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RESIDUALLY FINITE GROUPS WITH THE SAME FINITE IMAGES

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Section 1

The object of this note is to describe a new way of constructing finitely generated residually finite groups with the same finite images which are not isomorphic (see [1], [2], [4] and [9]). It is easy to construct examples of this kind unless severe restrictions are placed on the groups concerned – in the works cited above they are either finitely generated nilpotent or polycyclic. Here we shall derive a recipe for constructing some surprising simple additional examples. In particular this recipe leads to the

THEOREM: *Let F be a finite cyclic group with an automorphism of order n, where n is different from 1, 2, 3, 4 and 6. Then there are at least two non-isomorphic cyclic extensions of F with the same finite images.*

It is, perhaps, worth emphasizing that the groups provided by the theorem are all metacyclic i.e., extensions of cyclic groups by cyclic groups (and hence residually finite [5]). Thus even metacyclic groups are not determined by their finite images. In fact it is easy to extract from the proof of the theorem the somewhat surprising

COROLLARY: *The metacyclic groups*

$$G = \langle a, b; a^{25} = 1, b^{-1}ab = a^{6}\rangle \text{ and } H = \langle c, d; c^{25} = 1, d^{-1}cd = c^{11}\rangle$$

have the same finite images and are nilpotent of class two, but they are not isomorphic.

This corollary establishes the existence of non-isomorphic finitely generated nilpotent groups of class two with the same finite images.

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Section 2

The proof of Theorem 1 depends on the following simple

PROPOSITION: *Let A, B, C and D be finitely generated groups. If B and D have precisely the same finite images and if*

$$A \times B \simeq C \times D,$$

then A and C also have the same finite images.

PROOF: Let $\underline{V}$ be any variety which is generated by a finite group. If $V(X)$ denotes the verbal subgroup of the group X defined by $\underline{V}$ (see [7]) then it follows that

$$V(A \times B) = V(A) \times V(B) \simeq V(C) \times V(D) = V(C \times D).$$

Hence

$$A/V(A) \times B/V(B) \simeq C/V(C) \times D/V(D). \tag{1}$$

Now the finitely generated groups in a variety generated by a finite group are finite (see [7], p. 18). Thus all of the groups in equation (1) are finite. Moreover $B/V(B) \simeq D/V(D)$ since, by hypothesis, B and D have the same finite images. Therefore, by the well-known theorem of R. Remak [8], $A/V(A)$ and $C/V(C)$ are isomorphic. Since $\underline{V}$ is any variety generated by a finite group, it follows that A and C have the same finite images.

This proposition may be viewed as a recipe for constructing non-isomorphic finitely generated residually finite groups with the same finite images. We need only choose A and C to be finitely generated residually finite groups which are not isomorphic but admit a choice of two finitely generated groups B and D such that $A \times B \simeq C \times D$. This is not difficult (see [10] and [6]). The theorem is proved in this way by allying the proposition with Hirshon's remarks in [6].

Bearing these comments in mind we shall proceed now with the details of the proof of the theorem. Thus we suppose that $F = gp(a)$ is a finite cyclic group with an automorphism α of order n, n different from 1, 2, 3, 4 and 6. Since $\phi(n) > 2$, where $\phi(n)$ is the number of positive integers less than and prime to n (cf. Hardy and Wright [3]), we can find a power α^l of α with the properties

(i) $\alpha^l \neq \alpha$, $\alpha^l \neq \alpha^{-1}$ and
(ii) $(l, n) = 1$.

Let A be the split extension of F by an infinite cyclic group which induces α on F and let C be the split extension of F by an infinite cyclic group which induces α^l on F. If $a\alpha = a^r$ we may present A and C as follows:

$A = \langle a, b; a^m = 1, b^{-1}ab = a^r \rangle$ and

$$C = \langle a, c; a^m = 1, c^{-1}ac = a^{r^l} \rangle.$$

We shall prove

LEMMA 1: $A \ncong C$

and

LEMMA 2: *A and C have the same finite images.*

The proof of Lemma 1 is straightforward while that of Lemma 2, which can be proved directly, makes use of the proposition. First we prove Lemma 1. Thus suppose, if possible, that $\theta : A \to C$ is an isomorphism. Now F is the set of elements of finite order in both A and C. Therefore θ induces an automorphism of F. Hence

$$a\theta = a^s$$

where s and m are coprime. Moreover since A/F and C/F are both infinite cyclic we either have

$$b\theta = ca^t \quad \text{or} \quad b\theta = c^{-1}a^t$$

where t is a suitably chosen integer. This implies that either $\alpha = \alpha^l$ or that $\alpha^{-1} = \alpha^l$ contradicting the choice of l in (i). To see this suppose that $b\theta = ca^t$. Then

$$a^s\alpha = a^{rs} = (a^r)\theta = (b^{-1}ab)\theta = (b\theta)^{-1}a\theta b\theta = (ca^t)^{-1}a^s(ca^t)$$
$$= c^{-1}a^sc = a^s\alpha^l$$

But $(s, m) = 1$ which means that $\alpha = \alpha^l$. A similar argument yields $\alpha^{-1} = \alpha^l$ in the case where $b\theta = c^{-1}a^t$. This completes the proof of Lemma 1.

In order to prove Lemma 2 it suffices, by the proposition, to prove that $P = A \times Z$, where Z is an infinite cyclic group generated by z, has a second direct decomposition $P = C^* \times Z^*$ where $C^* \cong C$ and $Z^* \cong Z$. This is done by following, essentially verbatim, the argument given by Hirshon in [6]. For completeness we give the details here. By (ii) we can find integers u and v such that $ul - vn = 1$. Put $Z^* = gp(b^nz^u)$ and $C^* = gp(a, b^lz^v)$. Observe that Z^* is central in P and that $P = C^* \times Z^*$ because

$$(b^nz^u)^l(b^lz^u)^{-n} = z^{ul-vn} = z.$$

This completes the proof of Lemma 2.

Putting Lemma 1 and Lemma 2 together now proves the Theorem.

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