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THE POINT-OUTERTHICKNESS OF COMPLETE n -PARTITE GRAPHS

John Mitchem

A graph G is said to have property F_n , $n \geq 1$, if G has no subgraphs homeomorphic from the complete graph K_{n+1} or the complete bipartite graph $K([(n+2)/2], \{(n+2)/2\})$. For a real number x , $[x]$ denotes the greatest integer not exceeding x , and $\{x\}$ is the least integer not less than x . For $n = 1, 2, 3, 4$ graphs with property F_n correspond respectively with totally disconnected, acyclic, outerplanar, and planar graphs. In [3] Chartrand, Geller, and Hedetniemi defined the *point-partition number* $f_n(G)$, $n \geq 1$, of a graph G as the minimum number of pairwise disjoint subsets into which the point set of G can be partitioned such that each set induces a graph with property F_n . Such a partition is called an F_n partition. The parameter f_1 is the famous chromatic number, and f_2 is the more recently introduced point-arboricity. (See, for example, [4], [5], or [8].) In this paper we consider f_3 , the point-outerthickness.

By replacing the word 'point' in the definition of $f_n(G)$, $n \geq 2$, with 'line' we obtain the line-partition number $f'_n(G)$. Nash-Williams [9] developed an exact formula for $f'_2(G)$, the arboricity of G . The parameter $f'_4(G)$ is called the thickness of G . The precise value of $f'_4(K_p)$ is known for all p (See [7] and [6]). Beineke, Harary, and Moon [2] and Beineke [1] have determined $f'_4(K(m, n))$ for most, but not all, values of m and n .

Before beginning our investigation of $f_3(G)$, which henceforth is denoted simply $f(G)$ we need some additional definitions and notation. The cardinality of set S is denoted by $|S|$. Let $V_1, V_2, \dots, V_n$ be finite, non-void, mutually disjoint sets with $|V_i| = p_i$, $1 \leq i \leq n$, and $p_1 \leq p_2 \leq \dots \leq p_n$, the *complete n -partite graph* $G = K(p_1, p_2, \dots, p_n)$ has point set $\bigcup_{i=1}^n V_i$ and two points of G are adjacent if and only if they are in different V_i . The V_i are called *partite sets* of G . The complete bipartite graph $K(1, n)$ is called a *star*. Now, in four theorems we develop an exact formula for the point-outerthickness of any complete n -partite graph and also give the desired decomposition. Chartrand, Kronk, and Wall, [4], developed the analogous formula for point-arboricity.

We begin with a number of observations.

REMARK 1: For every positive integer p , $f(K_p) = \{p/3\}$.

REMARK 2: A complete n -partite graph G , $n \geq 2$, is outerplanar if and only if G is isomorphic to one of the following: $K(1, 1, 2)$, $K(2, 2)$, $K(1, 1, 1)$, or $K(1, m)$ where m is any positive integer.

REMARK 3: Let S be a set of at least five points of a complete n -partite graph G . If the graph induced by S is outerplanar, then it either has no lines or is a star, and S has all but possibly one point from a single partite set.

Throughout the remainder of the paper we use the following notation:

$$G = K(p_1, p_2, \dots, p_n)$$

$$p_0 = 0$$

$$a = \text{least positive integer such that } \sum_{i=1}^a p_i \geq n - a.$$

$$r = \max \{i: p_i \leq 2\}$$

$$k = \max \{i: p_i \leq 1\}$$

$$s = \left\{ \left(\sum_{i=1}^r p_i + 3(n-r) \right) / 4 \right\} \text{ if } (k+r-n) \leq (2/3)(2r-n) \text{ and } p_{a+1} \leq 2.$$

$$s = \{(2n-r)/3\} \text{ if } (k+r-n) > (2/3)(2r-n) \text{ and } p_{a+1} \leq 2.$$

THEOREM 1: If $p_{a+1} \geq 3$, then $f(G) = n - \max \{b: \sum_{i=1}^b p_i \leq n - b\}$.

PROOF: We consider two cases and in each case show that the desired result is an upper bound for the point-outerthickness of G . Then, combining the two cases, we verify that there is no smaller outerplanar partition of $V(G)$.

Case (i) Suppose $\sum_{i=1}^a p_i = n - a$. We can partition $V(G)$ into $n - a$ sets $S_1, S_2, \dots, S_{n-a}$, where $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leq j \leq n - a$, and each v_j is an element of $\bigcup_{i=1}^a V_i$. Since each S_j induces a star we have that $f(G) \leq n - a = n - \max \{b: \sum_{i=1}^b p_i \leq n - b\}$.

Case (ii) Assume $\sum_{i=1}^a p_i > n - a$. Since $\sum_{i=1}^{a-1} p_i < n - a + 1$, the number of elements in $\bigcup_{i=1}^{a-1} V_i$ is less than the number of sets in the collection $\{V_a, V_{a+1}, \dots, V_n\}$. We form $r = \sum_{i=1}^{a-1} p_i$ mutually disjoint subsets $S_1, S_2, \dots, S_r$ of $V(G)$, with $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leq j \leq r$, and where each v_j is an element of $\bigcup_{i=1}^{a-1} V_i$. Next, form mutually disjoint point sets $S_{r+1}, \dots, S_{n-a}$ where, for $k = r+1, \dots, n - a$, $S_k = V_{n+1-k} \cup \{v_k\}$ and the v_k are distinct elements of V_a . Since $\sum_{i=1}^a p_i > n - a$, we have some points of V_a which are not in any S_j , $j = 1, \dots, n - a$. Call this set of points S_{n-a+1} . The sets $S_1, \dots, S_{n-a}$ each induce a star and the set S_{n-a+1} induces a totally disconnected graph. It follows that $f(G) \leq n - a + 1 = n - \max \{b: \sum_{i=1}^b p_i \leq n - b\}$.

In each of the aforementioned cases denote the upper bound by z and suppose $f(G) = t < z$. Then $V(G)$ has an outerplanar partition $T_1, T_2, \dots, T_t$ where $|T_i| \geq |T_{i+1}|$. Let h be the largest integer such that $|T_h| > |S_h|$.

Then

$$|\bigcup_1^h T_i| - h > |\bigcup_1^h S_i| - h.$$

From the formulation of the various S_i it follows that the cardinality of S_h is at least four. For $i < h$, $|T_i| \geq |T_h| > |S_h| \geq 4$. Remark 3 implies that each T_i , $i \leq h$, has all but at most one point from a single partite set. If such a point exists for a given T_i , denote it by w_i . Then, for $i \leq h$, define $T'_i = T_i - \{w_i\}$ for all i for which w_i exists and $T'_i = T_i$, otherwise. This implies that the set $\bigcup_1^h T'_i$ has all of its points in h or fewer partite sets. However,

$$|\bigcup_{n-h+1}^n V_i| = |\bigcup_1^h S_i| - h.$$

Thus the union of any h partite sets has at most $|\bigcup_1^h S_i| - h$ points, but

$$|\bigcup_1^h S_i| - h < |\bigcup_1^h T_i| - h \leq |\bigcup_1^h T'_i|$$

implies that $\bigcup_1^h T'_i$ cannot have all of its points in h or fewer partite sets. We have a contradiction and $f(G) = z$ in both cases.

THEOREM 2: *If $p_{a+1} \leq 2$, then $V(G)$ can be partitioned into outerplanar sets $S_1, S_2, \dots, S_s$, where $|S_i| \geq |S_{i+1}|$.*

PROOF: We exhibit an outerplanar partition of $V(G)$ into the desired number of subsets. The inequality $r > a$ implies that $\sum_1^a p_i \geq n - a > n - r$. Thus there are more elements in the set $\bigcup_1^a V_i$ than sets in the collection $\{V_{r+1}, V_{r+2}, \dots, V_n\}$. We form $n - r$ mutually disjoint sets $S_1, S_2, \dots, S_{n-r}$ where $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leq j \leq n - r$ and $v_j \in \bigcup_1^a V_i$. Moreover, the points v_j are always selected successively from the set V_i with i minimum such that V_i has points remaining.

Each of the S_i induces a star with at least four points, and there are $\sum_1^r p_i - (n - r) > 0$ points of G not in any S_i . Each of these points is contained in a partite set of G which consists of at most two elements.

Case (i) Suppose $k + r - n \leq (2/3)(2r - n)$. If $k - (n - r)$ is positive, we have $k + r - n$ unused one-point partite sets of G . In defining the S_i we used points from at most $2(n - r)$ partite sets of G . Thus, there are at least $n - 2(n - r) = 2r - n$ partite sets of G which are disjoint from each S_i , $i = 1, \dots, n - r$. Since $k + r - n \leq (2/3)(2r - n)$, we form mutually disjoint sets $S_{n-r+1}, \dots, S_q$, each consisting of two one-point partite sets and one two-point partite set until we have at most one unused singleton partite set. All remaining partite sets have precisely two points. If $k + r - n$ is not positive, then there are only two-point partite sets of G remaining and

perhaps one more point which is an element of a two-point partite set. Thus, in either case, we have two-point partite sets remaining, and possibly one extra point. With the remaining points, we may form mutually disjoint sets which consist of the unit of two of the remaining two-point partite sets until there are at most three points remaining. These points form an outerplanar set. Thus, we have partitioned $V(G)$ into

$$\{(\sum_1^r p_i + 3(n-r))/4\} = s$$

outerplanar sets, each of which, with at most one exception, has at least four points.

Case (ii) Suppose $k+r-n > (2/3)(2r-n)$. In this case, $2r-n$ is non-negative, and thus $k+r-n$, the number of unused singleton partite sets, is positive. This implies that for $1 \leq i \leq n-r$, $S_i = V_i \cup V_{n+1-i}$, and we have precisely $2r-n$ unused partite sets of G . In this case there are more than twice as many unused partite sets with one point as unused partite sets with two points. It follows that we can form disjoint sets $S_{n-r+1}, \dots, S_{n-k}$ in such a way that each set consists of four points from three of the remaining partite sets. When this is done, there are $3k-r-n$ points remaining in G . These points induce a complete subgraph and have an outerplanar partition into $\{(3k-r-n)/3\}$ sets. Let the sets in this partition be denoted by $S_{n-k+1}, \dots, S_s$, $s = n-k + \{(3k-r-n)/3\} = \{(2n-r)/3\}$.

THEOREM 3: *Let $p_{a+1} \leq 2$ and suppose that $V(G)$ has an outerplanar partition $T_1, \dots, T_t$ where $|T_i| \geq |T_{i+1}|$ and $t < s$. Then there exists a largest positive integer h such that $|T_h| > |S_h|$, and furthermore $|T_h| = 4$. Also if $m = \max \{i: p_i \leq 3\}$, then the T_i can be reordered if necessary so that T_h does not contain V_i , $m+1 \leq i \leq n$.*

PROOF: Since all but perhaps one of the S_i has at least three points, it follows that $|T_h| \geq 4$. In order to verify the first part of the theorem we assume that $|T_h| > 4$ and obtain a contradiction. Since $|T_h| > |S_h|$, we have

$$|\bigcup_{h+1}^t T_i| < |\bigcup_{h+1}^s S_i|,$$

which implies that

$$|\bigcup_1^h T_i| - h > |\bigcup_1^h S_i| - h.$$

For $i \leq h$, T_i has five or more points and Remark 3 implies that each such T_i has all but possibly one point from a single partite set. Define T'_i , $1 \leq i \leq h$ as in Theorem 1. Then the set $\bigcup_1^h T'_i$ has all of its points in h or fewer partite sets. We now consider two cases depending upon h .

Case (i) $h \leq n-r$. From the fact that each S_i , $1 \leq i \leq n-r$, consists of V_{n-i+1} together with one other point it follows that

$$\left| \bigcup_{n-h+1}^n V_i \right| = \left| \bigcup_1^h S_i \right| - h.$$

Hence, the union of any h partite sets has at most $|\bigcup_1^h S_i| - h$ points. However,

$$\left| \bigcup_1^h S_i \right| - h < \left| \bigcup_1^h T_i \right| - h \leq \left| \bigcup_1^h T'_i \right|.$$

Thus, $|\bigcup_1^h T'_i|$ cannot have all of its points in h or fewer partite sets, a contradiction.

Case (ii) $h > n-r$. The sets $S_1, \dots, S_{n-r}$ exhaust all partite sets with three or more points. Since h is necessarily less than s , the sets $S_{n-r+1}, \dots, S_h$ each use partite sets with one or two points. Without loss of generality, we may assume that these are the partite sets $V_{n+1-(n-r+1)}, \dots, V_{n+1-h}$. This implies that

$$\left| \bigcup_{n-h+1}^n V_i \right| < \left| \bigcup_1^h S_i \right| - h.$$

The union of any h partite sets has at most $|\bigcup_{n-h+1}^n V_i|$ points. However, the fact that

$$\left| \bigcup_{n-h+1}^n V_i \right| < \left| \bigcup_1^h S_i \right| - h < \left| \bigcup_1^h T_i \right| - h \leq \left| \bigcup_1^h T'_i \right|$$

is again a contradiction. Thus $|T_h| = 4$.

For the second part of the Theorem we reorder the T_i , $1 \leq i \leq t$, so that, if $|T_i| = |T_j|$ and T_i has more points from some partite set than T_j has from any partite set, then $i < j$.

We now suppose there exists V_{i_1} , $m < i_1 \leq n$, which is contained in T_h and obtain a contradiction. Since $|T_h| = 4$ and $|V_{i_1}| \geq 4$, we know that $T_h = V_{i_1}$. From our ordering on the partition $T_1, \dots, T_t$, it follows that the sets $T_1, \dots, T_h$ have at most $h-1$ points from one-point partite sets of G . The sets $T_{h+1}, \dots, T_t$ have at most $|\bigcup_{h+1}^t T_i|$ points from one-point partite sets of G . The partition $T_1, \dots, T_t$ uses all one-point partite sets of G , and the number used must be not more than $h-1 + |\bigcup_{h+1}^t T_i|$. Thus,

$$(1) \quad h-1 + \left| \bigcup_{h+1}^t T_i \right| \geq k.$$

The set S_h is the union of three one-point partite sets of G , and thus the sets $S_{h+1}, \dots, S_s$ each consist of only points from one-point partite sets; that is, the sets $S_{h+1}, \dots, S_s$ contain $|\bigcup_{h+1}^s S_i|$ points from one-point

partite sets. However, each of the sets $S_1, \dots, S_h$ contains at least one point from a one-point partite set. Thus, the partition $S_1, \dots, S_s$ contains at least $h + |\bigcup_{h+1}^s S_i|$ points from one-point partite sets. It follows that

$$(2) \quad k \geq h + \left| \bigcup_{h+1}^s S_i \right|.$$

The fact that $|\bigcup_{h+1}^s S_i| > |\bigcup_{h+1}^t T_i|$, together with (1) and (2), yields a contradiction and completes the proof of Theorem 3.

THEOREM 4: *If $p_{a+1} \leq 2$, then $f(G) = s$.*

PROOF: Suppose that $V(G)$ has an outerplanar partition $T_1, T_2, \dots, T_t$, $t < s$, with $|T_i| \geq |T_{i+1}|$. Then the set T_h as given in Theorem 3 has cardinality 4. If $(k+r-n) \leq (2/3)(2r-n)$, then by the construction in Theorem 2, $4 \leq |S_h| < |T_h| = 4$. Since this is impossible we need only consider $(k+r-n) > (2/3)(2r-n)$.

Among the outerplanar partitions of $V(G)$ into t sets, select one which has a maximum number, say M , of V_i , $m < i \leq n$, with the property that each is contained in some set of the partition. Call this partition $T_1, \dots, T_t$, and order the sets as in the second part of Theorem 3. According to Theorem 3, $|T_h| = 4$. Again let $m = \max \{i: p_i \leq 3\}$ and consider two cases.

Case (i) Each of the sets $V_{m+1}, \dots, V_n$ is contained in some T_i . We may assume, without loss of generality, that $V_i \subset T_{n+1-i}$, for $i = m+1, \dots, n$. From the facts that, for $1 \leq i \leq n-k$, $S_i = V_{n+1-i} \cup W_i$ where W_i consists of one or two points and S_h consists of three points from three different partite sets, we have that

$$(1) \quad h > n-k.$$

The sets $T_{n-m+1}, T_{n-m+2}, \dots, T_h$ each have at least four points and therefore at least two points from one partite set. However, all partite sets with at least four points are used in sets $T_1, \dots, T_{n-m}$. Thus, we need $h - (n-m+1) + 1$ partite sets with two or three points, and there are only $m-k$ such partite sets. Hence, using inequality (1) we have a contradiction.

Case (ii) At least one of the partite sets with four or more points, say V_{i_0} , has points in two or more of the sets T_i .

If V_{i_0} has at least three points in one T_j , say T_b , we add all other points of V_{i_0} to T_b . We now have an outerplanar partition of $V(G)$ into t sets such that $M+1$ partite sets with at least four points are contained in various T_j . This is a contradiction.

If V_{i_0} has exactly two points in some T_i , say T_b , then V_{i_0} has one or two points in T_c , $c \neq b$. We add the points of $T_c \cap V_{i_0}$ to T_b and add one point of $T_b - V_{i_0}$ (if such a point exists) to T_c . We have an outerplanar partition

of $V(G)$ into t sets such that V_{i_0} has three or more points in one set, and M partite sets V_i , $m < i \leq n$, are each contained in some T_j . According to the previous paragraph, this leads to a contradiction.

We now suppose that V_{i_0} has each point in a different T_j . Then T_h has at most one point of V_{i_0} . Let w_1, w_2 , and w_3 be points in $T_h - V_{i_0}$. Add all points of V_{i_0} to T_h . Since V_{i_0} has at least four points, three of these points must be in distinct T_j different from T_h , say T_{i_1}, T_{i_2} , and T_{i_3} . For $k = 1, 2, 3$, insert w_k into T_{i_k} . As before, this yields a new outerplanar partition of $V(G)$ into t sets. By the second part of Theorem 3, T_h did not contain any partite sets with four or more points, and hence this new partition has $M + 1$ sets, each of which contains a V_i , $m < i \leq n$. This is a contradiction and we have shown that $f(G) = s$.

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