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A CHARACTERIZATION OF THE B -REALCOMPACT SPACE

David K. Hsieh

1. Introduction

Realcompact spaces are most often studied and characterized in the framework of the ring $C(X)$ of all continuous functions, both bounded and unbounded, defined on the space X . We abandon this traditional approach to realcompact spaces in favor of a more general yet simpler setting made available by the λ -compactification theory. Thus instead of the ring of all continuous functions, we consider a Banach algebra of functions; instead of both bounded and unbounded functions, we need only to work with the bounded functions. Frolik introduced in [1] the concept of a complete family for unbounded continuous functions and properties of Q -spaces (i.e. realcompact spaces) were derived in terms of complete families. Since we deal with only the bounded functions in the λ -compactification theory, Frolik's definition of complete family cannot be adopted here. In this article, we introduce without using unbounded functions the definition of B -complete family of bounded functions in terms of positive singular functions in a given Banach algebra B , and then characterize the B -realcompact space, which is a generalization of the realcompact space, in terms of the B -complete families.

2. Preliminaries

In general we adopt the notation and definitions in [3] and [4]. Thus we study an arbitrary set E and a Banach algebra B of bounded real-valued functions defined on E . The norm in B is defined by $\|f\| = \sup_{x \in E} |f(x)|$ for each f in B . B is called an admissible Banach algebra if B contains constants and B separates points in E . A positive cone in B is a collection of positive singular functions in B which is closed under addition, and multiplication by positive scalars. A maximal positive cone (m.p.c.) is a positive cone not properly contained in any other positive cone. A m.p.c. M is said to be strong if there exists a function f in M such that $f(x) \neq 0$ for each $x \in E$, otherwise M is said to be weak. A m.p.c. M is said to be free if there exists no point x in E such that $f(x) = 0$ for every f in M .

3. B-realcompact spaces and B-complete family of functions

In the following discussions, E will always denote a set and B an admissible Banach algebra on E .

3.1 DEFINITION: Let A be a subset of E and f be a non-negative function in B . f is said to be strongly bounded on A provided that either there is a point $\bar{x}$ in A such that $f(\bar{x}) = 0$ or there exists an $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for each x in A .

NOTATION: When f is strongly bounded on A , we write $f \not\equiv 0$ on A .

3.2 DEFINITION: Let $\mathcal{F}$ be a filter of zero sets of functions in B . $\mathcal{F}$ is said to be a maximal ε -filter in (E, B) provided that the set $\mathcal{F}^- = \{f \in B: f \geq 0, [f, \varepsilon] \in \mathcal{F} \text{ for each } \varepsilon > 0\}$ is a m.p.c. (maximal positive cone) in B where $[f, \varepsilon]$ denotes the set $\{x \in E: |f(x)| \leq \varepsilon\}$.

For latter discussion, we shall need the following lemma which is proved in [4].

3.3 LEMMA: $\mathcal{F}$ is a maximal ε -filter if and only if given a non-negative function f in B if for every $\varepsilon > 0$ the set $[f, \varepsilon] = \{x \in E: |f(x)| \leq \varepsilon\}$ meets every member of $\mathcal{F}$ then $[f, \varepsilon] \in \mathcal{F}$ for every $\varepsilon > 0$.

3.4 LEMMA: Let M be a m.p.c. in B . Let $M^* = \{[f, \varepsilon]: \varepsilon > 0, f \in M\}$ where $[f, \varepsilon]$ denotes the set $\{x \in E: |f(x)| \leq \varepsilon\}$. Then M^* is a maximal ε -filter.

PROOF: Since M is a positive cone, by definition $[f, \varepsilon] \neq \emptyset$ for each $f \in M$. Now let $[f, \varepsilon]$ and $[g, \delta]$ be in M^* . Clearly $[f, \varepsilon] \cap [g, \delta] \supset [f+g, \varepsilon \wedge \delta]$.

Finally we show the following: if $Z(h) \supset [f, \varepsilon]$ for some $[f, \varepsilon]$ in M^* then $Z(h) \in M^*$. First we define a sequence g_n in B as below; for each positive integer n , let

$$g_n = \left[|h(x)| + \frac{\varepsilon}{f(x) \vee 1/n} \right] \wedge [|h(x)| + 1].$$

It can be readily verified that g_n converges uniformly to a function $g(x)$ where $g(x) = 1$ if $x \in [f, \varepsilon]$ and $g(x) = |h(x)| + \varepsilon/f(x)$ if $x \notin [f, \varepsilon]$. Since B is a Banach algebra with the norm defined by $\|f\| = \sup \{|f(x)|: x \in E\}$, g is in B . In fact g is a non-negative function in B . We now claim gf is in M . This claim follows from the observation: (i) $gf \leq \|g\|f$; (ii) $\|g\|f \in M$; and (iii) M is a m.p.c. However $Z(h)$ is precisely the set $[gf, \varepsilon]$. Hence $Z(h) \in M$. This completes the proof that M^* is a filter of zero sets. Since M is a m.p.c., M^* is a maximal ε -filter by definition.

3.5 DEFINITION: A subfamily of non-negative functions $H \subset B$ is said to be complete provided that given a maximal ε -filter $\mathcal{F}$, if for every $f \in H$ there exists a $Z_f \in \mathcal{F}$ such that $f \not\triangleright 0$ on Z_f then $\bigcap \mathcal{F} \neq \emptyset$.

3.6 THEOREM: Suppose $\mathcal{F}$ is a maximal ε -filter, and suppose that for each sequence $\{F_n\}$ in $\mathcal{F}$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Then for each non-negative function f in B there is a Z_f in $\mathcal{F}$ such that $f \not\triangleright 0$ on Z_f .

PROOF: For each positive integer n , let $[f, 1/n]$ denote the set $\{x \in E : |f(x)| \leq 1/n\}$. We consider two cases.

Case I: Suppose there is an integer k such that $[f, 1/k] \notin \mathcal{F}$. By Lemma 3.3, there is some integer t and there is a $Z_f \in \mathcal{F}$ such that $[f, 1/t] \cap Z_f = \emptyset$. Hence $f > 1/t$ on Z_f .

Case II: On the other hand, suppose $[f, 1/n] \in \mathcal{F}$ for each n . Since each sequence in $\mathcal{F}$ has non-empty intersection, there is a point $\bar{x}$ in $\bigcap_{n=1}^{\infty} [f, 1/n]$. Clearly $f(\bar{x}) = 0$. Therefore in either case $f \not\triangleright 0$ on some member of $\mathcal{F}$.

3.7 DEFINITION: E is said to be B -realcompact provided that every free m.p.s. in B is strong.

REMARK: It can be readily verified that a topological space X is realcompact if and only if X is $C^*(X)$ -realcompact where $C^*(X)$ is the Banach algebra of all bounded real-valued continuous functions.

3.8 THEOREM: The following statements are equivalent:

- (i) E is B -realcompact.
- (ii) If $\mathcal{F}$ is a maximal ε -filter in (E, B) such that each sequence in $\mathcal{F}$ has non-empty intersection, then $\bigcap \mathcal{F} \neq \emptyset$.
- (iii) The set $B^+ = \{f \in B : f \geq 0\}$ is a B -complete family.
- (iv) $G = \{f \in B : f \triangleright 0 \text{ on } E\}$ is B -complete, where $f \triangleright 0$ means that $f > 0$ and for each $\varepsilon > 0$ there exists $x \in E$ such that $f(x) < \varepsilon$.
- (v) There exists a B -complete family H in B .

PROOF:

(i) $\Rightarrow$ (ii). Suppose there exists a maximal ε -filter $\mathcal{F}$ such that each sequence in $\mathcal{F}$ has non-empty intersection, but $\bigcap \mathcal{F} = \emptyset$. By definition 3.2, the set $\mathcal{F}^- = \{f \in B : f \geq 0, [f, \varepsilon] \in \mathcal{F} \text{ for each } \varepsilon > 0\}$ is a m.p.c. in B . Since $\bigcap \mathcal{F} = \emptyset$, $\mathcal{F}^-$ is a free m.p.c. For each $f \in \mathcal{F}^-$, $[f, 1/n] \in \mathcal{F}$ for every integer n . Thus $\bigcap_{n=1}^{\infty} [f, 1/n] \neq \emptyset$. Suppose that y is in $\bigcap_{n=1}^{\infty} [f, 1/n]$. Clearly $f(y) = 0$. That is $\mathcal{F}^-$ is a weak free m.p.c. in B . Hence E is not B -realcompact.

(ii) $\Rightarrow$ (i). Suppose E is not B -realcompact. There exists a weak free m.p.c. M in B . Let $M^* = \{[f, \varepsilon] : \varepsilon > 0, f \in M\}$ where $[f, \varepsilon]$ denotes the

set $\{x \in E: |f(x)| \leq \varepsilon\}$. It follows from Lemma 3.4 that M^* is a maximal ε -filter. Let $\{[f_n, \varepsilon_n]\}$ be a sequence in M^* . We may assume $\|f_n\| = 1$ for each n . Hence $f = \sum_{n=1}^{\infty} f_n/2^n$ is in M . Since M is a weak m.p.c., there exists $\bar{x}$ in E such that $f(\bar{x}) = 0$. Hence $f_n(\bar{x}) = 0$ for each n . It follows that $\bar{x} \in \bigcap_{n=1}^{\infty} [f_n, \varepsilon_n] \neq \emptyset$. But $\bigcap M^* = \emptyset$ as M is a free m.p.c.

(iii) $\Rightarrow$ (ii). It follows directly from Theorem 3.6 and Definition 3.5.

(ii) $\Rightarrow$ (iii). Let $\mathcal{F}$ be a maximal ε -filter. Suppose for each $f \in B^+$, there exists Z_f in $\mathcal{F}$ such that $f \not\geq 0$ on Z_f . To show that B^+ is B -complete, it suffices, in view of (ii), to show each sequence in $\mathcal{F}$ has non-empty intersection. Assume the contrary: suppose that there exists a sequence $\{Z_n\}$ in $\mathcal{F}$ such that $\bigcap_{n=1}^{\infty} Z_n = \emptyset$. Since $\mathcal{F}$ is a maximal ε -filter, $\mathcal{F}^- = \{f \in B: f \geq 0, [f, \varepsilon] \in \mathcal{F} \text{ for every } \varepsilon > 0\}$ is a m.p.c. in B . Write $Z_n = Z(f_n)$, the zero set of f_n , with $f_n \in B^+$ and $\|f_n\| \leq 1$. Obviously $[f_n, \varepsilon] \in \mathcal{F}$ for each $\varepsilon > 0$ and each n . Thus $f_n \in \mathcal{F}^-$ for each n . Define $f = \sum_{n=1}^{\infty} f_n/2^n$. Clearly f is in $\mathcal{F}^-$ as $\mathcal{F}^-$ is a m.p.c. But $\bigcap_{n=1}^{\infty} Z(f_n) = \emptyset$, it follows that $f > 0$. Now recall that $Z_f \in \mathcal{F}$, and let $\varepsilon > 0$ be given. Choose n so large that $1/2^n < \varepsilon$. Since $\mathcal{F}$ is a filter, there exists $\bar{x}$ in $Z_1 \cap \cdots \cap Z_n \cap Z_f$. Clearly $f(\bar{x}) \leq 1/2^n < \varepsilon$. This contradicts that $f \not\geq 0$ on Z_f .

(iii) $\Rightarrow$ (iv). Suppose $\mathcal{F}$ is a maximal ε -filter and that for each $f \in G$ there exists a $Z_f \in \mathcal{F}$ with $f \not\geq 0$ on Z_f . To show $\bigcap \mathcal{F} \neq \emptyset$, let $g \in B^+ \sim G$. Being the zero set of the zero function, E is in $\mathcal{F}$. Clearly $g \not\geq 0$ on E which is in $\mathcal{F}$. Since B^+ is B -complete $\bigcap \mathcal{F} \neq \emptyset$.

(iv) $\Rightarrow$ (v). Trivial.

(v) $\Rightarrow$ (ii). It follows immediately from Theorem 3.6 and Definition 3.5.

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