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ON ENGEL-LIKE CONGRUENCES

Paul M. Weichsel

1. Introduction

In this note we investigate the commutator-subgroup structure of groups that satisfy congruences and laws that are similar to Engel laws. We begin with the necessary notation. If G is a group and α a positive integer, then $(G)^\alpha$ is the subgroup generated by $\{g^\alpha | g \in G\}$. A *left-normed commutator* $(x_1, \dots, x_n)$ of weight n on $x_1, \dots, x_n$ is defined inductively for $n \geq 2$ by $(x_1, x_2) = x_1^{-1}x_2^{-1}x_1x_2$ and $(x_1, \dots, x_n) = ((x_1, \dots, x_{n-1}), x_n)$. The r th term of the *lower central series* of a group G , denoted by G_r , is the subgroup of G generated by commutators of the form $(x_1, \dots, x_r)$, all $x_i \in G$, $G_1 = G$. The terms of the derived series are defined by $G^{(0)} = G$, $G^{(1)} = G_2$ and $G^{(l)} = (G^{(l-1)})_2$. A group G is called *metabelian* if $G^{(2)} = 1$. If $A_1, \dots, A_s$ are normal subgroups of G , $s \geq 2$, then $(A_1, \dots, A_s)$ is the subgroup of G generated by $\{(a_1, \dots, a_s) | a_i \in A_i, i = 1, \dots, s\}$. If $w = (x_{\alpha_1}, \dots, x_{\alpha_r})$ with $x_{\alpha_i} \in \{x_1, \dots, x_\alpha\}$, then $w(G)$ is the subgroup generated by $\{(g_{\alpha_1}, \dots, g_{\alpha_r}) | g_{\alpha_i} \in G, i = 1, \dots, r\}$ (α_i may be equal to α_j for some pairs i, j , $i \neq j$). If G is a group, then $\text{var } G$ is the variety generated by G , i.e., the intersection of all varieties containing G .

DEFINITION: Let $w(x_1, \dots, x_n)$ be a left-normed commutator of weight d on $x_1, \dots, x_n$. The group G is said to satisfy the *w-congruence* if $w(g_1, \dots, g_n) \in G_{d+1}$ for all $g_i \in G$, $i = 1, \dots, n$. G is said to satisfy the *strong w-congruence* if $w(g_1, \dots, g_n) \in A_{d+1}$, with A the subgroup generated by $\{g_1, \dots, g_n\}$ for each set $\{g_1, \dots, g_n\}$ and corresponding subgroup A . w is said to be a law of G if $w(G) = 1$. An important example of a *w-congruence* is the *Engel congruence*: $w = (x, y, y, \dots, y)$.

The main theorem of this note (2.5) shows that in a group which satisfies a *w-congruence* the descending central series and the derived series are linked in a special way. Two consequences are derived. The first (3.3) states that a p -group G satisfying a strong *w-congruence*, w of weight $d < p$ is nilpotent of class at most $(d-1)^{l-1}$ if it is solvable of derived length at most l . The second (4.1) characterizes those finite p -groups of class $c < p$, satisfying the *c-weight Engel law*.

The proof of the main theorem depends on the observation that a

result of Gupta and Newman [1. Theorem] on metabelian groups can be modified to apply to a much larger class of groups.

2. The main theorem

We begin by quoting a weakened version of the theorem of Gupta and Newman.

PROPOSITION: *Let w be a left-normed commutator of weight d . If G is metabelian and $w(G) = 1$, then*

$(G_{d+1})^\alpha = 1$ with α an integer whose prime divisors are less than d , and $(G_d/G_{d+1})^\beta = 1$ with β an integer whose prime divisors are less than $d+1$.

The proof of this theorem depends on a number of properties of commutators in metabelian groups. They are:

- (i) $(b, a_1, \dots, a_t) = (b, a_{\sigma_1}, \dots, a_{\sigma_t})$ for $b \in G_2$, $a_1, \dots, a_t \in G$ and σ an arbitrary permutation on the set $\{1, \dots, t\}$.
- (ii) $(b^i, a) = (b, a)^i$ for every integer i , whenever $b \in G_2$, and $a \in G$.

On the other hand, once the weight of w is given, then the only commutators which actually occur in the proof are those of weight d or greater. Thus if the weight of w is d , and G is any group, then the theorem will hold for the group $\bar{G} = G / \bigcup_{r,s} (G_r, G_s)$, $r+s = d$ and $r, s \geq 2$.

We first verify that properties (i) and (ii) hold in the group $\bar{G}$.

2.1 LEMMA: *If G is any group and $i, j \geq 2$, then*

$$(G_i, G_j, G, \dots, G) \subseteq \bigcup_{k, r, s} (G_r, G_s),$$

$r+s = i+j+k$, and $r, s \geq 2$.

PROOF: Induction on k . If $k = 1$, then the lemma follows from the 3-subgroup-lemma of P. Hall, [3. Theorem 3.4.7], since

$$(G_i, G_j, G) \subseteq (G_j, G, G_i)(G, G_i, G_j) = (G_{j+1}, G_i)(G_{i+1}, G_j).$$

We now recall that if $A, B, C \triangleleft G$, then

$$(AB, C) \subseteq (A, C)(B, C).$$

Hence $(\bigcup_{r,s} (G_r, G_s), G) \subseteq \bigcup_{r,s} (G_r, G_s, G) \subseteq \bigcup_{u,v} (G_u, G_v)$ with $r+s=n$, $r, s \geq 2$ and $u+v = n+1$, $u, v \geq 2$, and the lemma follows by induction.

2.2 LEMMA: *Let $a \in G_d$, $d \geq 2$ and $b, c \in G$. Then $(a, b, c) \in (a, c, b)(G_d, G_2)$.*

PROOF: The proof is identical to the usual one for metabelian groups.

2.3 LEMMA: *Let $a_i \in G$, $i = 1, \dots, n$ and $b \in G_m$, $m \geq 2$. Then*

$$(b, a_1, a_2, a_3, \dots, a_n) \in (b, a_2, a_1, a_3, \dots, a_n) \bigcup_{r,s} (G_r, G_s), \quad r+s = n+m, \quad r, s \geq 2.$$

PROOF

Case I. Let $n = 2$. Then $(b, a_1, a_2) \in (b, a_2, a_1)(G_m, G_2)$ by (2.2).

Case II. Let $n > 2$ and induct on n . Thus assume the lemma for n and consider $(b, a_1, a_2, a_3, \dots, a_{n+1})$. By induction $(b, a_1, a_2, a_3, \dots, a_n) = (b, a_2, a_1, a_3, \dots, a_n)c$, with $c \in \bigcup_{r,s}(G_r, G_s)$, $r+s = n+m$, $r, s \geq 2$. Hence $(b, a_1, a_2, a_3, \dots, a_{n+1}) = ((b, a_2, a_1, a_3, \dots, a_n)c, a_{n+1}) = (b, a_2, a_3, \dots, a_n, a_{n+1})ef$, with $e \in (G_{n+m+1}, G_2)$ and $f \in (G_{n+m}, G_2)$, both subgroups of $\bigcup_{r,s}(G_r, G_s)$, $r+s = n+m+1$, $r, s \geq 2$. This completes the proof.

It now follows easily that property (i) holds in the group

$$\bar{G} = G / \bigcup_{r,s} (G_r, G_s), \quad r+s = n, \quad r, s \geq 2$$

for commutators of total weight greater than or equal to n .

2.4 LEMMA: Let $b \in G_t$ and $a \in G$. Then for all integers i ,

$$(b^i, a) \in (b, a)^i \bigcup_{r,s} (G_r, G_s), \quad r+s = 2t+1, \quad r, s \geq 2.$$

PROOF: If $i = -1$, then $(b^{-1}, a) \in (b, a)^{-1}(G_r, G_s)$, for $r+s = 2t+1$. We now induct on i for $i \geq 1$. If $i = 1$, the result is trivial. If $(b^n, a) \in (b, a)^n \bigcup_{r,s}(G_r, G_s)$, $r+s = 2t+1$, $r, s \geq 2$, then $(b^{n+1}, a) = (b^n, a)(b^n, a, b) \times (b, a)$ and so $(b^{n+1}, a) \in (b, a)^n(b, a) \bigcup_{r,s}(G_r, G_s) = (b, a)^{n+1} \bigcup_{r,s}(G_r, G_s)$, $r+s = 2t+1$, $r, s \geq 2$.

We will now state the main theorem in two different forms.

2.5 THEOREM: Let w be a left-normed commutator of weight d and G a group satisfying the w -congruence. Then

$$(G_d)^\alpha \subseteq \bigcup_{r,s} (G_r, G_s)G_{d+1},$$

with $r+s = d$, $r, s \geq 2$ and α an integer whose prime divisors are less than $d+1$. Furthermore, if $(G_d)^q = G_d$, for every prime $q < d+1$, then

$$G_d = \bigcup_{r,s} (G_r, G_s)G_t \quad r, s \text{ as above}$$

and t every integer greater than or equal to $d+1$.

PROOF: Let $\bar{G} = G / \bigcup_{r,s}(G_r, G_s)$, $r+s = d$, $r, s \geq 2$. Then commutators of weight d in $\bar{G}$ satisfy conditions (i) and (ii) needed in the proof of the Gupta-Newman Theorem. Now let $\bar{G} = \bar{G}/w(\bar{G})$ and since $w(\bar{G}) = 1$, we conclude that $(\bar{G}_d)^\alpha = 1$ with α as described in the hypothesis. That is, $(G_d)^\alpha \subseteq \bigcup_{r,s}(G_r, G_s)G_{d+1}$, $r+s = d$, $r, s \geq 2$.

Now if $(G_d)^q = G_d$ for every prime $q < d+1$ we get

$$G_d = \bigcup_{r,s} (G_r, G_s)G_{d+1}$$

since $\bigcup_{r,s} (G_r, G_s)G_{d+1} \subseteq G_d$. But this relation remains true if d is replaced by $d+1$ since

$$\begin{aligned} G_{d+1} &= (G_d, G) = \left(\bigcup_{r,s} (G_r, G_s)G_{d+1}, G \right) \subseteq \bigcup_{r,s} (G_r, G_s, G)G_{d+2} \subseteq \\ &\bigcup_{a,b} (G_a, G_b)G_{d+2} \subseteq G_{d+1}, \quad r+s = d, \quad a+b = d+a, \quad r, s, a, b \geq 2. \end{aligned}$$

Thus

$$G_{d+1} = \bigcup_{a,b} (G_a, G_b)G_{d+2}, \quad a+b = d+1, \quad a, b \geq 2.$$

Hence

$$G_d = \bigcup_{r,s} (G_r, G_s) \bigcup_{a,b} (G_a, G_b)G_{d+2} = \bigcup_{r,s} (G_r, G_s)G_{d+2},$$

$r+s = d$, $a+b = d+1$, $r, s, a, b \geq 2$, and the conclusion follows by induction.

2.6 THEOREM: *Let w be a left-normed commutator of weight d and G a group satisfying $w(G) = 1$. Then $(G_d)^\alpha \subseteq \bigcup_{r,s} (G_r, G_s)$, $r+s = d$, $r, s \geq 2$ and α an integer whose prime divisors are less than $d+1$.*

Furthermore if $(G_d)^q = G_d$, for every prime $q < d+1$, then

$$G_d = \bigcup_{r,s} (G_r, G_s), \quad r+s = d, \quad r, s \geq 2.$$

PROOF: Since $w(G) = 1$, $w(\bar{G}) = 1$ with $\bar{G} = G / \bigcup_{r,s} (G_r, G_s)$, $r+s = d$, $r, s \geq 2$. Now applying the conclusions of the Gupta-Newman theorem we get that $(\bar{G}_{d+1})^\gamma = 1$ and $(\bar{G}_d / \bar{G}_{d+1})^\beta = 1$ with β, γ integers whose prime divisors are less than $d+1$. Therefore $(G_{d+1})^\gamma \subseteq \bigcup_{r,s} (G_r, G_s)$, $r+s = d$, $r, s \geq 2$, and $(\bar{G}_d)^\beta \subseteq \bar{G}_{d+1}$. Thus $(G_d)^{\beta\gamma} \subseteq (G_{d+1})^\gamma \subseteq \bigcup_{r,s} (G_r, G_s)$, $r+s = d$, $r, s \geq 2$ and $\beta\gamma$ satisfies the requirements of α in the theorem.

Now if $(G_d)^q = (G_d)$ for all primes $q < d+1$, we get $G_d = \bigcup_{r,s} (G_r, G_s)$, $r+s = d$, $r, s \geq 2$.

3. p -groups satisfying a small congruence

We say that a p -group G satisfies a small congruence if $w(G) \subseteq G_{d+1}$ with w a left-normed commutator of weight $d < p$. In this section we will show that a p -group satisfying a small strong congruence is nilpotent if it is solvable and derive a bound on its nilpotency class in terms of its derived length.

3.1. LEMMA: Let G be a group in which the relation $G_d = \bigcup_{r,s} (G_r, G_s)$, $r+s = d$, $r, s \geq 2$ holds for some fixed $d \geq 4$. Then

$$G_{r(d-1)+1} = \bigcup_{a_i} (G_{a_1}, \dots, G_{a_{r+1}}), \quad \sum_{i=1}^{r+1} a_i = r(d-1)+1,$$

$a_i \geq 2$ all i .

PROOF: Induction on r . If $r = 1$, the conclusion is the hypothesis. Suppose that

$$G_{r(d-1)+1} = \bigcup_{a_i} (G_{a_1}, \dots, G_{a_{r+1}}), \quad \sum_{i=1}^{r+1} a_i = r(d-1)+1,$$

$a_i \geq 2$ all i . Then

$$G_{(r+1)(d-1)+1} = (G_{r(d-1)+1}, \underbrace{G, \dots, G}_{d-1}) = (\bigcup_{a_i} (G_{a_1}, \dots, G_{a_{r+1}}),$$

$$\underbrace{G, \dots, G}_{d-1}) \subseteq \bigcup_{b_i} (G_{b_1}, \dots, G_{b_{r+1}}) \subseteq G_{(r+1)(d-1)+1},$$

$$\sum_{i=1}^{r+1} a_i = r(d-1)+1, \quad \sum_{i=1}^{r+1} b_i = \sum_{i=1}^{r+1} a_i + (d-1),$$

$a_i, b_i \geq 2$ all i by 2.1. Hence we have $G_{(r+1)(d-1)+1} = \bigcup_{b_i} (G_{b_1}, \dots, G_{b_{r+1}})$ and the lemma follows by induction.

3.2 LEMMA: Let G be a group in which the relation $H_d = \bigcup_{r,s} (H_r, H_s)$, $r+s = d$, $r, s \geq 2$, d a fixed integer, $d \geq 4$ holds for all subgroups H of G . Then

$$H_{(d-1)^l+1} \subseteq H^{(l+1)}.$$

PROOF: If $l = 1$, then $(d-1)^l+1 = d$ and by hypothesis

$$H_d = \bigcup_{r,s} (H_r, H_s) \subseteq H^{(2)}, \quad r+s = d, \quad r, s \geq 2.$$

Now suppose that $H_{(d-1)^l+1} \subseteq H^{(l+1)}$. Then by replacing H by H' , we get $(H')_{(d-1)^l+1} \subseteq H^{(l+2)}$. But according to 3.1

$$H_{(d-1)^{(l+1)}+1} = H_{(d-1)^l(d-1)+1} = \bigcup_{a_i} (H_{a_1}, \dots, H_{a_{(d-1)^l+1}}) \subseteq (H')_{(d-1)^l+1} \subseteq H^{(l+2)}$$

since $a_j \geq 2$ and $H_{a_j} \subseteq H'$. Hence $H_{(d-1)^l+1} \subseteq H^{(l+2)}$ which proves the lemma.

3.3 THEOREM: *Let w be a left-normed commutator of weight d , and let G be a p -group with $d < p$. If G satisfies the strong w -congruence and G is solvable of derived length l , then G is nilpotent of class at most $(d-1)^{l-1}$.*

PROOF: We may assume without loss of generality that G is finitely generated and therefore finite. Thus G is nilpotent and we must derive a bound for the nilpotency of G independent of the number of its generators. Since $d < p$ and G is a p -group it follows from 2.5 that every subgroup H of G satisfies

$$H_d = \bigcup_{r,s} (H_r, H_s), \quad r+s = d, \quad r, s \geq 2.$$

Thus by 3.2, $G_{(d-1)^{l-1}+1} \subseteq G^{(l+1)}$, and since G is solvable of length l , H is nilpotent of class at most $(d-1)^{l-1}$. This completes the proof.

REMARK: The version of the Gupta-Newman theorem that we have used is a relatively crude version of the original. In particular, the prime divisor properties of the integers α and β are determined not only by the weight of the commutator w but by the multiplicities of the variables which occur in w . In fact, if we know that w involves at least 3 variables, then the bound $d < p$ in the theorem above can be improved to $d \leq p$. A particularly interesting case of this occurs when G is solvable of derived length l and has exponent p . For then G satisfies the strong $(p-1)$ -Engel congruence and Gupta has shown [2, Theorem 7.18] that G is nilpotent of class at most $(p-1)^{l-1} + \cdots + (p-1) + 1$.

If on the other hand G is a solvable-of-length- l p -group satisfying the strong w -congruence with w of weight p and involving at least 3 variables, then the class of G is at most $(p-1)^{l-1}$.

4. Nilpotent p -groups

In this section we will characterize those nilpotent p -groups of class $c < p$ which satisfy the Engel law of weight c .

4.1 THEOREM: *Let G be a nilpotent p -group of class $c < p$ and let*

$$w = (x, y, \underbrace{\cdots}_{c-1}, y).$$

Then G satisfies the law $w = 1$ if and only if $H_c = \bigcup_{r,s} (H_r, H_s)$, $r+s = c$, $r, s \geq 2$ for all groups $H \in \text{var } G$.

PROOF: Suppose $w(G) = 1$ with

$$w = (x, y, \underbrace{\cdots}_{c-1}, y).$$

Then by 2.6) $G_c = \bigcup_{r,s} (G_r, G_s)$, $r+s=c$, $r, s \geq 2$.

Thus since G satisfies the law $w=1$, every group $H \in \text{var } G$ satisfies it and the theorem follows in one direction.

Now suppose that $H_c = \bigcup_{r,s} (H_r, H_s)$, $r+s \geq c$, $r, s \geq 2$ for all $H \in \text{var } G$. It follows that this relation holds for the relatively free groups in $\text{var } G$. Thus every group in $\text{var } G$ satisfies a law: $(x_1, \dots, x_c) = \prod d_j^{y_j}$ with each d_j an element of (F_r, F_s) , F the relatively free group generated by $\{x_1, \dots, x_c, \dots\}$. Furthermore we may assume that each factor on the right involves each of the variables $x_1, \dots, x_c$. Now we set $x = x_1$ and $y = x_2 = \dots = x_c$ on both sides of the equation. Thus

$$(*) \quad w = (x, \underbrace{y, \dots, y}_{c-1}) = \prod_j d_j^{y_j}$$

We now utilize a standard argument based on the facts that each commutator of weight c is a bilinear form and that each non-trivial factor on the right involves at least 2 occurrences of x . Let l be a primitive root of p and replace x by x^l in $(*)$. Then we get

$$w^l = \prod d_j^{y_j l^{r(j)}}$$

where d_j has $r(j)$ occurrences of x . Then raising both sides of $(*)$ to the power $-l$ and multiplying we get

$$(**) \quad 1 = \prod d_j^{y_j(l^{r(j)}-l)}.$$

We continue this process with $(**)$ thereby eliminating those factors of $(**)$ containing the minimum number of occurrences of x . In this way we eventually get a law of the form

$$1 = (\prod d_j^{y_j})^m$$

in which each factor contains the same number of occurrences of x and y , and with m an integer relatively prime to p . Now working backwards we can conclude that

$$w = (x, \underbrace{y, \dots, y}_{c-1}) = 1.$$

Thus G satisfies the law $w=1$.

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