem to be of help as far as this question is concerned. The first question which should be answered in this connection is the following: Is every complemented subspace of \( S_p \) isomorphic to either \( l_p \) or \( S_p \)? The results and the methods of proof of section 5 may possibly be of some help in this connection.

(iii) We did not check whether Propositions 6 and 7 are valid also for \( p > 2 \).

We conclude this section with the proof of another result showing the impossibility of a certain embedding. This result is not concerned with ranges of projections but with the relation between \( C_p \) and \( L_p \). As we mentioned in the introduction it is known that for \( 1 \leq p < \infty, p \neq 2 \), \( C_p \) is not isomorphic to a subspace of \( L_p \). What about the other direction, i.e., is \( L_p \) isomorphic to a subspace of \( C_p \)? the answer is trivially no if \( 1 \leq p < 2 \) (since e.g., \( L_p \) contains for every \( p < r < 2 \) a subspace isomorphic to \( l_r \) and by Proposition 4, \( C_p \) has no such subspaces). For
2 < p < ∞ the answer to the question we posed is also negative but this is a less trivial fact.

**Theorem 6:** For 2 < p < ∞ the space $L_p$ is not isomorphic to a subspace of $C_p$.

This theorem generalizes a result stated in [12]. It is stated there that $L_p$ is not isomorphic to a subspace of $(l_2 \oplus l_2 \oplus \cdots)_p$. The proof of this statement as presented in [12] is over simplified and therefore incomplete. The proof we give here for Theorem 6 is however based on the idea of the argument presented in [12]. We shall need first two lemmas.

**Lemma 4:** Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements in $C_p$, $2 < p < \infty$, so that for some constant $M$

\[
\sum_{n=1}^{\infty} |\lambda_n u_n| \leq M^{-1} (\sum_{n=1}^{\infty} |\lambda_n|^2)^{1/2}.
\]

Let $0 < \gamma < (MK_p)^{-1}$, where $K_p$ is the constant in (1.5). Then there is an integer $m$ so that

\[
\|E_m u_n\| \geq \gamma \quad n = 1, 2, 3, \cdots.
\]

**Proof:** If (6.13) fails to hold we can choose a sequence of positive numbers $\{\epsilon_i\}_{i=1}^{\infty}$, and increasing sequences of integers $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ (with $m_0 = 0$) so that

\[
||(1 - P_{m_i})u_{n_i}|| \leq \epsilon_i, \quad \|E_{m_{i-1}} u_{n_i}\| < \gamma, \quad \sum_{i=1}^{\infty} \epsilon_i^2 < \infty
\]

for $i = 1, 2, \cdots$.

By (6.12) and (6.14) we get that for every integer $k$ (here, as before, $\{r_i\}_{i=1}^{\infty}$ are the Rademacher functions in [0, 1])

\[
M^{-1} \cdot k^{1/2} \leq \left( \int_0^1 \left( \sum_{i=1}^{k} |r_i(t) u_{n_i}|^p dt \right)^{1/p} \right)
\]

\[
\leq \left( \int_0^1 \left( \sum_{i=1}^{k} |r_i(t)P_{m_i}(1 - E_{m_{i-1}})u_{n_i}|^p dt \right)^{1/p} \right)
\]

\[+ \left( \int_0^1 \left( \sum_{i=1}^{k} |r_i(t)[u_{n_i} - P_{m_i}(1 - E_{m_{i-1}})u_{n_i}]|^p dt \right)^{1/p} \right)
\]

\[\leq M k^{1/p} + K_p \left[ \sum_{i=1}^{k} (||u_{n_i}|| + ||P_{m_i} E_{m_{i-1}} u_{n_i}||)^2 \right]^{1/2}
\]

\[\leq M k^{1/p} + K_p (\sum_{i=1}^{\infty} \epsilon_i^2)^{1/2} + K_p \cdot \gamma \cdot k^{1/2}.
\]
So \((M^{-1} - K_p \gamma)k \leq M \cdot k^{1/p} + \text{const.}\) and by the definition of \(\gamma\), this is a contradiction for large enough \(k\). \(\square\)

The second lemma which we put down here for easy reference is trivial and well known.

**Lemma 5**: Let \(p > 2\) and \(\varepsilon > 0\) be given. Then there is a \(k(p, \varepsilon)\) so that for every \(k > k(p, \varepsilon)\) and every operator \(V\) from \(l_p^k\) into \(l_2\) with \(\|V\| = 1\) we have

\[
|\{1 \leq i \leq k, \|V e_i\| \geq \varepsilon\}| \leq \varepsilon \cdot k
\]

where \(\{e_i\}_{i=1}^k\) denote the unit vectors in \(l_p^k\) and the notation \(|A|\) is used to denote the cardinality of the set \(A\).

**Proof of Theorem 6**: Let \(\{\varphi_i\}_{i=1}^\infty\) be the normalized Haar basis of \(L_p(0, 1)\), i.e., for \(n = 0, 1, \cdots\)

\[
\varphi_{2n+i}(t) = \begin{cases} 
2^{n/p}, & t \in [i \cdot 2^{-n}, (i+\frac{1}{2})2^{-n}] \\
-2^{n/p}, & t \in ((i+\frac{1}{2})2^{-n}, (i+1)2^{-n}) \\
0, & \text{otherwise}
\end{cases}
\]

The functions \(\{\varphi_{2n+i}\}_{i=0}^{2^n-1}\) will be called the Haar functions of the \(n\)'th level. Let us recall that the functions

(6.16) \[ r_n(t) = \left( \sum_{i=0}^{2^n-1} \varphi_{2n+i}(t) \right)/2^{n/p} \]

form the Rademacher functions on \([0, 1]\) and so by Khintchine's inequality are equivalent to the unit vector basis in \(l_2\). We shall use also the Rademacher systems on dyadic subinterval of \([0, 1]\) which are defined as follows. Let \(k\) be an integer and let \(0 \leq j \leq 2^k - 1\). For every \(n \geq k\) we let \(r_{n,k,j}\) be the normalized averages of the Haar functions of the \(n\)'th level which are supported on \([j/2^k, (j+1)/2^k]\), i.e.,

(6.17) \[ r_{n,k,j}(t) = \left( \sum_{i=0}^{2^n-1} \varphi_{2n+j2^{n-k}+i}(t) \right)/2^{(n-k)/p}. \]

Observe that \(r_n = r_{n,0,0}\) and

(6.18) \[ r_n = \left( \sum_{j=0}^{2^k-1} r_{n,k,j} \right)/2^{k/p}, \quad k \leq n < \infty. \]
The natural isometry from \( L_p[2^{-k}, (j+1)2^{-k}] \) onto \( L_p(0, 1) \) takes \( r_{n,k,j} \) into \( r_n \) and therefore we have

\[
\| \sum_{n=k}^{\infty} \lambda_n r_{n,k,j} \|_p = \| \sum_{n=k}^{\infty} \lambda_n r_n \|_p \geq \| \sum_{n=k}^{\infty} \lambda_n r_n \|_2 = \left( \sum_{n=k}^{\infty} |\lambda_n|^2 \right)^{1/2}
\]

for all choices of scalars \( \{\lambda_n\} \) and for every \( j \) and \( k \).

Assume now that there is an isomorphism from \( L_p \) into \( C_p \). It will be a little more convenient to work with \( T_p \) instead of \( C_p \) and we can do this in view of Proposition 1. Since the Haar basis is a reproducing basis of \( L_p \) (cf. [12]) it follows from our assumption that there is a bounded linear operator \( U \) from \( L_p[0, 1] \) into \( T_p \) with \( \|U^{-1}\| = 1 \) and an increasing sequence of integers \( \{b_{ij}\}_{i=1}^{\infty} \) so that

\[
U \varphi_i = (P_{b_{i+1}} - P_{b_i}) U \varphi_i \quad 1 \leq i < \infty
\]

and thus in particular the sequence \( \{U \varphi_i\}_{i=1}^{\infty} \) forms an unconditional basic sequence (with unconditional constant equal to 1).

In view of (6.19) (for \( j = k = 0 \)) and Lemma 4, it follows that if \( \gamma = (2\|U\|K_p)^{-1} \) then there is an integer \( m_1 \) so that

\[
\|E_{m_1} U r_n\| \geq \gamma, \quad n = 1, 2, \ldots.
\]

Since \( E_{m_1} T_p \) is isomorphic to \( l_2 \) and for all \( n \geq k \) the vectors \( \{r_{n,k,j}\}_{j=0}^{2^k-1} \) are isometrically equivalent to the unit vector basis in \( l_p^k \) it follows from Lemma 5 that there is a \( k_1 \) so that for \( n \geq k_1 \)

\[
|A_n| \geq 2^{k_1-1} \quad \text{where } A_n = \{0 \leq j < 2^{k_1}; \|E_{m_1} U r_{n,k_1,j}\| \leq \gamma/2\}.
\]

In view of (6.19) we may apply Lemma 4 to each of the \( 2^{k_1} \) sequences \( \{U r_{n,k_1,j}\}_{n=k_1}^{\infty}, 0 \leq j < 2^{k_1} \), and find an integer \( m_2 > m_1 \) so that

\[
\|E_{m_2} U r_{n,k_1,j}\| \geq \gamma, \quad k_1 \leq n < \infty, \quad 0 \leq j < 2^{k_1}.
\]

By (6.22) and (6.23) we get that

\[
\|(E_{m_2} - E_{m_1}) U r_{n,k_1,j}\| \geq \gamma/2, \quad j \in A_n, \quad k_1 \leq n < \infty.
\]

By (6.20) we get that the sequence \( \{(E_{m_2} - E_{m_1}) U r_{n,k_1,j}\}_{j=0}^{2^{k_1-1}} \) is for every \( n \) an unconditional sequence (with unconditional constant 1) and hence in view of (1.5), (6.18) and (6.24) we get that for \( n \geq k_1 \)

\[
\|(E_{m_2} - E_{m_1}) U r_n\| \geq \left( \sum_{j=0}^{2^{k_1-1}} \|(E_{m_2} - E_{m_1}) U r_{n,k_1,j}\|^p \right)^{1/p}/2^{k_1/p} \geq (\gamma/2) \cdot |A_n|^{1/p}/2^{k_1/p} \geq \gamma/2 \cdot 2^{1/p} \geq \gamma/4.
\]
By repeating exactly the same argument with a sufficiently large $k_2$ (whose size is determined via Lemma 5 by $d(E_{m_2}C_p, l_2)$) we can find an $m_3 > m_2$ so that $||(E_{m_3} - E_{m_2})Ur_n|| \geq \gamma/4$ for $n \geq k_2$. In general we can find this procedure increasing sequences $\{k_i\}$ and $\{m_i\}$ of integers so that

\[(6.26) \quad ||(E_{m_{i+1}} - E_{m_i})Ur_n|| \geq \gamma/4 \quad 1 \leq i, \quad n \geq k_i.\]

By (1.5) and (6.26) we get for any integer $s$ and $n \geq k_s$

\[(6.27) \quad ||U|| \geq ||Ur_n|| \geq ||E_{m_s} Ur_n||
\]

\[\geq (||E_{m_s} Ur_n||^p + ||(E_{m_{s-1}} - E_{m_s}) Ur_n||^p + \cdots + ||(E_{m_1} - E_{m_s}) Ur_n||^p)^{1/p}
\]

\[\geq s^{1/p} \cdot \gamma/4.\]

This is obviously impossible for large $s$ and thus the assumption that $L_p$ embeds into $C_p$ led to a contradiction.

REFERENCES


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