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SOME LINEAR TOPOLOGICAL PROPERTIES OF THE SPACES C_p OF OPERATORS ON HILBERT SPACE

J. Arazy and J. Lindenstrauss*

1. Introduction

The purpose of this paper is to investigate some linear topological properties of the Banach spaces C_p, $1 \leqq p < \infty$, consisting of all compact operators x on the separable infinite dimensional Hilbert space l_2 for which $\|x\|_p = (\text{trace } (x^* x)^{p/2})^{1/p} < \infty$. Our main interest is in comparing the properties of these spaces with some known results concerning the structure of the function spaces $L_p = L_p(0, 1)$.

In section 2, we make some preliminary observations (which follow directly from known results) concerning projections in C_p. We discuss in particular the subspace T_p of C_p of those operators whose matrix representation with respect to a fixed orthonormal basis is triangular (the role of T_p in C_p is quite analogue to that of the Hardy space H_p in L_p). We mention an analogy between the Haar basis in L_p and an unconditional *Schauder* decomposition of C_p and single out a subspace S_p of C_p which plays a central role in the study of C_p. The space S_p is the direct sum

$$S_p = (C_p^1 \oplus C_p^2 \oplus \cdots \oplus C_p^n \oplus \cdots)_p \tag{1.1}$$

where C_p^n denotes the space of all operators x on the n-dimensional Hilbert space l_2^n with $\|x\| = (\text{trace } (x^* x)^{p/2})^{1/p}$.

In section 3 we study basic sequences in C_p. For $2 \leqq p < \infty$ the situation is simple and very similar to that in L_p (as presented in [10]). Every normalized basic sequence has a subsequence equivalent either to the unit vector basis in l_2 or l_p. It turns out that the same result is valid (but the proof somewhat less simple) also for $1 \leqq p < 2$. This is in marked contrast to the situation in L_p, $1 \leqq p < 2$, where the structure of basic

* The contribution of the first named author is part of his Ph.D. thesis prepared at the Hebrew University of Jerusalem under the supervision of the second named author.

sequences is known to be far more involved (and interesting).

In section 4 we study the structure of 'small' subspaces of C_p. The situation is analogous to that discussed by Johnson and Odell [9] in the case of L_p. For $\infty > p > 2$ we show that a subspace of C_p which does not contain a subspace isomorphic to l_2 is isomorphic to a subspace of S_p. For $1 \leqq p < 2$ this is no longer true. The 'right' result for $1 < p < 2$ turns out to be the following. A subspace X of C_p embeds into S_p if and only if every normalized basic sequence in X has a subsequence which is K-equivalent to the unit vector basis of l_p (where the constant K depends only on X). This result is more complete than the analogue obtained in [9] for L_p.

In section 5 we study subspaces of C_p which contain a copy of S_p. These spaces are characterized by their local structure.

In section 6 we classify up to isomorphism all the 'obvious' complemented subspaces of C_p, by showing the non-existence of certain embeddings. In this connection we also prove that for $1 \leqq p < \infty$ $p \neq 2$, L_p cannot be isomorphically embedded in C_p.

The main point in the study we present here is in the comparison between the properties of C_p and the other two familiar spaces associated with the index p, i.e., the sequence space l_p and the function space L_p. The space C_p can be viewed as the natural matrix space associated to p and as in the case of L_p its structure is governed by an interplay between l_2 and l_p. From the results proved here combined with known results (several of those will be quoted in this and in the next section) there emerges what we think is a quite interesting picture. These results should motivate on the one hand a deeper study of the structure of C_p and on the other hand a careful study of matrix spaces associated to other symmetric sequence spaces (e.g. Orlicz spaces l_M). The results proved in the present paper depend on properties which are well known to characterize l_p among sequence spaces (perfectly homogeneous bases for example) and thus their analogues for e.g. l_M are false. What we have in mind when we speak of the study of more general matrix spaces is the study in the context of matrices of questions which are usually considered in the sequence space or function space settings. Here is one such example. Can two matrix spaces associated e.g. with Orlicz spaces l_{M_1} and l_{M_2} be isomorphic without being identical? It is known that for sequence spaces this happens quite often while known partial results suggest that for function spaces this cannot happen. For a background to the comments made in this paragraph and for a general reference to the terminology of Banach space theory we refer to [13].

The basic properties of C_p as a Banach space are presented in [1], [4] and [15]. Let us recall that C_2 is the space of Hilbert Schmidt operators. As a Banach space it is isometric to l_2 and so its structure is simple and well known. Therefore the case $p = 2$ will often be excluded from the discussion below. The space C_1 is the space of nuclear operators on l_2 and is also called the space of the trace class operators. We shall use the notation C_∞ to denote the space of all compact operators on l_2 with the usual operator norm. Most of our attention will be given to the spaces C_p with $1 < p < \infty$ and $p \neq 2$. It is well known that if $1 < p \leqq \infty$ and $p^{-1} + q^{-1} = 1$ then C_p^* is isometric to C_q, the pairing between these spaces is given by $\langle x, y \rangle = \operatorname{trace}(y^* x)$. The space C_p is reflexive iff $1 < p < \infty$.

Here are some known results concerning the structure of C_p which we shall need in the future.

(i) The spaces C_p are uniformly convex for $1 < p < \infty$ and their modulus of convexity is up to a bounded factor the same as that for L_p (cf. [15] and [16]). More precisely there are positive constants α_p and β_p so that

$$\alpha_p \varepsilon^p \leqq \delta_{C_p}(\varepsilon) \leqq \beta_p \varepsilon^p \quad \text{if} \quad \infty > p > 2,\ 0 < \varepsilon \leqq 1 \tag{1.2}$$

$$\alpha_p \varepsilon^2 \leqq \delta_{C_p}(\varepsilon) \leqq \beta_p \varepsilon^2 \quad \text{if} \quad 1 < p < 2,\ 0 < \varepsilon \leqq 1. \tag{1.3}$$

Closely related to (1.2) and (1.3) is the behaviour of terms of the form 'average over all choices of signs of $\|\sum_{j=1}^n \pm x_j\|$, $\{x_j\}_{j=1}^n \in C_p$'. The following inequalities concerning these expressions are proved in [15] and [16]: For $1 \leqq p < \infty$ there is a constant K_p such that for every integer n and every choice of $\{x_j\}_{j=1}^n$ in C_p

$$K_p\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac{1}{2}} \leqq \left(\int_0^1 \Big\|\sum_{j=1}^n r_j(t) x_j\Big\|^p \mathrm{d}t\right)^{1/p} \leqq \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p}, \quad 1 \leqq p \leqq 2 \tag{1.4}$$

$$\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \leqq \left(\int_0^1 \Big\|\sum_{j=1}^n r_j(t) x_j\Big\|^p \mathrm{d}t\right)^{1/p} \leqq K_p\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac{1}{2}}, \quad 2 \leqq p < \infty \tag{1.5}$$

where the $r_j(t)$ denote the Rademacher functions.

(ii) The spaces $C_p, p \neq 2$, have no local unconditional structure (cf. [6]). We do not need here the definition of this notion but only the following consequence of the result of [6]. There is a sequence $\lambda(p, n)$ with $\lim_{n \to \infty} \lambda(p, n) = \infty$ for $p \neq 2$ so that if X is Banach space with an unconditional basis (with unconditional constant equal to 1) if $T : C_p^n \to X$

is an isomorphism into and if P is a projection from X onto TC_p^n then $\|T\| \cdot \|T^{-1}\| \cdot \|P\| \geqq \lambda(p, n)$.

(iii) The space C_p is not isomorphic to a subspace of L_p if $1 \leqq p < \infty$, $p \neq 2$. This fact is due to McCarthy [15] (for $p > 2$ the proof given by McCarthy is complete. For $1 \leqq p < 2$ his basic idea works but some details have to be changed. Another proof for the case $1 \leqq p < 2$ which gives also more precise information is given in [6]).

2. Preliminary observations

In this section we make some essentially known preliminary observations concerning some projections in C_p and their ranges.

The elements of C_p are by definition operators on l_2. We shall often work with the matrix representation of these elements with respect to a fixed orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of l_2. The matrix $x(i, j)$ representing the element $x \in C_p$ is defined by $x(i, j) = (xe_i, e_j)$ $1 \leqq i, j < \infty$. We shall often use the elements $u_{i,j} \in C_p$ defined by

$$u_{i,j}(k, l) = \delta_i^k \delta_j^l \qquad 1 \leqq i, j, k, l < \infty, \tag{2.1}$$

i.e., $u_{i,j}$ is the operator whose matrix has only one non-zero entry and this is 1 in the (i, j)th place. We mention in passing that in a suitable ordering the $\{u_{i,j}\}_{i,j=1}^{\infty}$ form a Schauder basis of C_p (cf. [3], [11]).

The first projection we consider is the triangular projection P_T defined by

$$P_T x(i, j) = \begin{cases} x(i, j) & i \geqq j \\ 0 & j > i \end{cases} \tag{2.2}$$

This projection is known (cf. [14] and for a much simpler proof [5, pp. 118–120]) to be bounded in C_p if $1 < p < \infty$ and not bounded in C_1 and C_∞ (actually for $p = 1$ or $p = \infty$ (2.2) is defined only for x belonging to the linear span of the $u_{i,j}$ and since P_T is not bounded there it cannot be extended to the whole space). The range $P_T C_p$ of the projection P_T is denoted by T_p. More precisely (in order to take into account also $p = 1, \infty$) we denote by T_p the subspace of C_p consisting of those x for which $x(i, j) = 0$ for $j > i$.

Another important projection, actually a whole class of projections, in C_p is obtained as follows. Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be two collections (n is finite or ∞) of subsets of the integers such that $A_k \cap A_l = B_k \cap B_l = \emptyset$ if $k \neq l$. Corresponding to these families of subsets of the integers we define a projection as follows:

$$(2.3)\quad P(\{A_k\}, \{B_k\})x(i,j) = \begin{cases} x(i,j) & \text{if } \ i \in A_k,\ j \in B_k,\ k = 1, \cdots \\ 0 & \text{otherwise} \end{cases}$$

It is trivial to check that for every $1 \leqq p \leqq \infty$ and every choice of $\{A_k\}$ and $\{B_k\}$ the projection $P(\{A_k\}, \{B_k\})$ has norm 1. It is also evident that

$$(2.4)\qquad \|P(\{A_k\}, \{B_k\})x\| = (\sum_k \|x_k\|^p)^{1/p}$$

where $x_k(i,j) = x(i,j)$ if $(i,j) \in A_k \times B_k$ and $= 0$ otherwise. (If $p = \infty$ the sum in the right hand side will be replaced by $\sup_k \|x_k\|$. We shall use the same convention in the future without mentioning it specifically. Whenever we use l_p in the context of $p = \infty$ we shall mean the space c_0. Also direct sums in the l_p sense will mean in case $p = \infty$ direct sums in the sense of c_0.) In particular if $A_k = B_k = \{k\}$ for $k = 1, 2, \cdots$ then $P(\{A_k\}, \{B_k\})$ is equal to the projection of a matrix onto its diagonal, and its range is isometric to l_p.

PROPOSITION 1: *The space C_p is isomorphic to its subspace T_p if and only if* $1 < p < \infty$.

PROOF: Assume that $1 < p < \infty$. Since P_T is bounded we have

$$C_p = P_T C_p \oplus (I - P_T)C_p = T_p \oplus (I - P_T)C_p.$$

It is clear that $(I - P_T)C_p$ is isometric to T_p, and hence ($\approx$ denotes isomorphism)

$$(2.5)\qquad C_p \approx T_p \oplus T_p$$

Let now $\{A_k\}_{k=1}^{\infty}$ be a sequence of disjoint infinite subsets of the integers. By (2.4), $P(\{A_k\}, \{A_k\})T_p$ is isometric to $(T_p \oplus T_p \oplus \cdots)_{l_p}$, i.e.,

$$(2.6)\qquad T_p \approx (T_p \oplus T_p \oplus \cdots)_p \oplus X.$$

It follows from (2.6) that $T_p \approx T_p \oplus T_p$ and thus by (2.5) $C_p \approx T_p$.

If $p = 1$ or $p = \infty$ the space C_p is not isomorphic to T_p since C_p does not have an unconditional Schauder decomposition into finite dimensional spaces (cf. [11]) while T_p has such a decomposition. In fact for every $1 \leqq p \leqq \infty$ we have $T_p = \sum_{j=1}^{\infty} \oplus E_j$ where $E_j = \text{span}\ \{u_{i,j}, 1 \leqq i \leqq j\}$ and the unconditional constant of this decomposition is evidently equal to 1.

Let us make some remarks concerning the notion of finite-dimensional Schauder decompositions which entered into the proof of Proposition 1. In [11] it was shown that C_p (or more generally a symmetric matrix space in the terminology of [11]) has an unconditional finite-dimensional Schauder decomposition if and only if the triangular projection is bounded. The 'if' part follows also from the argument of Proposition 1. The interesting part is the 'only if' part. Its proof in [11] actually shows somewhat more which we would like to point out here (in view of the analogy with the Haar basis in L_p cf. [12]).

DEFINITION:

(i) A Schauder decomposition $X = \sum_n \oplus E_n$ is said to be equivalent to a Schauder decomposition $Y = \sum \oplus F_n$ if there is an isomorphism T from X onto Y so that $TE_n = F_n$ for all n.

(ii) If $X = \sum \oplus E_n$ is a Schauder decomposition of X, if $n_1 = 1 < n_2 < n_3 < \cdots$ is an increasing sequence of integers and

$$F_j \subset E_{n_j} \oplus E_{n_j+1} \oplus \cdots \oplus E_{n_{j+1}-1}$$

then the $\{F_j\}_{j=1}^{\infty}$ are called a block decomposition of $\{E_n\}_{n=1}^{\infty}$. (The $\{F_j\}_{j=1}^{\infty}$ form a Schauder decomposition of their closed linear span.)

(iii) A finite dimensional Schauder decomposition $X = \sum \oplus E_n$ is said to be reproducible if whenever $X \subset Y$ and Y has a finite dimensional Schauder decomposition $\sum \oplus F_n$ then the $\{E_n\}_{n=1}^{\infty}$ are equivalent to a block decomposition of the $\{F_n\}_{n=1}^{\infty}$.

We introduced here this definition of a reproducible decomposition since the matrix spaces provide natural examples for such decompositions which are not already bases.

For every integer n let P_n be the projection on C_p defined by

$$P_n x(i,j) = \begin{cases} x(i,j) & \text{if } \; 1 \leqq i,j \leqq n \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Observe that P_n is a special case of the family of projections defined in (2.3) (take $A = B = \{1, 2, \cdots, n\}$) and that $P_n C_p$ is isometric to the space we denoted in section 1 by C_p^n.

PROPOSITION 2:

(i) *For* $1 \leqq p \leqq \infty$ *the decomposition* $C_p = \sum_{n=1}^{\infty} \oplus (P_n - P_{n-1})C_p$ *is reproducible.*

(ii) *For* $1 \leqq p \leqq \infty$ *the decomposition* $T_p = \sum_{n=1}^{\infty} \oplus (P_n - P_{n-1})T_p$ *is reproducible.*

This proposition is actually valid in the more general context of symmetric matrix spaces. The proof of (i) is essentially given in [11] and we shall not repeat it here. The proof of (ii) is similar and even simpler. Observe that the decomposition in (i) is unconditional only if $1 < p < \infty$. The decomposition in (ii) is exactly the decomposition which appeared in the proof of Proposition 1 and it is unconditional for every p.

The ranges of the projections (2.3) will be classified in section 6. We deal here only with one special case dealing with the space S_p which plays a central role in section 4.

PROPOSITION 3: *Let $1 \leqq p \leqq \infty$ and let $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be sequences of integers so that $\sup_k (\min (n_k, m_k)) = \infty$. Let $\{A_k\}$ and $\{B_k\}$ be families of disjoint subsets of the integers so that $|A_k| = n_k$ and $|B_k| = m_k$ for every k. Then $P(\{A_k\}, \{B_k\})C_p$ is isomorphic to $S_p = (\sum \oplus C_p^n)_p$.*

PROOF: Let $X = P(\{A_k\}, \{B_k\})C_p$. Let $h \to k_h$ be a one to one map of the integers into themselves so that $h \leqq \min (n_{k_h}, m_{k_h})$. Let A'_h and B'_h be subsets of A_{k_h} and B_{k_h} respectively so that $|A'_h| = |B'_h| = h$. Then $P(\{A'_h\}, \{B'_h\})X$ is a complemented subspace of X which is isometric to S_p. Hence $X \approx S_p \oplus Y$ for some space Y. A similar remark shows that $S_p \approx X \oplus W$ and $S_p \approx (S_p \oplus S_p \oplus \cdots)_p \oplus Z$ for some spaces W and Z. A simple application of the decomposition method shows that $S_p \approx X$.

Observe in particular that Proposition 3 implies that S_p is isomorphic to $(S_p \oplus S_p \oplus \cdots)_p$.

Another projection which will be of great use in the sequel is the projection E_n, $n = 1 \cdots$ defined by

$$E_n x(i, j) = \begin{cases} x(i, j) & \min (i, j) \leqq n \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

The projection E_n is the sum of two projections of the type (2.3) and thus in particular $\|E_n\| \leqq 2$. Its use in classifying subspaces of C_p is demonstrated in

PROPOSITION 4:

(i) *A subspace X of C_p, $1 \leqq p \leqq \infty$ $p \neq 2$ is isomorphic to l_2 if and only if the restriction $E_{n|X}$ of E_n to X is an isomorphism for some n. Consequently every subspace of C_p which is isomorphic to l_2 is complemented in C_p.*

(ii) *If X is subspace of C_p such that $E_{n|X}$ fails to be an isomorphism for every n, then for every $\varepsilon > 0$ there is a subspace Z of X such that $d(Z, l_p) \leqq 1+\varepsilon$ and there is a projection of norm $\leqq 1+\varepsilon$ from C_p onto Z.*

In particular every subspace X of C_p has a subspace which is isomorphic either to l_2 or to l_p.

This proposition was observed by several mathematicians independently. A proof of it is given in [2] (and in [8] for $p = 1, \infty$) and will not be given here.

3. Basic sequences

This section is devoted to the proof of the following theorem.

THEOREM 1: *Let $\{x_n\}_{n=1}^{\infty}$ be a normalized basic sequence in C_p, $1 \leqq p < \infty$. Then there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ which is equivalent to the unit vector basis in l_2 or in l_p. Moreover the subsequence may be chosen so that the span of $\{x_{n_k}\}$ is complemented in C_p.*

PROOF: The following three cases $2 < p < \infty$, $1 < p < 2$, and $p = 1$ will be discussed separately.

Consider first the reflexive case i.e., $1 < p < \infty$. The sequence $\{x_n\}$ tends weakly to 0 and thus by a standard perturbation argument (and passing to a subsequence if necessary) we may assume that there is a sequence of integers $m_1 < m_2 < \cdots$ so that

$$x_n \in (P_{m_{n+1}} - P_{m_n})C_p \qquad n = 1, 2, \cdots \tag{3.1}$$

and thus in particular (since $X = \sum \oplus (P_{m+1} - P_m)X$ is an unconditional decomposition) that the sequence $\{x_n\}$ is an unconditional basic sequence.

Assume now that $2 < p < \infty$. If there is an integer m, a subsequence $\{n_k\}$ of the integers and a $\delta > 0$ so that

$$\|E_m x_{n_k}\| > \delta \qquad k = 1, 2, \cdots \tag{3.2}$$

holds then $\{x_{n_k}\}$ is equivalent to the unit vector basis in l_2. Indeed, since the projections E_m and P_l commute, the sequence $\{E_m x_{n_k}\}_{k=1}^{\infty}$ is an unconditional semi-normalized basic sequence in $E_m C_p$ which is isomorphic to a Hilbert space. Hence there is a $\rho > 0$ such that for every choice of $\{\lambda_k\}$

$$\|\sum_k \lambda_k x_{n_k}\| \geqq 2^{-1}\|\sum_k \lambda_k E_m x_{n_k}\| \geqq \rho(\sum_k |\lambda_k|^2)^{\frac{1}{2}}. \tag{3.3}$$

On the other hand by (1.5) and the fact that $\{x_{n_k}\}$ is unconditional we get that

$$\|\sum_k \lambda_k x_{n_k}\| \leqq R(\sum_k |\lambda_k|^2 \|x_{n_k}\|^2)^{\frac{1}{2}} = R(\sum_k |\lambda_k|^2)^{\frac{1}{2}}$$

for some R. The span of $\{x_{n_k}\}$ is complemented in C_p by the general fact that every isomorph of l_2 is complemented in C_p (see Proposition 4).

If (3.2) does not hold then $\lim_{n\to\infty}\|E_m x_n\| = 0$ for every m. A standard argument shows now that for every given sequence ε_k of positive numbers there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, vectors $\{y_k\}$ in C_p and a sequence t_k of integers so that for $k = 1, 2, \cdots$

$$(3.4) \qquad \|y_k - x_{n_k}\| \leqq \varepsilon_k, \qquad y_k = P_{t_k} y_k, \qquad E_{t_{k-1}} y_k = 0, \qquad \|y_k\| = 1.$$

Hence for every choice of $\{\lambda_k\}$, $\|\sum \lambda_k y_k\| = (\sum_k |\lambda_k|^p)^{1/p}$ (this is a special case of (2.4)) and there is a projection of norm 1 from C_p onto $\overline{\text{span}}\,\{y_k\}$. By the usual perturbation argument it follows that if the $\{\varepsilon_k\}$ are small enough the sequence $\{x_{n_k}\}$ is equivalent to the unit vector basis in l_p and its span is complemented in C_p.

We pass now to the case $1 < p < 2$. If there is a $\delta > 0$ such that for every m there is an n with

$$(3.5) \qquad \|(I - E_m)x_n\| > \delta$$

then there is a subsequence of $\{x_n\}$ which is equivalent to the unit vector basis in l_p. Indeed, if (3.5) holds we can in view of (3.1) find increasing sequences of integers $\{n_k\}$ and $\{t_k\}$ so that $\|Q_k x_{n_k}\| > \delta$ for every k and $Q_k x_{n_l} = 0$ if $k \neq l$ where $Q_k = (I - E_{t_{k-1}})P_{t_k}$. The projection $\sum_k Q_k$ is a projection of norm 1 in C_p (it is a projection of the type (2.3)) and hence for every choice of $\{\lambda_k\}$ we have

$$(3.6) \qquad \|\sum_k \lambda_k x_{n_k}\| \geqq \|(\sum_l Q_l)(\sum_k \lambda_k x_{n_k})\| = \|\sum_k \lambda_k Q_k x_{n_k}\| \geqq \delta(\sum_k |\lambda_k|^p)^{1/p}.$$

On the other hand since $\{x_{n_k}\}$ is an unconditional normalized basic sequence it follows from (1.4) that

$$(3.7) \qquad \|\sum_k \lambda_k x_{n_k}\| \leqq R(\sum_k |\lambda_k|^p)^{1/p}$$

for some R. The relations (3.6) and (3.7) show that $\{x_{n_k}\}$ is equivalent to the unit vector basis in l_p. The operator P defined by

$$(3.8) \qquad Px = \sum_{k=1}^{\infty} \operatorname{trace}\,(Q_k x \cdot y_k) x_{n_k}$$

where $y_k = |Q_k x_{n_k}|^{p-2}(Q_k x_{n_k})^* / \|Q_k x_{n_k}\|^p$, is, as can be easily checked, a bounded linear projection from C_p onto $\overline{\text{span}}\,\{x_{n_k}\}$.

We turn now to the case where (3.5) fails, i.e.,

$$\lim_{m\to\infty} \|(I-E_m)x_n\| = 0 \text{ uniformly in } n \tag{3.9}$$

and show that in this case there is a subsequence of $\{x_n\}$ which is equivalent to the unit vector basis in l_2 (as before, $1 < p < 2$). In this case the proof is a little less trivial and requires the following lemma.

Lemma 1: *Let $1 < p < \infty$, let m be an integer and let $\{y_n\}_{n=1}^{\infty}$ be a normalized basic sequence in $E_m C_p$. Then there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ so that for all $\{\lambda_k\}$*

$$4^{-1}\Big(\sum_k |\lambda_k|^2\Big)^{\frac{1}{2}} \leqq \Big\|\sum_k \lambda_k y_{n_k}\Big\| \leqq 4\Big(\sum_k |\lambda_k|^2\Big)^{\frac{1}{2}}. \tag{3.10}$$

The point in the lemma is the fact that in spite of the fact that $d(E_m C_p, l_2)$ tends to infinity with m the constants in (3.10) are independent of m.

Proof of the Lemma: There is no loss of generality to assume that $y_n \in (P_{t_{n+1}} - P_{t_n})C_p$ for some increasing sequence of integers. We shall assume in addition that $y_n(i,j) = 0$ if $i > m$ and prove that in this case (3.10) holds with 4 replaced by 2. This will prove (3.10) in the general case since each $y_n \in E_m C_p$ has a natural decomposition of the form $y_n = y_n' + y_n''$ with $y_n'(i,j) = 0$ if $j > m$ and $y_n''(i,j) = 0$ if $i > m$. From our assumptions it follows that $y_n^* y_k = 0$ if $n \neq k$ and that $y_n(e_i) = 0$ if $i > m$. For every n let u_n be a unitary operator in l_2 such that $u_n(e_i) = e_i$ for $1 \leqq i \leqq m$ and $u_n y_n(e_i) \subset \text{span}\,\{e_j\}_{j=1}^{2m}$ for $1 \leqq i \leqq m$ (and thus for all i). By the compactness of the unit ball in C_p^{2m} there is a sequence of integers $\{n_k\}$ so that $u_{n_k} y_{n_k}$ converges in norm to some operator x_0. Assume for the moment that $u_{n_k} y_{n_k} = x_0$ for all k. Then for all scalars $\{\lambda_k\}$

$$\Big(\sum \bar{\lambda}_k y_{n_k}^*\Big)\Big(\sum \lambda_k y_{n_k}\Big) = \sum_k |\lambda_k|^2 x_0^* u_{n_k} u_{n_k}^* x_0 = \sum_k |\lambda_k|^2 x_0^* x_0$$

and hence

$$\Big\|\sum_k \lambda_k y_{n_k}\Big\| = \Big(\sum_k |\lambda_k|^2\Big)^{\frac{1}{2}}. \tag{3.11}$$

If instead of $u_{n_k} y_{n_k} = x_0$ we have only $\|u_{n_k} y_{n_k} - x_0\| \leqq \varepsilon_k$ with ε_k sufficiently small we get instead of (3.11) the relation (3.10) with 2 say, instead of 4.

We return to the proof of Theorem 1. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be any sequence of positive numbers. By (3.9) there is an integer t_1 such that

$$\|(I-E_{t_1})x_n\| < \varepsilon_1 \qquad n = 1, 2, \cdots. \tag{3.12}$$

By Lemma 1 there is a subsequence $\{n_k^1\}_{k=1}^{\infty}$ of the integers so that

$$4^{-1}\Big(\sum_k |\lambda_k|^2 \|E_{t_1} x_{n_k^1}\|^2\Big)^{\frac{1}{2}} \leqq \Big\|E_{t_1}\Big(\sum_k \lambda_k x_{n_k^1}\Big)\Big\| \leqq 4\Big(\sum_k |\lambda_k|^2 \|E_{t_1} x_{n_k^1}\|^2\Big)^{\frac{1}{2}}. \tag{3.13}$$

By (3.1) and (3.9) we can choose an integer t_2 so that

$$E_{t_2} x_{n_1^1} = x_{n_1^1}, \qquad \|(I-E_{t_2})x_n\| < \varepsilon_2, \qquad n = 1, 2, \cdots. \tag{3.14}$$

By Lemma 1 there is a subsequence $\{n_k^2\}$ of $\{n_k^1\}$ so that

$$\Big\|(E_{t_2}-E_{t_1}) \sum_k \lambda_k x_{n_k^2}\Big\| \leqq 4\Big(\sum_k |\lambda_k|^2 \|(E_{t_2}-E_{t_1})x_{n_k^2}\|^2\Big)^{\frac{1}{2}}. \tag{3.15}$$

We continue inductively to construct an increasing sequence of integers $\{t_k\}_{k=1}^{\infty}$ and sequences $\{n_k^l\}_{k=1}^{\infty}$ of integers so that $\{n_k^{l+1}\}_{k=1}^{\infty}$ is for every l a subsequence of $\{n_k^l\}_{k=1}^{\infty}$ and

$$E_{t_{k+1}} x_{n_k^k} = x_{n_k^k}, \qquad \|(I-E_{t_{k+1}})x_n\| < \varepsilon_{k+1}, \qquad n = 1, 2, \cdots \tag{3.16}$$

$$\Big\|(E_{t_l}-E_{t_{l-1}}) \sum_k \lambda_k x_{n_k^l}\Big\| \leqq 4\Big(\sum_k |\lambda_k|^2 \|(E_{t_l}-E_{t_{l-1}})x_{n_k^l}\|^2\Big)^{\frac{1}{2}}. \tag{3.17}$$

Let $n_k = n_k^k$ be the diagonal sequence. We claim that $\{x_{n_k}\}$ is equivalent to the unit vector basis in l_2. Indeed, we have (with $E_{t_0} = 0$) by (3.16) and (3.17) that

$$\begin{aligned}
\Big\|\sum_{k=1}^{\infty} \lambda_k x_{n_k}\Big\| &= \Big\|\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \lambda_k (E_{t_l}-E_{t_{l-1}}) x_{n_k}\Big\| \\
&= \Big\|\sum_{k=1}^{\infty} \sum_{l=1}^{k+1} \lambda_k (E_{t_l}-E_{t_{l-1}}) x_{n_k}\Big\| \\
&\leqq \Big\|\sum_{k=1}^{\infty} \lambda_k (E_{t_{k+1}}-E_{t_k}) x_{n_k}\Big\| + \sum_{l=1}^{\infty} \Big\|\sum_{k=l}^{\infty} \lambda_k (E_{t_l}-E_{t_{l-1}}) x_{n_k}\Big\| \\
&\leqq 2\sum_{k=1}^{\infty} |\lambda_k|\varepsilon_k + 4\Big(\sum_{k=1}^{\infty} |\lambda_k|^2\Big)^{\frac{1}{2}} + 8\sum_{l=2}^{\infty} \Big(\sum_{k=l}^{\infty} |\lambda_k|^2 \varepsilon_{l-1}^2\Big)^{\frac{1}{2}} \\
&\leqq \Big(\sum_{k=1}^{\infty} |\lambda_k|^2\Big)^{\frac{1}{2}}\Big(4 + 10\sum_{l=1}^{\infty} \varepsilon_l\Big).
\end{aligned} \tag{3.18}$$

An estimate of $\|\sum_k \lambda_k x_{n_k}\|$ from below (to show that it is greater than

$\rho(\sum_k|\lambda_k|^2)^{\frac{1}{2}}$ for some $\rho > 0$) can be obtained either by using the fact that $\{x_{n_k}\}_{k=1}^{\infty}$ is an unconditional basic sequence and (1.4) or by using the left hand side inequality of (3.13) which combined with computation in (3.18) gives $\|\sum_k \lambda_k x_{n_k}\| \geqq (4^{-1}-10\sum_l \varepsilon_l)(\sum_k|\lambda_k|^2)^{\frac{1}{2}}$.

It remains to consider the case $p = 1$. If the sequence $\{x_n\}$ satisfies (3.1) then the proof works just as in the case $1 < p < 2$. The only difference between the case $1 < p < 2$ and $p = 1$ is that the sequence $\{x_n\}$ need not be an unconditional basic sequence. However the use of unconditionality can be avoided in the two places it was used, since (3.7) is trivial for $p = 1$ and as we have just observed the use of the left hand inequality of (3.13) avoids the use of unconditionality at the end of the proof. A general normalized basic sequence $\{x_n\}_{n=1}^{\infty}$ in C_1 has a subsequence of the form $x_{n_k} = y+z_{n_k}$ where $z_{n_k} \to 0$ in the w^* topology of C_1 and $\|z_{n_k}\| \geqq \delta > 0$ for all k and some δ. By the preceding observation we may assume also that $\{z_{n_k}\}$ is equivalent either to the unit vector basis in l_1 and then the same is true for $\{x_{n_k}\}$ or to the unit vector basis in l_2 and then (since x_{n_k} is a basic sequence) y must be equal to 0. This concludes the proof of the theorem for all $1 \leqq p < \infty$.

Remark: Instead of assuming in the statement of Theorem 1 that $\{x_n\}$ is a normalized basic sequence we could assume of course that $x_n \to 0$ weakly (w^* if $p = 1$) and that $\|x_n\| \not\to 0$.

4. Subspaces of S_p

The first theorem in this section deals with subspaces of C_p $2 < p < \infty$ which embeds into S_p. The theorem and its proof is an adaptation of the work of Johnson and Odell [9] to the setting of C_p.

Theorem 2: *A subspace X of $C_p, 2 < p < \infty$, is isomorphic to a subspace of S_p if and only if X has no subspace isomorphic to l_2.*

Proof: The 'only if' part is obvious. In order to prove the 'if' part we observe first that by our assumption

$$\lim_{n\to\infty} \|E_m(I-P_n)_{|X}\| = 0 \qquad m = 1, 2, \cdots. \tag{4.1}$$

Indeed, if (4.1) fails it follows easily that there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X which tends weakly to 0 but for which $\inf_n\|E_m x_n\| > 0$ for some m. By the proof of Theorem 1 such a sequence has a subsequence which is equivalent to the unit vector basis in l_2 contradicting our assumption.

Let $\delta > 0$ be given. In view of (4.1) there is a sequence of integers $\{n_k\}_{k=1}^{\infty}$ so that

$$\|(I-P_{n_{k+1}})E_{n_k}x\| \leqq 2^{-k}\delta, \qquad k = 1, 2, \cdots. \tag{4.2}$$

Put $R_k = P_{n_{k+1}}-P_{n_k}$ and $Q_k = R_k(I-E_{n_{k-1}})$ for $k = 0, 1, 2, \cdots$ (where $P_{n_0} = 0$, $E_{n_0} = E_{n_{-1}} = 0$). Observe that if $|k-h| > 1$, then Q_k and Q_h map into disjoint rectangles, i.e., there are sets $\{A_k\}$ and $\{B_k\}$ of integers so that $Q_k x(i, j) = 0$ unless $(i, j) \in A_k \times B_k$ and $A_k \cap A_h = B_k \cap B_h = \emptyset$ if $|k-h| > 1$. Let us consider first the odd indices. We have by (2.4) and the preceding remark for every $x \in C_p$

$$\|\sum_{k=0}^{\infty} Q_{2k+1}x\| = (\sum_{k=0}^{\infty} \|Q_{2k+1}x\|^p)^{1/p} \tag{4.3}$$

and by (4.2) for $x \in X$

$$\|\sum_{k=0}^{\infty} (R_{2k+1}-Q_{2k+1})x\| \leqq \|x\| \sum_{k=0}^{\infty} 2^{-2k}\cdot \delta \leqq 2\delta\|x\|. \tag{4.4}$$

Hence for $x \in X$

$$\begin{aligned} \|\sum_k R_{2k+1}x\| &\leqq \|\sum_k Q_{2k+1}x\|+2\delta\|x\| \\ &= (\sum_k \|Q_{2k+1}x\|^p)^{1/p}+2\delta\|x\| \leqq (\sum_k \|R_{2k+1}x\|^p)^{1/p} \\ &\quad +\|x\|(\sum_k (2^{-2k}\delta)^p)^{1/p}+2\delta\|x\| \leqq (\sum_k \|R_{2k+1}x\|^p)^{1/p}+4\delta\|x\|. \end{aligned} \tag{4.5}$$

A similar computation shows that

$$\|\sum_k R_{2k+1}x\| \geqq (\sum_k \|R_{2k+1}x\|^p)^{1/p}-4\delta\|x\| \tag{4.6}$$

and working with the even indices we get similarly

$$(\sum_{k=0}^{\infty} \|R_{2k}x\|^p)^{1/p}-6\delta\|x\| \leqq \|\sum_{k=0}^{\infty} R_{2k}x\| \leqq (\sum_{k=0}^{\infty} \|R_{2k}x\|^p)^{1/p}+6\delta\|x\|. \tag{4.7}$$

Combining (4.5) and (4.7) we get for $x \in X$

$$\begin{aligned} \|x\| = \|\sum_k R_k x\| &\leqq \|\sum_k R_{2k}x\|+\|\sum_k R_{2k+1}x\| \\ &\leqq 10\delta\|x\|+2^{1/q}(\sum_k \|R_k x\|^p)^{1/p} \end{aligned}$$

and hence

$$\|x\| \leqq 2^{1/q}(1-10\delta)^{-1}(\sum_k \|R_k x\|^p)^{1/p}. \tag{4.8}$$

By (4.6), (4.7) and the fact that the decomposition $C_p = \sum_k \oplus R_k C_p$ is unconditional (with an unconditional constant M_p) we get that

$$2M_p\|x\| \geqq \|\sum_k R_{2k}x\| + \|\sum_k R_{2k+1}x\| \geqq (\sum_k \|R_k x\|^p)^{1/p} - 10\delta\|x\|. \tag{4.9}$$

It follows from (4.8) and (4.9) that if δ is chosen small enough then for $x \in X$

$$(\sum_k \|R_k x\|^p)^{1/p}/3M_p \leqq \|x\| \leqq 2(\sum_k \|R_k x\|^p)^{1/p}. \tag{4.10}$$

Since $R_k C_p \subset C_p^{n_{k+1}}$ it follows from (4.10) that the map $x \to (R_1 x, R_2 x, \cdots)$ defines an isomorphism from X into S_p.

The direct extension of Theorem 2 to the case $1 < p < 2$ fails to be true. There is a subspace X of C_p, $1 < p < 2$, which does not contain a copy of l_2 but which does not embed into S_p. This counter-example is identical to that used in [9] for L_p and we recall it briefly. Let $\lambda > 0$ and let X_λ be the subspace of $(l_p \oplus l_2)_p$ which is spanned by the vectors $g_{i,\lambda} = e_i + \lambda f_i$ $i = 1, 2, \cdots$ where e_i and f_i denote the unit vectors in l_2 and l_p respectively. The sequence $\{g_{i,\lambda}\}_{i=1}^\infty$ is a symmetric basic sequence which is equivalent to the $\{f_i\}_{i=1}^\infty$ but the equivalence constant is large if λ is small (if $T : l_p \to X_\lambda$ is defined by $Tf_i = g_{i,\lambda}$ then $\|T\| \cdot \|T^{-1}\| \geqq \lambda^{-1}$). It is trivial to verify that every normalized basic sequence in S_p has for every $\varepsilon > 0$ a subsequence which is $(1+\varepsilon)$ equivalent to the unit vector basis l_p. It follows from this observation that the distance from X_λ to any subspace of S_p is $\geqq \lambda^{-1}$, and thus $X = (\sum_n \oplus X_{1/n})_p$ is not isomorphic to a subspace of S_p. The space X is clearly isometric to a subspace of $(l_2 \oplus l_2 \oplus l_2 \oplus \cdots)_p$ which in turn is isometric to a subspace of C_p (it is the range of a projection of the type (2.3) if we take $A_k = \{k\}$ and $\{B_k\}$ a family of disjoint infinite subsets of the integers). It is also evident that X has no subspace isomorphic to l_2. Let us also remark that by Theorem 1 (or as can be seen directly) every normalized basic sequence in X has a subsequence which is equivalent to the unit vector basis in l_p.

Theorem 2 may be restated as follows: A subspace X of C_p, $2 \leqq p < \infty$, is isomorphic to a subspace of S_p if and only if every normalized basic sequence in X has a subsequence which is equivalent to the unit vector basis in l_p. It turns out that by adding a uniformity condition on the equivalence we get a result which is valid also for $1 < p \leqq 2$.

THEOREM 3: *Let X be a subspace of C_p, $1 < p < \infty$. Then X is isomorphic to a subspace of S_p if and only if there is a constant K so that every normalized basic sequence in X has a subsequence which is K equivalent to the unit vector basis in l_p.*

The 'only if' part of the theorem is obvious. As remarked above the 'if' part for $2 \leqq p < \infty$ is contained already in Theorem 2 and the uniformity assumption (i.e. the existence of K) need not be assumed a-priori. On the other hand the example given above shows that for $1 < p < 2$ the uniformity assumption is essential. The proof we give to the 'if' part of Theorem 3 will show also that the distance coefficient of X from a suitable subspace of S_p can be estimated from above by a function of K. In [9] Johnson and Odell prove an analogue of Theorem 3 in the case of L_p. However, they use there an extra assumption on X and their result is thus less complete. The proof we give here, which avoids any extra assumption, is completely different from the argument used in [9]. Let us also point out that in contrast to the situation in L_p, $1 < p < 2$, the assumption of the 'if' part of Theorem 3 is, in view of Theorem 1, quite close to the assumption that X has no subspace isomorphic to l_2.

For the proof of Theorem 3 we need two lemmas.

LEMMA 2: *Let $n_0 = 1 < n_1 < n_2 < \cdots$ be a finite or infinite sequence of integers. Consider the set of pairs of integers*

$$D(n_1, n_2, \cdots) = \bigcup_{k \geqq 1} \{(i,j);\ n_k \leqq i < \infty,\ 1 \leqq j \leqq n_{k-1}\}$$

and let $Q(n_1, n_2 \cdots)$ be the projection on C_p $1 < p < \infty$ defined by

$$(4.11) \qquad Q(n_1, n_2, \cdots)x(i,j) = \begin{cases} x(i,j) & \text{if } (i,j) \in D(n_1, n_2 \cdots) \\ 0 & \text{otherwise} \end{cases}$$

Then there is a number M (depending on p but not on the integers $n_1, n_2, \cdots$) so that $\|Q(n_1, n_2, \cdots)\| \leqq M$.

PROOF: We assume that the sequence $\{n_k\}$ is infinite (for finite sequences the verification is the same but with a little different notation). Put $A_k = \{i;\, n_{k-1} \leqq i < n_k\}$ and $B_k = \{i;\, n_{k-1} < i \leqq n_k\}$, $k = 1, 2, \cdots$ and $B_0 = \{1\}$. The lemma is a consequence from the boundedness of the triangular projection P_T and the following easily verified identity

$$(4.12) \quad Q(n_1, n_2, \cdots) + P(\{A_k\}, \{B_{k-1}\}) + P_T P(\{A_k\}, \{B_k\}) = P_T.$$

LEMMA 3: *Let $1 < p < 2$ and let X be a subspace of C_p for which there is a constant K as in the statement of Theorem 3. Assume that $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences of elements in $T_p = P_T C_p$ so that*

$$u_n + v_n \in X, \quad \|u_n\| = 1, \quad n = 1, 2, \cdots \tag{4.13}$$

$$u_n, v_n \to 0 \text{ weakly} \tag{4.14}$$

$$\|\sum_n \lambda_n u_n\| \leqq 4(\sum_n |\lambda_n|^2)^{\frac{1}{2}} \qquad \text{for all } \{\lambda_n\}. \tag{4.15}$$

Then for all but finitely many indices n we have

$$\|v_n\| \geqq 1/(2K+2). \tag{4.16}$$

PROOF: By (4.14) we may assume that (after passing to a suitable subsequence) $\{v_n\}$ is an unconditional basic sequence with an unconditional constant as close to 1 as we wish (recall that $v_n \in T_p$, and use the decomposition of T_p appearing in Proposition 2 (ii)). Hence in view of (1.4) we may assume without loss of generality that for all $\{\lambda_n\}$

$$\|\sum_n \lambda_n v_n\| \leqq 2(\sum |\lambda_n|^p \|v_n\|^p)^{1/p}. \tag{4.17}$$

By our assumption on K and (4.14) we may assume also that for all $\{\lambda_n\}$

$$(\sum |\lambda_n|^p \|u_n + v_n\|^p)^{1/p}/(K+\tfrac{1}{3}) \leqq \|\sum \lambda_n(u_n + v_n)\|. \tag{4.18}$$

Finally, if (4.16) fails, there is no loss of generality to assume that in addition to all the above we have that $\|v_n\| \leqq 1/(2K+2)$ for all n and thus also

$$\|u_n + v_n\| \geqq 1 - \|v_n\| \geqq (2K+1)/(2K+2) \qquad n = 1, \cdots \tag{4.19}$$

By (4.15), (4.17), (4.18) and (4.19) we get for all integers m

$$m^{1/p}(2K+1)/(K+\tfrac{1}{3})(2K+2) \leqq (\sum_{n=1}^{m} \|u_n + v_n\|^p)^{1/p}/(K+\tfrac{1}{3}) \tag{4.20}$$

$$\leqq \|\sum_{n=1}^{m} u_n\| + \|\sum_{n=1}^{m} v_n\| \leqq 4m^{\frac{1}{2}} + 2m^{1/p}/(2K+2).$$

Evidently (4.20) is false if m is large enough and this contradiction proves the lemma.

We now turn to the proof of Theorem 3 itself, $(1 < p < 2)$. We assume, as we may in view of Proposition 1, that $X \subset T_p$, i.e., that for all $x \in X$, $x(i, j) = 0$ if $j > i$.

The main step of the proof is the construction of an infinite sequence $n_0 = 1 < n_1 < n_2 < \cdots$ of integers so that for all $x \in X$

$$\|(I - Q(n_1, n_2, \cdots))x\| \geqq \|Q(n_1, n_2, \cdots)x\|/2M(2K+2). \tag{4.21}$$

The sequence will be constructed inductively so that for every k and every $x \in X$

$$\|(I - Q(n_1, n_2, \cdots, n_k))x\| \geqq \|Q(n_1, n_2, \cdots, n_k)x\|/(2-k^{-1})M(2K+2). \tag{4.22}$$

The choice of n_1 is similar to the induction step so we present here only the induction step. Let us assume that $n_1, n_2, \cdots, n_k$ have already been chosen so that (4.22) holds. If no suitable n_{k+1} exists then for all $n > n_k$ there is an $x_n \in X$ with $\|x_n\| = 1$ for which

$$x_n = y_n + z_n, \qquad y_n = Q(n_1, n_2, \cdots, n_k, n)x_n, \qquad \|z_n\| \leqq \|y_n\|/(2-(k+1)^{-1})M(2K+2). \tag{4.23}$$

By passing to a subsequence we may assume that $\{y_n\}$ and $\{z_n\}$ converge weakly to y and z respectively. Clearly $y + z \in X$ and it follows from (4.23) that

$$y = Q(n_1, n_2, \cdots, n_k)y \qquad Q(n_1, n_2, \cdots, n_k)\, z = 0. \tag{4.24}$$

Hence, by (4.22)

$$\|z\| \geqq \|y\|/(2-k^{-1})M(2K+2). \tag{4.25}$$

By comparing (4.23) with (4.25) we infer in particular that it is not true that $\underline{\lim}\,(\|y_n - y\| + \|z_n - z\|) = 0$ and thus

$$\|y + z - y_n - z_n\| \geqq \delta \tag{4.26}$$

for some $\delta > 0$ (recall that by (4.24) $\lim_n \|Q(n_1, n_2, \cdots, n_k, n)z\| = 0$ and thus $\|y - y_n\| + \|z - z_n\| \leqq (2M+2)\|y + z - y_n - z_n\|$ for large n).

Let $\varepsilon > 0$ (its precise choice will be specified later) and choose an integer $m > n_k$ so that

$$\|P_m z - z\| < \varepsilon \tag{4.27}$$

(P_m is the projection defined by (2.2)). Put

$$t_n = y - y_n, \qquad s_n = z - z_n, \qquad w_n = E_m s_n \tag{4.28}$$

where E_m is the projection defined by (2.8). We claim that for infinitely many integers n

$$\|s_n - w_n\| \geqq \|t_n + w_n\|/(2K+2). \tag{4.29}$$

Indeed, since $(s_n - w_n) + (t_n + w_n) = y + z - y_n - z_n$, (4.29) is a consequence of (4.26) if $\underline{\lim}\, \|t_n + w_n\| = 0$. If $\|t_n + w_n\|$ is bounded from below then (4.29) is a consequence of Lemma 3 when applied to $u_n = (t_n + w_n)/\|t_n + w_n\|$ and $v_n = (s_n - w_n)/\|t_n + w_n\|$. The sequence $\{u_n\}$ and $\{v_n\}$ certainly satisfy (4.13) and (4.14) while (4.15) can be obtained by passing to a suitable subsequence (use Lemma 1 and the fact that $u_n = E_m u_n$ for all n).

Since $\lim_n \|Q(n_1, n_2, \cdots, n_k, n)z\| = 0$ it follows from (4.23) and (4.28) that

$$\lim_n \|Q(n_1, n_2, \cdots, n_k, n)w_n\| = 0, \quad Q(n_1, n_2, \cdots, n)t_n = t_n, \quad \text{all } n. \tag{4.30}$$

By (4.29) and (4.30) and Lemma 2 we get that for infinitely many indices n

$$\|s_n - w_n\|^p \geqq \|t_n\|^p \cdot (2K+2)^{-p} M^{-p} - \varepsilon^p. \tag{4.31}$$

Observe next that on T_p, $P_m + I - E_m$ is a projection of norm 1 of the type (2.3) and thus for all n

$$\begin{aligned} \|z_n\|^p &\geqq \|(P_m + I - E_m)z_n\|^p = \|P_m z_n\|^p + \|(I - E_m)z_n\|^p \\ &= \|P_m z - P_m s_n\|^p + \|(I - E_m)z - s_n + w_n\|^p. \end{aligned} \tag{4.32}$$

Now, $\lim_n \|P_m s_n\| \to 0$ (since $s_n \to 0$ weakly and $\dim P_m C_p < \infty$) and thus by (4.27) and (4.32) we get that for sufficiently large n

$$\|z_n\|^p \geqq \|z\|^p + \|s_n - w_n\|^p - 4\varepsilon^p. \tag{4.33}$$

Also since $t_n \to 0$ weakly it follows from (1.4) (or more precisely its special case called the Clarkson inequality) that for sufficiently large n

$$\|y_n\|^p \leqq \|y\|^p + \|t_n\|^p + \varepsilon^p. \tag{4.34}$$

Combining (4.25), (4.31), (4.33) and (4.34) we get that there exist infinitely many integers n for *which*

$$(4.35)\quad \begin{aligned} \|z_n\|^p &\geqq \|z\|^p + \|s_n - w_n\|^p - 4\varepsilon^p \\ &\geqq \|y\|^p((2-k^{-1})M(2K+2))^{-p} + \|t_n\|^p(M(2K+2))^{-p} - 5\varepsilon^p \\ &\geqq (\|y\|^p + \|t_n\|^p)((2-k^{-1})M(2K+2))^{-p} - 5\varepsilon^p \\ &\geqq \|y_n\|^p((2-k^{-1})M(2K+2))^{-p} - \varepsilon^p(5 + ((2-k^{-1})M(2K+2))^{-p}). \end{aligned}$$

But (4.35) contradicts (4.23) if ε is chosen small enough. This contradiction establishes the existence of a suitable n_{k+1}. Thus we can choose the infinite sequence $\{n_k\}_{k=1}^{\infty}$ so that (4.22) holds for all k and therefore (4.21) holds as well.

It follows from (4.21) that the restriction of $I - Q(n_1, n_2, \cdots)$ to X is an isomorphism. The proof of Theorem 3 is now completed by showing that $Z = (I - Q(n_1, n_2, \cdots))T_p$ is isomorphic to a subspace of S_p. The proof of this fact is similar to some arguments which were used in the proof of Theorem 2 and we repeat them here only briefly. Denote by $\hat{Q}_k$ $k = 1, 2, \cdots$ the projection on T_p defined by

$$\hat{Q}_k x(i,j) = \begin{cases} x(i,j) & \text{if } n_{k-1} \leqq i < n_k,\ n_{k-2} < j \leqq n_k \\ 0 & \text{otherwise} \end{cases}$$

where $n_{-1} = 0$. Then $Z = \sum_k \oplus \hat{Q}_k Z$ and this decomposition is unconditional, with unconditionality constant equal to 1. Moreover for every $z \in Z$

$$\|\sum_k \hat{Q}_{2k} z\|^p = \sum_k \|\hat{Q}_{2k} z\|^p, \qquad \|\sum_k \hat{Q}_{2k+1} z\|^p = \sum_k \|\hat{Q}_{2k+1} z\|^p$$

and this implies that Z is isomorphic to $(\sum_k \oplus \hat{Q}_k Z)_p$ which in turn is isometric to a subspace of S_p.

Note: The above isomorphism of Z into S_p has constant $\leqq 4$. Using this fact and (4.22) we get an isomorphism of X into S_p with constant $\leqq \alpha \cdot M \cdot K$, where α is an absolute constant, M depends only on p, and K is the constant which appears in the statement of Theorem 3.

5. Subspaces of C_p which contain S_p

The purpose of this section is to prove that the answer to the question whether or not a given subspace X of C_p contains in it a copy of S_p depends only on the local structure of it.

THEOREM 4: *Let X be a subspace of C_p, $1 \leqq p \leqq \infty$. Then S_p is isomorphic to a subspace of X if and only if there is a constant K so that for every n there is an isomorphism T from C_p^n into X with $\|T\| \cdot \|T^{-1}\| \leqq K$.*

PROOF: The 'only if' part is obvious and so we have to prove the 'if' part for $2 < p \leqq \infty$ and $1 \leqq p < 2$. Like in many other situations these two cases have to be considered separately.

Assume first that $2 < p \leqq \infty$. We shall prove that for every given sequence of positive $\{\varepsilon_n\}$ there exist an increasing sequence of integers m_n and operators $T_n : C_p^n \to X$ so that for $n = 1, 2, \cdots$

$$\|P_{m_n} T_n - T_n\| \leqq \varepsilon_n \tag{5.1}$$

$$\|E_{m_n} T_{n+1}\| \leqq \varepsilon_n \tag{5.2}$$

$$\|x\| \leqq \|T_n x\| \leqq K\|x\| \qquad x \in C_p^n. \tag{5.3}$$

Once (5.1), (5.2) and (5.3) are established the result follows easily. Indeed if the $\{\varepsilon_n\}$ are small enough we get from (5.1) and (5.2) that for any choice of $x_n \in C_p^n$

$$\tfrac{1}{2}(\|T_n x_n\|^p)^{1/p} \leqq \|\sum T_n x_n\| \leqq 2(\sum \|T_n x_n\|^p)^{1/p} \tag{5.4}$$

and thus in view also of (5.3) the mapping $\hat{T} : S_p \to X$ defined by

$$\hat{T}(x_1, x_2, \cdots) = \sum_n T_n x_n \tag{5.5}$$

is an isomorphism.

We construct the operators $\{T_n\}$ and the integers $\{m_n\}$ inductively in the following order $T_1, m_1, T_2, m_2 \cdots$. The operator T_1 is any operator satisfying (5.3). Once T_n is constructed we choose m_n so that (5.1) holds. The main point in the proof is to show given m_n how to construct T_{n+1} so that (5.2) holds as well as (5.3) (with n replaced by $n+1$).

Let $N \geqq 2n+2$ be such that

$$2K(N-1)^{1/p} < (N-1)^{\frac{1}{2}}\varepsilon_n/m_n(n+1)^2 \tag{5.6}$$

where $(N-1)^{1/p}$ is replaced by 1 if $p = \infty$. By Ramsey's theorem [7], there is an $M = M(N)$ such that whenever the set of pairs of integers (i, j), $1 \leqq i, j \leqq M$, $i < j$, is divided into two parts then there is a set $A \subset \{1, \cdots, M\}$ of cardinality $|A| = N$ so that all pairs (i, j) with $i, j \in A$ (and $i < j$) belong to one of these parts. By our assumption there is an operator $T : C_p^M \to X$ so that

$$\|x\| \leqq \|Tx\| \leqq K\|x\| \qquad \text{for all } x \in C_p^M. \tag{5.7}$$

Let $u_{i,j}$ denote the usual basis vectors of C_p^M. We show now that there is no subset A of $\{1, \cdots, M\}$ with $|A| = N$ so that

$$\|E_{m_n} Tu_{i,j}\| \geqq \varepsilon_n/(n+1)^2 \qquad i, j \in A, \quad i < j. \tag{5.8}$$

Indeed assume that $A = \{i_1, i_2, \cdots, i_N\}$ were such a set and consider the vectors $v_k = u_{i_k, i_{k+1}}$ $k = 1, 2, \cdots N-1$ in C_p^M. The $\{v_k\}_{k=1}^{N-1}$ are isometrically equivalent to the unit vector basis of l_p^{N-1} and hence by (5.7) we have for every choice of signs

$$\|\sum_{k=1}^{N-1} \pm E_{m_n} Tv_k\| \leqq 2K(N-1)^{1/p}. \tag{5.9}$$

Since $d(E_{m_n} C_p, l_2) \leqq m_n$ we get by the generalized parallelogram equality in l_2 (i.e., (1.4) and (1.5) which reduce to equalities with $K_p = 1$ if $p = 2$) that

$$\int_0^1 \|\sum_k r_k(t) E_{m_n} Tv_k\|^2 dt \geqq \sum_k \|E_{m_n} Tv_k\|^2/m_n^2 \geqq (N-1)\varepsilon_n^2/m_n^2(n+1)^4 \tag{5.10}$$

but this together with (5.9) contradicts the choice of N in (5.6).

In view of what we just proved and Ramsey's theorem it follows that there is a subset A of $\{1, 2, \cdots, M\}$ with $|A| = N$ so that

$$\|E_{m_n} Tu_{i,j}\| \leqq \varepsilon_n/(n+1)^2 \qquad i, j \in A \quad i < j. \tag{5.11}$$

Since $N \geqq 2(n+1)$ there are disjoint subsets A_1 and A_2 of A so that $|A_1| = |A_2| = n+1$. Clearly (5.11) holds for all $(i, j) \in A_1 \times A_2$ and $Y = \operatorname{span}\{u_{i,j}, (i, j) \in A_1 \times A_2\}$ is isometric to C_p^{n+1}. Clearly every $y \in Y$ has the form $y = \sum_{i,j} \alpha_{i,j} u_{i,j}$ with $|\alpha_{i,j}| \leqq \|y\|$ for all $(i, j) \in A_1 \times A_2$. Hence by (5.11) we get that $\|E_{m_n} Ty\| \leqq \varepsilon_n \|y\|$ for all $y \in Y$. In view of this fact and (5.7) we deduce that we can take as T_{n+1} the restriction of T to Y. This concludes the proof for $2 < p \leqq \infty$.

Assume now that $1 \leqq p < 2$. In this case we shall prove that given any sequence $\{\varepsilon_n\}$ of positive numbers it is possible to choose an increasing sequence of integers m_n and operators $T_n : C_p^n \to X$ so that for $n = 1, 2, \cdots$

$$\|P_{m_n} T_n - T_n\| \leqq \varepsilon_n \tag{5.12}$$

(5.13) $$\|P_{m_n} T_{n+1}\| \leqq \varepsilon_n$$

(5.14) $$\|(I-E_{m_n})T_{n+1}x\| \geqq \|x\|/2, \qquad x \in C_p^{n+1}$$

(5.15) $$\|x\| \leqq \|T_n x\| \leqq K\|x\|, \qquad x \in C_p^n.$$

As in the proof of the case $p > 2$ we show first how to conclude the proof once (5.12)–(5.15) are satisfied. Put $Q_1 = P_{m_1}$ and $Q_n = P_{m_n}(I-E_{m_{n-1}})$ for $n \geqq 2$. It follows from (5.12) and (5.13) that if the ε_n are small enough then

(5.16) $$\|\sum_n T_n x_n\| \leqq M(\sum_n \|T_n x_n\|^p)^{1/p}$$

for some constant M and all choices of $x_n \in C_p^n$ (if $1 < p < 2$ use the fact that the decomposition $P_{m_{n+1}} - P_{m_n}$ is unconditional as well as (1.4). For $p = 1$ (5.16) is trivial). It follows from (5.12), (5.13) and (5.14) that if the ε_n are small enough we have for all $x_n \in C_p^n$

(5.17) $$\begin{aligned} \|\sum_n T_n x_n\| &\geqq \|(\sum_j Q_j)(\sum_n T_n x_n)\| \\ &\geqq \|\sum_n Q_n T_n x_n\| - \sum_n \|(\sum_{j<n} Q_j)T_n x_n\| - \sum_n \|(\sum_{j>n} Q_j)T_n x_n\| \\ &\geqq (\sum_n \|Q_n T_n x_n\|^p)^{1/p} - 2\sum_n \varepsilon_n\|x_n\| \geqq 4^{-1}(\sum_n \|x_n\|^p)^{1/p}. \end{aligned}$$

(5.15), (5.16) and (5.17) combine to show that (5.5) defines an isomorphism $\hat{T}$ from S_p into X.

We now pass to the inductive construction of $T_1, m_1, T_2, m_2, \cdots$. As in the case $p > 2$ the only fact which needs verification is that given m_n we can find a T_{n+1} so that (5.13), (5.14) and (5.15) (for $n+1$) hold. Let N be an integer so that

(5.18) $$KN^{\frac{1}{2}} < N\varepsilon_n/3(n+1)^2$$

(5.19) $$N^{1/p} > 4Km_n N^{\frac{1}{2}}.$$

Since the space $P_{m_n}C_p$ is finite dimensional there is a finite set of points which is an $\varepsilon_n/3(n+1)^2$ net in the ball of radius K and center 0 in $P_{m_n}C_p$. By Ramsey's theorem and by applying the assumption in the theorem to C_p^M for a suitable M (much larger than N) it follows that there is an operator $T : C_p^{(n+1)N} \to X$ and an element $x_0 \in P_{m_n}C_p$ so that

$$(5.20)\qquad \|x\| \leqq \|Tx\| \leqq K\|x\|, \qquad x \in C_p^{(n+1)N}$$

$$(5.21)\qquad \|P_{m_n} Tu_{i,j} - x_0\| \leqq \varepsilon_n/3(n+1)^2 \qquad 1 \leqq i, j \leqq (n+1)N.$$

Since $\|\sum_{j=1}^N u_{1,j}\| \leqq N^{\frac{1}{2}}$ we get from (5.20) and (5.21) that

$$(5.22)\qquad N\|x_0\| \leqq \sum_{j=1}^N \|P_{m_n} Tu_{1,j} - x_0\| + \|P_{m_n}\| \cdot \|T\| \cdot \|\sum_{j=1}^N u_{1,j}\| \leqq N\varepsilon_n/3(n+1)^2 + KN^{\frac{1}{2}}$$

and hence by (5.18), $\|x_0\| \leqq 2\varepsilon_n/3(n+1)^2$, and therefore (5.21) implies

$$(5.23)\qquad \|P_{m_n} Tu_{i,j}\| \leqq \varepsilon_n/(n+1)^2 \qquad 1 \leqq i, j \leqq (n+1)N.$$

We represent now the set $\{1, 2, \cdots, (n+1)N\}$ as a union of N disjoint sets $\{A_k\}_{k=1}^N$ with $|A_k| = n+1$ for every k. Let $Y_k = \operatorname{span}\{u_{i,j}, (i,j) \in A_k \times A_k\}$. Each Y_k is isometric to C_p^{n+1} and the restriction of T to every Y_k satisfies (5.13) and (5.15) (with $T_{n+1} = T_{|Y_k}$). We show that for at least one k also (5.14) holds and this will conclude the proof. Assume that (5.14) fails for every k, i.e., there is a $y_k \in Y_k$ with $\|y_k\| = 1$ and

$$(5.24)\qquad \|(I - E_{m_n})Ty_k\| \leqq \tfrac{1}{2} \qquad k = 1, 2, \cdots, N.$$

Since the $\{y_k\}_{k=1}^N$ are equivalent to the unit vector basis in l_p^N, we get by using (1.4), (5.24), the fact that $d(E_{m_n} C_p, l_2) \leqq m_n$ that

$$(5.25)\qquad N^{1/p} = \left(\int_0^1 \|\sum_{k=1}^N r_k(t) y_k\|^p dt \right)^{1/p} \leqq \left(\int_0^1 \|\sum_{k=1}^N r_k(t) Ty_k\|^p dt \right)^{1/p}$$

$$\leqq \left(\int_0^1 \|\sum_{k=1}^N r_k(t) E_{m_n} Ty_k\|^p dt \right)^{1/p} + \left(\int_0^1 \|\sum_{k=1}^N r_k(t)(I - E_{m_n}) Ty_k\|^p dt \right)^{1/p}$$

$$\leqq \left(\int_0^1 \|\sum_{k=1}^N r_k(t) E_{m_n} Ty_k\|^2 dt \right)^{\frac{1}{2}} + \left(\sum_{k=1}^N \|(I - E_{m_n}) Ty_k\|^p \right)^{1/p}$$

$$\leqq m_n \left(\sum_{k=1}^N \|E_{m_n} Ty_k\|^2 \right)^{\frac{1}{2}} + N^{1/p}/2 \leqq m_n \cdot 2KN^{\frac{1}{2}} + N^{1/p}/2.$$

However (5.25) contradicts (5.19) and this concludes the proof.

6. Complemented subspaces of C_p

In this section we classify up to isomorphism the ranges of the projections $P(\{A_k\}, \{B_k\})$ and obtain some related result concerning the possibility of embedding certain of these spaces into each other. We first introduce some more notations. By $C_p^{n,\infty}$ we denote the subspace of C_p consisting of all those x for which $x(i,j) = 0$ if $j > n$. Clearly for every n $C_p^{n,\infty}$ is isomorphic to l_2 but $d(C_p^{n,\infty}, l_2) \to \infty$ as $n \to \infty$ (unless $p = 2$). The space $(\sum_{n=1}^{\infty} \oplus C_p^{n,\infty})_p$ is denoted by S_p^{∞}.

THEOREM 5: *Let $1 < p < \infty$, $p \neq 2$. The infinite-dimensional ranges of projections of the type $P(\{A_k\}, \{B_k\})$ (defined by (2.3)) are isomorphic to one of the following spaces:*

$$l_2,\quad l_p,\quad l_2 \oplus l_p,\quad (l_2 \oplus l_2 \oplus \cdots)_p,\quad S_p,\quad S_p \oplus l_2,$$
$$S_p \oplus (l_2 \oplus l_2 \oplus \cdots)_p,\quad S_p^{\infty},\quad C_p.$$

All the spaces in this list are of a different isomorphism type.

PROOF: It is easy to see that all the spaces in the list are even isometric to $P(\{A_k\}, \{B_k\})C_p$ for a suitable choice of $\{A_k\}$ and $\{B_k\}$. A simple application of the decomposition method (similar to that used in the proof of Proposition 3 which is a special case of the present theorem) shows that for every choice of $\{A_k\}$ and $\{B_k\}$ the range of $P(\{A_k\}, \{B_k\})$ is isomorphic to one of the spaces in the list. We omit the easy details.

We pass to the proof of the second statement of Theorem 5 which is less trivial. It is a well known fact that the first four spaces in the list are of different isomorphism types. None of them is isomorphic to any of the last five spaces (the first four spaces have an unconditional basis while the last five fail even to have a local unconditional structure [6], cf. the end of section 1).

We now consider the last five spaces in the list. They are written in increasing order; each one is obviously isometric to a subspace of the space following it in the list. Among them the space S_p is singled out by the fact that it does not contain a subspace isomorphic to l_2. To conclude the proof of Theorem 5 it is enough to prove the following three propositions (observe that non-isomorphism for $1 < p < 2$ implies non-isomorphism for $2 < p < \infty$ by duality):

PROPOSITION 5: *For $1 \leqq p < \infty$, $p \neq 2$, $S_p \oplus (l_2 \oplus l_2 \oplus \cdots)_p$ is not isomorphic to a subspace of $S_p \oplus l_2$.*

PROPOSITION 6: *For $1 \leqq p < 2$, S_p^∞ is not isomorphic to a subspace of $S_p \oplus (l_2 \oplus l_2 \oplus \cdots)_p$.*

PROPOSITION 7: *For $1 \leqq p < 2$, C_p is not isomorphic to a subspace of S_p^∞.*

The proof of Propositions 5 and 6 is routine. We give e.g. a brief outline of the proof of Proposition 6. Let T be an operator from S_p^∞ into $S_p \oplus (l_2 \oplus \cdots \oplus l_2 \oplus \cdots)_p$ and let P be the natural projection of the latter space onto its summand S_p. Since $C_p^{n,\infty}$ is isomorphic to l_2 and any operator from l_2 to l_p is compact (here $p < 2$ is used) it follows easily that the restriction of PT to every summand $C_p^{n,\infty}$ of S_p^∞ is compact. Using this observation we deduce easily that if T is an isomorphism it is possible to find also an isomorphism from S_p^∞ into $(l_2 \oplus l_2 \oplus \cdots)_p$ which is known to be impossible.

The proof of Proposition 7 is less routine and we present it in detail.

PROOF OF PROPOSITION 7: Assume that there is an isomorphism T from C_p into $S_p^\infty = (C_p^{1,\infty} \oplus C_p^{2,\infty} \oplus \cdots)_p$ and let $u_{i,j}$ be the usual basis vectors in C_p. Put

$$Tu_{i,j} = (x_{i,j}(1), x_{i,j}(2), \cdots) \tag{6.1}$$

with $x_{i,j}(n) \in C_p^{n,\infty}$ for all i, j, n. Put also $\alpha_{i,j}(n) = \|x_{i,j}(n)\|$ and

$$\alpha_{i,j} = (\alpha_{i,j}(1), \quad \alpha_{i,j}(2), \cdots) \in l_p \qquad 1 \leqq i, j < \infty. \tag{6.2}$$

Clearly $\|\alpha_{i,j}\| = \|Tu_{i,j}\| \leqq \|T\|$ (the norm of $\alpha_{i,j}$ and all the norms of other α vectors in the sequel is taken in l_p). The bounded sequence $\{\alpha_{1,j}\}_{j=1}^\infty$ of vectors in l_p has a subsequence $\{\alpha_{1,j_k}\}_{k=1}^\infty$ converging weakly (if $p = 1$, w^*) to some vector $\alpha_1 \in l_p$. We claim that

$$\lim_{k\to\infty} \|\alpha_{1,j_k} - \alpha_1\| = 0. \tag{6.3}$$

Indeed if (6.3) fails there is *a* $\rho > 0$ so that (after passing again to a subsequence if necessary) $\alpha_{1,j_k} = \alpha_1 + \beta_k$ where $\|\beta_k\| \geqq \rho$ and the $\{\beta_k\}$ have almost disjoint supports in l_p. Thus for every integer m

$$m^{\frac{1}{2}}\|T\| = \|\sum_{k=1}^m u_{1,j_k}\| \ \|T\| \geqq \|\sum_{k=1}^m Tu_{1,j_k}\| \geqq \tfrac{1}{2}\|\sum_{k=1}^m \beta_k\| \geqq \rho m^{1/p}/4 \tag{6.4}$$

but this is a contradiction for large m.

The same argument which we gave for $\alpha_{1,j}$ clearly works for $\{\alpha_{i,j}\}_{j=1}^\infty$

for every i and thus by the diagonal procedure we may find a subsequence of the integers (which in order to avoid more indices we continue to call $\{j_k\}_{k=1}^{\infty}$) so that

$$(6.5) \qquad \lim_{k\to\infty} \|\alpha_{i,j_k} - \alpha_i\| = 0 \qquad i = 1, 2, \cdots.$$

Using once again the same argument this time for the vectors $\{\alpha_i\}_{i=1}^{\infty}$ we get that there is a subsequence $\{i_l\}_{l=1}^{\infty}$ of the integers and a vector α in l_p so that

$$(6.6) \qquad \lim_{l\to\infty} \|\alpha_{i_l} - \alpha\| = 0$$

(we use here the fact that for every k $\overline{\text{span}}\, \{u_{i,j_k}\}_{i=1}^{\infty}$ is isometric to l_2).

By (6.5) and (6.6) it follows that there are increasing sequences of integers $\{i_h\}_{h=1}^{\infty}$ and $\{j_h\}_{h=1}^{\infty}$ so that

$$(6.7) \qquad \|\alpha_{i_h, j_h} - \alpha\| \leqq 2^{-h} \qquad h = 1, 2, \cdots.$$

Put $v_h = u_{i_h, j_h} = h = 1, 2, \cdots$. The vectors $\{v_h\}_{h=1}^{\infty}$ are isometrically equivalent to the unit vector basis in l_p and by (6.7) we have

$$(6.8) \qquad Tv_h = (y_h(1), y_h(2), \cdots) + z_h$$

with $y_h(n) \in C_p^{n,\infty}$, $\|y_h(n)\| = \alpha(n)$, $z_h \in S_p^{\infty}$, $\|z_h\| \leqq 2^{-h}$. Since $d(C_p^{n,\infty}, l_2) \leqq n$ we get for every m and n

$$(6.9) \qquad \left(\int_0^1 \|\sum_{h=1}^m r_h(t) y_h(n)\|^p \mathrm{d}t\right)^{1/p} \leqq \left(\int_0^1 \|\sum_{h=1}^m r_h(t) y_h(n)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leqq n(\sum_{h=1}^m \|y_h(n)\|^2)^{\frac{1}{2}} = n\alpha(n) m^{\frac{1}{2}}.$$

By using (1.4) we get another estimate for the same expression

$$(6.10) \qquad \left(\int_0^1 \|\sum_{h=1}^m r_h(t) y_h(n)\|^p \mathrm{d}t\right)^{1/p} \leqq (\sum_{h=1}^m \|y_h(n)\|^p)^{1/p} = \alpha(n) m^{1/p}.$$

Combining (6.9) and (6.10) we get for any m and n

$$(6.11) \qquad m^{1/p}\|T^{-1}\|^{-1} \leqq \|T^{-1}\|^{-1} \left(\int_0^1 \|\sum_{h=1}^m r_h(t) v_h\|^p \mathrm{d}t\right)^{1/p}$$
$$\leqq \left(\int_0^1 \|\sum_{h=1}^m r_h(t) T v_h\|^p \mathrm{d}t\right)^{1/p} \leqq \sum_{h=1}^m \|z_h\| + \left(\int_0^1 \sum_{k=1}^{\infty} \|\sum_{h=1}^m r_h(t) y_h(k)\|^p \mathrm{d}t\right)^{1/p}$$
$$\leqq 2 + \sum_{k=1}^n \left(\int_0^1 \|\sum_{h=1}^m r_h(t) y_h(k)\|^p \mathrm{d}t\right)^{1/p} + \left(\int_0^1 \sum_{k=n+1}^{\infty} \|\sum_{h=1}^m r_h(t) y_h(k)\|^p \mathrm{d}t\right)^{1/p}$$
$$\leqq 2 + m^{\frac{1}{2}} \sum_{k=1}^n k\alpha(k) + m^{1/p}(\sum_{k=n+1}^{\infty} \alpha^p(k))^{1/p}.$$

However if n is chosen so that $\|T^{-1}\|(\sum_{k=n+1}^{\infty}\alpha^p(k))^{1/p}$ is smaller than 1 then (6.11) fails to hold for large enough m. This contradiction concludes the proof of Proposition 7 and thus of Theorem 5.

REMARK:

(i) Theorem 5 is true also in the cases $p = 1$ and $p = \infty$, provided we add to the list given in the statement of theorem 5 two more spaces:

$$(\sum_{k=1}^{\infty} \oplus\, l_2^k)_p; \qquad (\sum_{k=1}^{\infty} \oplus\, l_2^k)_p \oplus l_2$$

(Here, as before, for $p = \infty$ the direct sum is taken in the sense of c_0, and l_∞ means c_0.)

When showing that all the spaces in the enlarged list are of different isomorphic types, after some trivial observations, we need only to compare two pairs of spaces: l_p with $(\sum_{k=1}^{\infty} \oplus\, l_2^k)_p$, and $l_2 \oplus l_p$ with $l_2 \oplus (\sum_{k=1}^{\infty} \oplus\, l_2^k)_p$. Since l_p $(p = 1, \infty)$ has unique unconditional basis (see [13], I.2.c), and the natural basis of $(\sum_{k=1}^{\infty} \oplus\, l_2^k)_p$ is unconditional but is not equivalent to the unit-vector-basis of l_p, we see that $l_p \not\approx (\sum_{k=1}^{\infty} \oplus\, l_2^k)_p$.

To prove that $l_2 \oplus l_p \not\approx l_2 \oplus (\sum_{k=1}^{\infty} \oplus\, l_2^k)_p$, it is enough to prove that $(\sum_{k=1}^{\infty} \oplus\, l_2^k)_p$ is not isomorphic to a complemented subspace of $l_2 \oplus l_p$. But the infinite-dimensional complemented subspaces of $l_2 \oplus l_p$ are known to be the trivial ones: l_2, l_p and $l_2 \oplus l_p$ (see[17]), so the proof is complete.

(ii) We do not know whether the list given in Theorem 5 exhausts all the possible isomorphism types of complemented subspaces of C_p. A comparison with L_p does not seem to be of help as far as this question is concerned. The first question which should be answered in this connection is the following: Is every complemented subspace of S_p isomorphic to either l_p or S_p? The results and the methods of proof of section 5 may possibly be of some help in this connection.

(iii) We did not check whether Propositions 6 and 7 are valid also for $p > 2$.

We conclude this section with the proof of another result showing the impossibility of a certain embedding. This result is not concerned with ranges of projections but with the relation between C_p and L_p. As we mentioned in the introduction it is known that for $1 \leqq p < \infty$, $p \neq 2$, C_p is not isomorphic to a subspace of L_p. What about the other direction, i.e., is L_p isomorphic to a subspace of C_p? the answer is trivially no if $1 \leqq p < 2$ (since e.g., L_p contains for every $p < r < 2$ a subspace isomorphic to l_r and by Proposition 4, C_p has no such subspaces). For

$2 < p < \infty$ the answer to the question we posed is also negative but this is a less trivial fact.

THEOREM 6: *For $2 < p < \infty$ the space L_p is not isomorphic to a subspace of C_p.*

This theorem generalizes a result stated in [12]. It is stated there that L_p is not isomorphic to a subspace of $(l_2 \oplus l_2 \oplus \cdots)_p$. The proof of this statement as presented in [12] is over simplified and therefore incomplete. The proof we give here for Theorem 6 is however based on the idea of the argument presented in [12]. We shall need first two lemmas.

LEMMA 4: *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements in C_p, $2 < p < \infty$, so that for some constant M*

$$(6.12)\quad \|u_n\| \leqq M, \qquad n = 1, 2, \cdots; \qquad \|\sum_{n=1}^{\infty} \lambda_n u_n\| \geqq M^{-1}(\sum_{n=1}^{\infty} |\lambda_n|^2)^{\frac{1}{2}}$$

Let $0 < \gamma < (MK_p)^{-1}$, where K_p is the constant in (1.5). Then there is an integer m so that

$$(6.13)\qquad \|E_m u_n\| \geqq \gamma \qquad n = 1, 2, 3, \cdots.$$

PROOF: If (6.13) fails to hold we can choose a sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$, and increasing sequences of integers $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ (with $m_0 = 0$) *so that*

$$(6.14)\quad \|(1-P_{m_i})u_{n_i}\| \leqq \varepsilon_i, \qquad \|E_{m_{i-1}} u_{n_i}\| < \gamma, \qquad \sum_{i=1}^{\infty} \varepsilon_i^2 < \infty$$

for $i = 1, 2, \cdots$.

By (6.12) and (6.14) we get that for every integer k (here, as before, $\{r_i\}_{i=1}^{\infty}$ are the Rademacher functions in [0, 1])

$$\begin{aligned}
(6.15)\quad M^{-1} \cdot k^{\frac{1}{2}} &\leqq \left(\int_0^1 \|\sum_{i=1}^{k} r_i(t) u_{n_i}\|^p \mathrm{d}t\right)^{1/p} \\
&\leqq \left(\int_0^1 \|\sum_{i=1}^{k} r_i(t) P_{m_i}(1-E_{m_{i-1}})u_{n_i}\|^p \mathrm{d}t\right)^{1/p} \\
&\quad + \left(\int_0^1 \|\sum_{i=1}^{k} r_i(t)[u_{n_i} - P_{m_i}(1-E_{m_{i-1}})u_{n_i}]\|^p \mathrm{d}t\right)^{1/p} \\
&\leqq Mk^{1/p} + K_p[\sum_{i=1}^{k} (\|(1-P_{m_i})u_{n_i}\| + \|P_{m_i}E_{m_{i-1}}u_{n_i}\|)^2]^{\frac{1}{2}} \\
&\leqq Mk^{1/p} + K_p(\sum_{i=1}^{\infty} \varepsilon_i^2)^{\frac{1}{2}} + K_p \cdot \gamma \cdot k^{\frac{1}{2}}.
\end{aligned}$$

So $(M^{-1} - K_p\gamma)k^{\frac{1}{2}} \leqq M \cdot k^{1/p} + \text{const.}$ and by the definition of γ, this is a contradiction for large enough k. $\square$

The second lemma which we put down here for easy reference is trivial and well known.

LEMMA 5: *Let $p > 2$ and $\varepsilon > 0$ be given. Then there is a $k(p, \varepsilon)$ so that for every $k > k(p, \varepsilon)$ and every operator V from l_p^k into l_2 with $\|V\| = 1$ we have*

$$|\{1 \leqq i \leqq k, \|Ve_i\| \geqq \varepsilon\}| \leqq \varepsilon \cdot k$$

where $\{e_i\}_{i=1}^k$ denote the unit vectors in l_p^k and the notation $|A|$ is used to denote the cardinality of the set A.

PROOF OF THEOREM 6: Let $\{\varphi_i\}_{i=1}^{\infty}$ be the normalized Haar basis of $L_p(0, 1)$, i.e., for $n = 0, 1, \cdots$

$$\varphi_{2^n+i}(t) = \begin{cases} 2^{n/p}, & t \in [i \cdot 2^{-n}, (i+\frac{1}{2})2^{-n}] \\ -2^{n/p}, & t \in ((i+\frac{1}{2})2^{-n}, (i+1)2^{-n}) \\ 0, & \text{otherwise} \end{cases} \qquad 0 \leqq i \leqq 2^{n-1}$$

The functions $\{\varphi_{2^n+i}\}_{i=0}^{2^n-1}$ will be called the Haar functions of the n'th level. Let us recall that the functions

$$r_n(t) = (\sum_{i=0}^{2^n-1} \varphi_{2^n+i}(t))/2^{n/p} \tag{6.16}$$

form the Rademacher functions on $[0, 1]$ and so by Khintchine's inequality are equivalent to the unit vector basis in l_2. We shall use also the Rademacher systems on dyadic subinterval of $[0, 1]$ which are defined as follows. Let k be an integer and let $0 \leqq j \leqq 2^k - 1$. For every $n \geqq k$ we let $r_{n,k,j}$ be the normalized averages of the Haar functions of the n'th level which are supported on $[j/2^k, (j+1)/2^k]$, i.e.,

$$r_{n,k,j}(t) = (\sum_{i=0}^{2^{n-k}-1} \varphi_{2^n+j2^{n-k}+i}(t))/2^{(n-k)/p}. \tag{6.17}$$

Observe that $r_n = r_{n,0,0}$ and

$$r_n = (\sum_{j=0}^{2^k-1} r_{n,k,j})/2^{k/p}, \qquad k \leqq n < \infty. \tag{6.18}$$

The natural isometry from $L_p[j2^{-k}, (j+1)2^{-k}]$ onto $L_p(0, 1)$ takes $r_{n,k,j}$ into r_n and therefore we have

$$(6.19) \qquad \|\sum_{n=k}^{\infty} \lambda_n r_{n,k,j}\|_p = \|\sum_{n=k}^{\infty} \lambda_n r_n\|_p \geqq \|\sum_{n=k}^{\infty} \lambda_n r_n\|_2 = (\sum_{n=k}^{\infty} |\lambda_n|^2)^{\frac{1}{2}}$$

for all choices of scalars $\{\lambda_n\}$ and for every j and k.

Assume now that there is an isomorphism from L_p into C_p. It will be a little more convenient to work with T_p instead of C_p and we can do this in view of Proposition 1. Since the Haar basis is a reproducing basis of L_p (cf. [12]) it follows from our assumption that there is a bounded linear operator U from $L_p[0, 1]$ into T_p with $\|U^{-1}\| = 1$ and an increasing sequence of integers $\{b_i\}_{i=1}^{\infty}$ so that

$$(6.20) \qquad U\varphi_i = (P_{b_{i+1}} - P_{b_i})U\varphi_i \qquad 1 \leqq i < \infty$$

and thus in particular the sequence $\{U\varphi_i\}_{i=1}^{\infty}$ forms an unconditional basic sequence (with unconditional constant equal to 1).

In view of (6.19) (for $j = k = 0$) and Lemma 4, it follows that if $\gamma = (2\|U\|K_p)^{-1}$ then there is an integer m_1 so that

$$(6.21) \qquad \|E_{m_1} U r_n\| \geqq \gamma, \qquad n = 1, 2, \cdots.$$

Since $E_{m_1} T_p$ is isomorphic to l_2 and for all $n \geqq k$ the vectors $\{r_{n,k,j}\}_{j=0}^{2^k-1}$ are isometrically equivalent to the unit vector basis in $l_p^{2^k}$ it follows from Lemma 5 that there is a k_1 so that for $n \geqq k_1$

$$(6.22) \quad |A_n| \geqq 2^{k_1-1} \qquad \text{where } A_n = \{0 \leqq j < 2^{k_1}; \|E_{m_1} U r_{n,k_1,j}\| \leqq \gamma/2\}.$$

In view of (6.19) we may apply Lemma 4 to each of the 2^{k_1} sequences $\{U r_{n,k_1,j}\}_{n=k_1}^{\infty}$, $0 \leqq j < 2^{k_1}$, and find an integer $m_2 > m_1$ so that

$$(6.23) \qquad \|E_{m_2} U r_{n,k_1,j}\| \geqq \gamma, \qquad k_1 \leqq n < \infty, \qquad 0 \leqq j < 2^{k_1}.$$

By (6.22) and (6.23) we get that

$$(6.24) \qquad \|(E_{m_2} - E_{m_1}) U r_{n,k_1,j}\| \geqq \gamma/2, \qquad j \in A_n, \qquad k_1 \leqq n < \infty.$$

By (6.20) we get that the sequence $\{(E_{m_2} - E_{m_1}) U r_{n,k_1,j}\}_{j=0}^{2^{k_1}-1}$ is for every n an unconditional sequence (with unconditional constant 1) and hence in view of (1.5), (6.18) and (6.24) we get that for $n \geqq k_1$

$$\begin{aligned}(6.25) \quad \|(E_{m_2} - E_{m_1}) U r_n\| &\geqq (\sum_{j=0}^{2^{k_1}-1} \|(E_{m_2} - E_{m_1}) U r_{n,k_1,j}\|^p)^{1/p} / 2^{k_1/p} \\ &\geqq (\gamma/2) \cdot |A_n|^{1/p} / 2^{k_1/p} \geqq \gamma/2 \cdot 2^{1/p} \geqq \gamma/4.\end{aligned}$$

By repeating exactly the same argument with a sufficiently large k_2 (whose size is determined via Lemma 5 by $d(E_{m_2}C_p, l_2)$) we can find an $m_3 > m_2$ so that $\|(E_{m_3}-E_{m_2})Ur_n\| \geqq \gamma/4$ for $n \geqq k_2$. In general we can find this procedure increasing sequences $\{k_i\}$ and $\{m_i\}$ of integers so that

$$(6.26) \qquad \|(E_{m_{i+1}}-E_{m_i})Ur_n\| \geqq \gamma/4 \qquad 1 \leqq i, \quad n \geqq k_i.$$

By (1.5) and (6.26) we get for any integer s and $n \geqq k_s$

$$(6.27) \quad \|U\| \geqq \|Ur_n\| \geqq \|E_{m_s}Ur_n\|$$
$$\geqq (\|E_{m_1}Ur_n\|^p + \|(E_{m_2}-E_{m_1})Ur_n\|^p + \cdots + \|(E_{m_s}-E_{m_{s-1}})Ur_n\|^p)^{1/p}$$
$$\geqq s^{1/p} \cdot \gamma/4.$$

This is obviously impossible for large s and thus the assumption that L_p embeds into C_p led to a contradiction.

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