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ISAAC CHAVEL
EDGAR A. FELDMAN

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THE FIRST EIGENVALUE OF THE LAPLACIAN ON MANIFOLDS OF NON-NEGATIVE CURVATURE

Isaac Chavel and Edgar A. Feldman [1]

Our purpose in this paper is to give upper bounds for the first non-zero eigenvalue (henceforth denoted by λ_1) of the Laplace-Beltrami operator on compact Riemannian manifolds without boundary as consequences of the non-negativity of the Ricci curvature of the given Riemannian metric. We give two separate results, which are obtained by the same method, modeled on the phenomena of zonal harmonics and their nodal sets on spheres of constant curvature. We first state our results:

THEOREM (1): *Let M be a Riemannian manifold diffeomorphic to the real projective plane, P^2, and assume the Gauss curvature, $K(p)$, $p \in M$, satisfies*

$$0 \leqq K(p) \leqq \kappa \tag{1}$$

for all $p \in M$. Set $\rho = \arcsin 1/\sqrt{3} \in [0, \pi/2]$. Then

$$\lambda_1 \leqq \pi^2\kappa/4\rho^2 < 3\pi^2\kappa/4. \tag{2}$$

THEOREM (2): *Let M be a compact Riemannian manifold without boundary of dimension $n \geqq 2$, with non-negative Ricci curvature on M. Let $d > 0$ be a fixed positive number, and assume there exist points p_-, $p_+ \in M$ with distance to one another $\geqq d$ and each having injectivity radius $> d/2$. Then*

$$\lambda_1 \leqq 4(j_{n/2-1})^2/d^2 \tag{3}$$

where $j_{n/2-1}$ is the first zero of the $n/2-1$st Bessel function.

By W. Klingenberg's result [11, Theorem 1(b)] we immediately obtain

COROLLARY (3): *Let M be a compact Riemannian manifold without boundary of even dimension $\geqq 2$. Assume that for every 2-section, σ, tangent to M the Riemannian sectional curvature of σ, $K(\sigma)$, satisfies*

$$0 < K(\sigma) \leqq \kappa; \tag{4}$$

then

$$\lambda_1 \leqq 4\kappa(j_{n/2-1})^2/\pi^2. \tag{5}$$

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We first note that if M is a Riemannian manifold diffeomorphic to the 2-sphere, S^2, then J. Hersch [10] has proven, via conformal mapping and the standard imbedding of S^2 in $\mathbb{R}^3$, Euclidean 3-space, that

$$\lambda_1 \leqq 8\pi/v_g \tag{6}$$

where v_g is the volume of M relative to Riemann metric g. Equality is obtained in (6) if and only if the Riemannian metric on M has constant curvature. Therefore, if we assume that the Gaussian curvature, $K(p)$, on M satisfies

$$K(p) \leqq \kappa$$

for all $p \in M$, then we obtain immediately by the Gauss-Bonnet theorem

$$\lambda_1 \leqq 2\kappa, \tag{7}$$

with equality if and only if M has constant curvature κ. Our methods also yield (7) for M diffeomorphic to S^2 but *only* under the assumption (1) where we *also* require the bound from below. For information on Hersch's method in higher dimensions we refer the reader to [2].

We do not believe the inequalities (2), (3), (5), are sharp; indeed Hersch's result suggests that 6κ is the sharp upper bound in (2) without assuming the curvature is bounded from below, and that $\lambda_1 = 6\kappa$ would at the same time characterize the real projective plane of constant curvature κ. We remark that if the Ricci curvature of M is bounded from below by a positive constant then one does have the appropriate sharp inequality to estimate λ_1 from below. The result was obtained by A. Lichnerowicz, and M. Obata, and one can find a complete exposition with references in [3, 179–185].

We next note that (cf. [14, 486])

$$(\mathrm{j}_{n/2-1})^2 < \mathrm{n}(\mathrm{n}/2+2)$$

which implies in Theorem 2 that we have

$$\lambda_1 < 4\mathrm{n}(\mathrm{n}/2+2)/\mathrm{d}^2.$$

If we set d to be the diameter of M this invites comparison to Cheeger's result [6]

$$\lambda_1 \leqq 16(n+1)^2(n+2)/d^2$$

where only the non-negativity of the Ricci curvature is assumed.

We now turn to our results: M will henceforth be a compact, C^∞, manifold of dimension $n \geqq 2$, without boundary, and with Riemannian metric g and its associated Laplace-Beltrami operator Δ acting on C^∞ functions on M, $C^\infty(M)$. A number $\lambda \in \mathbb{R}$ is called an *eigenvalue* of Δ if there exists a function $f \in C^\infty(M)$, not identically zero, satisfying $\Delta f = \lambda f$ on all of M. It is known that there exists a sequence of eigenvalues

$0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. It is also known that λ_1 is the solution of the following problem in the Calculus of Variations [3, 186]:

Let $\mathrm{d}v_g$ denote the Riemannian measure on M associated with g and $\mathscr{H}$ the set of continuous functions $f: M \to \mathbb{R}$ on M with square-summable weak first derivatives (i.e., we let $<, >$ denote the pointwise inner product of tensors on M, $\| \;\|$ the induced norm, relative to g and assume $\|\mathrm{d}f\|^2$ is summable relative to the measure $\mathrm{d}v_g$). Set $\mathscr{H}' = \{f \in \mathscr{H}: \int_M f\,\mathrm{d}v_g = 0\}$. Then

$$\lambda_1 = \inf_{\substack{f \in \mathscr{H}' \\ f \neq 0}} \frac{\int_M \|\mathrm{d}f\|^2\,\mathrm{d}v_g}{\int_M f^2\,\mathrm{d}v_g}. \tag{8}$$

Our method is to first write $M = M^+ \cup M^- \cup M^*$ where M^+, M^-, M^* are pairwise disjoint and M^+, M^- are open with reasonable boundaries. In M^+, M^- we introduce an $n-1$ field of geodesics via geodesic polar or parallel coordinates and consider subspaces $\mathscr{H}^+$ (resp. $\mathscr{H}^-$) of functions f in $\mathscr{H}$ with support of $\mathrm{d}f$ in M^+ (resp. M^-) and support of f in the complement of M^- (resp. M^+). Furthermore the restriction of f to M^+ (resp. M^-) will only be a function of arc length along the geodesics of the field and will not depend on the geodesic. Finally at the boundaries of M^+ (resp. M^-) f will either vanish or have vanishing normal derivative. We then let

$$\lambda_+ = \inf_{\substack{f \in \mathscr{H}^+ \\ f \neq 0}} \frac{\int_{M^+} \|\mathrm{d}f\|^2\,\mathrm{d}v_g}{\int_{M^+} f^2\,\mathrm{d}v_g}$$

$$\lambda_- = \inf_{\substack{f \in \mathscr{H}^- \\ f \neq 0}} \frac{\int_{M^-} \|\mathrm{d}f\|^2\,\mathrm{d}v_g}{\int_{M^-} f^2\,\mathrm{d}v_g}$$

and assume $f_+ \in \mathscr{H}^+$, $f_- \in \mathscr{H}^-$, neither identically zero, which satisfy

$$\lambda_+ = \int_{M^+} \|\mathrm{d}f_+\|^2\,\mathrm{d}v_g \Big/ \int_{M^+} f_+^2\,\mathrm{d}v_g$$

$$\lambda_- = \int_{M^-} \|\mathrm{d}f_-\|^2\,\mathrm{d}v_g \Big/ \int_{M^-} f_-^2\,\mathrm{d}v_g;$$

one can invoke the appropriate existence theory, or adapt the following argument to minimizing sequences. Then there exist numbers $\alpha_+, \alpha_- \in \mathbb{R}$ such that the function $F = \alpha_+ f_+ + \alpha_- f_- \in \mathscr{H}'$. By (10) we have

$$\lambda_1 \leqq \int_M \|dF\|^2 dv_g \Big/ \int_M F^2 dv_g$$

$$\leqq \frac{\lambda_+ \alpha_+^2 \int_{M^+} f_+^2 dv_g + \lambda_- \alpha_-^2 \int_{M^-} f_-^2 dv_g}{\alpha_+^2 \int_{M^+} f_+^2 dv_g + \alpha_-^2 \int_{M^-} f_-^2 dv_g}$$

$$= \text{convex linear combination of } \lambda_+, \lambda_-.$$

Thus our problem is to estimate λ_+, λ_- from above. But these lead to a one-dimensional Sturm-Liouville problem on a finite interval with boundary conditions: the function vanishes at one end of the interval and has zero derivative at the other. Our tool here is W. T. Reid's comparison theorm [13] which we quote for future use.

THEOREM (*W. T. Reid*): Let

$$Lu = (\Phi u')' + \lambda \Phi u$$

$$Mv = (\Psi v')' + \mu \Psi v$$

be two ordinary differential operators on $[\alpha, \beta]$, where Φ and Ψ are C^∞ on $[\alpha, \beta]$ and both positive on (α, β). Let u and v be respective smooth solutions of $Lu = 0$, $Mv = 0$ satisfying the boundary conditions

$$u(\alpha) = v(\alpha) = u'(\beta) = v'(\beta) = 0,$$

and furthermore assume that neither u nor v vanish on (α, β). If $(\Phi/\Psi)' \geqq 0$ on all of (α, β) then $\lambda \leqq \mu$, with equality if and only if $\Phi/\Psi \equiv$ const on (α, β). If the boundary conditions are replaced with

$$u'(\alpha) = v'(\alpha) = u(\beta) = v(\beta) = 0,$$

then we have that $(\Phi/\Psi)' \leqq 0$ on (α, β) implies $\lambda \leqq \mu$, with equality if and only if $\Phi/\Psi \equiv$ const. on (α, β).

We now collect for the reader notations used in the sequel: For $p \in M$ we let M_p denote the tangent space to M at p, and TM the tangent bundle of M (i.e., $\bigcup_{p \in M} M_p$ with standard differentiable structure). For any differentiable mapping of manifolds $f: M \to N$ we let $f_*: TM \to TN$ denote the differential of f – in particular, for each $p \in M$, f_* maps M_p linearly into$N_{f(p)}$.

As mentioned earlier, the pointwise inner products on tensor spaces induced by the metric tensor g will be denoted by $<, >$ and the induced norm by $\| \ \|$. For any continuous piecewise differentiable path $\omega: [\alpha, \beta] \to M$ we let ω' denote its velocity vector field, defined for all but at most a finite number of points, and $l(\omega) = \int_\alpha^\beta \|\omega'(t)\| \, \mathrm{d}t$ the length of ω. Also for points $p, q \in M$, $\mathrm{d}(p, q)$ will denote the distance from p to q induced by g; and for $p \in M$, $E \subseteq M$, $\mathrm{d}(p, E)$ will denote the induced distance of the point p to the set E.

It will be convenient to *set* $\kappa = 1$ for the remainder of the paper.

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1. Proof of Theorem 1

Here we construct M^+, M^-, and M^* as follows: Let $\gamma: [0, 1] \to M$ denote the simple closed geodesic minimizing the length of all paths $\sigma: [0, 1] \to M$ in the non-trivial free homotopy class of continuous sectionally smooth closed paths in M (for the existence of γ, cf: [1, 243]). For given $\varepsilon > 0$ let $\gamma_\varepsilon = \{q \in M : \mathrm{d}(q, \gamma) = \varepsilon\}$ and set $\omega = \gamma_\rho$ where, as in the statement of Theorem 1, $\rho = \arcsin 1/\sqrt{3} \in [0, \pi/2]$. We then set

$$M^- = \bigcup_{0 \leqq \varepsilon < \rho} \gamma_\varepsilon$$

$$M^+ = \bigcup_{\rho < \varepsilon < \pi/2} \gamma_\varepsilon$$

and M^* to be the complement of $M^- \cup M^+$.

Now N. Grossman's estimate [8] of the distance from a totally geodesic submanifold to its focal cut locus on compact manifolds of *strictly positive* sectional curvature can in the case of *surfaces* be proven valid if we instead assume only (1). Thus the exponential mapping of the Cartesian product of the unit normal bundle of γ with $(0, \pi/2)$ is a diffeomorphism onto $(M^- \cup \{\gamma\}) \cup \{\omega\} \cup M^+$. One easily sees that M^- is homeomorphic to a Mobius band and that ω is null-homotopic. Let ω be given by $\omega: [0, 1] \to M$, $\|\omega'(t)\| = l(\omega)$, for all t.

We now introduce geodesic coordinates in $M^- \cup \{\omega\} \cup M^+$. Let $\xi(t)$ be the unit vector field along ω perpendicular to $\omega'(t)$ and pointing into M^+ for all $t \in [0, 1]$. Define $v: [0, 1] \times [0, \pi/2] \to M$ by

$$v(s, t) = \operatorname{Exp} (s - \rho)\xi(t)$$

and set

$$\varphi_t(s) = \|v_*(\partial/\partial t)\|$$

$$\Phi(s) = \frac{1}{2l(\gamma)} \int_0^1 \varphi_t(s) \, \mathrm{d}t$$

Then $\Phi(0) = 1$ since γ is in the non-trivial homotopy class of M and is covered twice as $s \to 0$. To define the subspaces of real valued functions on$M$, $\mathscr{H}^-$ and $\mathscr{H}^+$, we first define

$$H^- = \{f\colon [0, \rho] \to \mathbb{R} \in C^1\colon f'(0) = f(\rho) = 0\}$$

and

$$H^+ = \{f\colon [\rho, \pi/2] \to \mathbb{R} \in C^1\colon f(\rho) = f'(\pi/2) = 0\}.$$

We define $\mathscr{H}^-$ by the following property: To each $F \in \mathscr{H}^-$ there exists a unique $f \in H^-$ such that

$$F(p) = \begin{cases} f(s) & p \in M^-,\ p = v(s, t) \\ 0 & p \in M - M^- \end{cases}$$

Similarly $\mathscr{H}^+$ will be defined by: To each $F \in \mathscr{H}^+$ there exists a unique $f \in H^+$ such that

$$F(p) = \begin{cases} 0 & p \in \overline{M^-} \\ f(s) & p \in M^+ \ \ p = v(s, t) \\ f(\pi/2) & p \in M^* - \{\omega\} \end{cases}$$

One now easily shows

$$\lambda_- = \inf_{\substack{f \in H^- \\ f \neq 0}} \frac{\displaystyle\int_0^\rho (f'(s))^2 \Phi(s)\, ds}{\displaystyle\int_0^\rho (f(s))^2 \Phi(s)\, ds}$$

$$\lambda_+ = \inf_{\substack{f \in H^+ \\ f \neq 0}} \frac{\displaystyle\int_\rho^{\pi/2} (f'(s))^2 \Phi(s) ds}{\displaystyle\int_\rho^{\pi/2} (f(s))^2 \Phi(s)\, ds}.$$

We set $\Psi(s) = \cos s$ and note that Sturmian arguments imply $(\Phi/\Psi)'(s) \geqq 0$ for all $s \in [0, \pi/2)$ which implies $\Phi(s) > 0$ for all $s \in [0, \pi/2)$. If $\Phi(\pi/2) = 0$ then we have that M is a projective plane of constant curvature 1 – so we assume $\Phi(\pi/2) > 0$. Then there exists a function $u\colon [\rho, \pi/2] \to \mathbb{R} \in C^2$ such that

$$\lambda_+ = \int_\rho^{\pi/2} (u'(s))^2 \Phi(s)\, ds \Big/ \int_\rho^{\pi/2} (u(s))^2 \Phi(s)\, ds$$

and u satisfies the Euler-Lagrange equation

$$(\Phi u')' + \lambda_+ \Phi u = 0$$

on $[\rho, \pi/2]$ with boundary conditions $u(\rho) = u'(\pi/2) = 0$. Also $u(s) \neq 0$ for all $s \in (\rho, \pi/2)$. For $\Psi(s)$ we have $\Psi(s) = \cos s$, and pick $v(s) = 3\sin^2 s - 1$. Then

$$(\Psi v')' + 6\Psi v = 0$$

on $[\rho, \pi/2]$ with boundary conditions $v(\rho) = v'(\pi/2) = 0$. Reid's theorem then implies $\lambda_+ \leqq 6$.

To estimate λ_- we have $u: [0, \rho] \to \mathbb{R} \in C^2$ satisfying the Euler-Lagrange equation

$$(\Phi u')' + \lambda_- \Phi u = 0$$

with boundary conditions $u'(0) = u(\rho) = 0$. Again, $u(s) \neq 0$ for all $s \in (0, \rho)$. Here we have, however, $(\Phi(s)/\cos s)' \geqq 0$ but need the opposite inequality for the given boundary conditions. So we set $\Psi(s) = 1$ for all $\Psi(s) = 1$ for all $s \in [0, \rho]$, an k $v(s) = \cos(s\pi/2\rho)^d$ Then

$$(\Psi v')' + (\pi^2/4\rho^2)\Psi v = 0$$

on $[0, \rho]$ with boundary conditions $v'(0) = v(\rho) = 0$. Also (1) implies $(\Phi/\Psi)'(s) \leqq 0$ for all $s \in [0, \rho]$, and from Reid's theorem we conclude $\lambda_- \leqq \pi^2/4\rho^2$. Thus λ_1 is less than or equal to a convex combination of 6 and $\pi^2/4\rho^2$ which implies Theorem 1.

2. Proof of Theorem 2

By hypotheses we have points $p_-, p_+ \in M$ with distance $d(p_-, p_+) \geqq d$ and such that the injectivity radius of p_+ is $> d/2$, and the injectivity radius of p_- is also $> d/2$. We set

$$\begin{aligned} M^- &= \{q \in M: d(q, p_-) < d/2\} \\ M^+ &= \{q \in M: d(q, p_+) < d/2\} \\ M^* &= M - (M^- \cup M^+) \\ B^- &= \{x \in M_{p_-}: \|x\| < d/2\} \\ B^+ &= \{x \in M_{p_+}: \|x\| < d/2\}. \end{aligned}$$

Also we set, for any $s \in [0, d/2]$

$$\begin{aligned} \mathscr{S}^-(s) &= \{q \in M: d(q, p_-) = s\} \\ \mathscr{S}^+(s) &= \{q \in M: d(q, p_+) = s\}. \end{aligned}$$

By hypothesis, Exp maps B^- diffeomorphically onto M^- and B^+ diffeomorphically onto M^+. Set $H = \{f: [0, d/2] \to \mathbb{R} \in C^1: f'(0) = f(d/2)\}$

$= 0\}$. The set $\mathscr{H}^-$ of functions $F: M \to \mathbb{R}$ is defined by: To each $F \in \mathscr{H}^-$ there exists a unique $f \in H$ such that

$$F(p) = \begin{cases} f(s) & p \in \mathscr{S}^-(s),\ 0 \leqq s \leqq d/2 \\ 0 & p \in M - M^-. \end{cases}$$

Similarly $\mathscr{H}^+$ is the set of real-valued functions on M defined by: To each $F \in \mathscr{H}^+$ there exists a unique $f \in H$ such that

$$F(p) = \begin{cases} 0 & p \in M - M^+ \\ f(s) & p \in \mathscr{S}^+(s),\ 0 \leqq s \leqq d/2. \end{cases}$$

Finally, let ω_{n-1} denote the volume of the standard unit $(n-1)$-sphere and define $\Phi^-, \Phi^+ : [0, d/2] \to \mathbb{R}$ by

$$\Phi^-(s) = (1/\omega_{n-1}) \operatorname{vol} \mathscr{S}^-(s)$$

$$\Phi^+(s) = (1/\omega_{n-1}) \operatorname{vol} \mathscr{S}^+(s)$$

where $\operatorname{vol} \mathscr{S}^-(s)$ is the $(n-1)$-dimensional volume of the submanifold $\mathscr{S}^-(s)$, and similarly for $\operatorname{vol} \mathscr{S}^+(s)$. Then

$$\lambda_- = \inf_{\substack{f \in H \\ f \not\equiv 0}} \frac{\int_0^{d/2} (f'(s))^2 \Phi^-(s) \, ds}{\int_0^{d/2} (f(s))^2 \Phi^-(s) \, ds}$$

$$\lambda_+ = \inf_{\substack{f \in H \\ f \not\equiv 0}} \frac{\int_0^{d/2} (f'(s))^2 \Phi^+(s) \, ds}{\int_0^{d/2} (f(s))^2 \Phi^+(s) \, ds}.$$

We work with λ_+. The proof for λ_- is exactly the same. Set $\Psi(s) = s^{n-1}$; then $(\Phi^+/\Psi)(s) \to 1$ as $s \to 0$, and by Rauch comparison arguments [5, p. 253], the non-negativity of the Ricci curvature implies $(\Phi^+/\Psi)'(s) \geqq 0$ for all $s \in [0, d/2]$.

Assume $u: [0, d/2] \to \mathbb{R}$, $v: [0, d/2] \to \mathbb{R}$ are functions in C^2 which satisfy

(9) $$(\Phi^+ u')' + \lambda_+ \Phi^+ u = 0$$

(10) $$(\Psi v')' + \mu \Psi v = 0$$

on $[0, d/2]$, with boundary condition $u'(0) = v'(0) = u(d/2) = v(d/2) = 0$. Also assume that neither u nor v vanish on $(0, d/2)$. Then to prove Theorem 2 it remains to discuss the existence of the functions u and v, and to identify the number μ.

Let $J_{n/2-1}(s)$ be the $n/2-1$st Bessel function, i.e., let $J_{n/2-1}(s)$ satisfy

$$s^2J''+sJ'+\left(s^2-\left(\frac{n}{2}-1\right)^2\right)J = 0.$$

Then the function $v_0(s) = s^{1-n/2}J_{n/2-1}(s)$ has a power series expansion and satisfies $v_0'(0) = 0$. Furthermore, v_0 satisfies the differential equation

$$sv_0''+(n-1)v_0'+sv_0 = 0.$$

Let $j_{n/2-1}$ be the first zero of $J_{n/2-1}$; it is also the first zero of v_0. If we set

$$v(s) = v_0(2j_{n/2-1}\, s/d)$$

then v satisfies (12) with $\mu = 4(j_{n/2-1})^2/d^2$. Also, $v'(0) = v(d/2) = 0$ and v does not vanish on $(0, d/2)$. It thus remains to consider u.

The existence question of u in H satisfying

$$\lambda_+ = \frac{\displaystyle\int_0^{d/2}(u'(s))^2\Phi^+(s)\,\mathrm{d}s}{\displaystyle\int_0^{d/2}(u(s))^2\Phi^+(s)\,\mathrm{d}s} \tag{11}$$

requires attention because of the singularity $\Phi(0) = 0$. One can remedy this by a more delicate 'Green's function' argument or can give an argument more in the geometric spirit of our approach. We present the second.

Let D be the closed disk in $\mathbb{R}^n$ of radius $d/2$ and (s, Θ) denote polar coordinates in D where $s \geqq 0$, $\Theta \in S^{n-1}$ the unit $n-1$-sphere. A Riemannian metric h on D is then uniquely determined by the following: At the orifin of D, h will be the Euclidean metric, and for each $\Theta \in S^{n-1}$, the ray $\gamma(s) = s\Theta$ is a geodesic. For any unit vector ζ orthogonal to Θ the vector field $Y(s) = \{\Phi^+(s)\}^{1/n-1}\zeta$ is declared to be a Jacobi field along γ orthogonal to γ with length $(\Phi^+(s))^{1/n-1}$ for each s. Then h is a smooth Riemannian metric in D. If $\square$ is the Laplacian of h acting on functions on D which vanish on the boundary of D, and $\mathscr{H}$ is the set of functions $f\colon D \to \mathbb{R}$ which are continuous on D with square-summable weak first derivatives and which vanish on the boundary of D, then the first non-zero eigenvalue of $\square$, σ_1, on D [4, Part II, Chapter 4] satisfies

$$\sigma_1 = \inf_{\substack{F\in\mathscr{H}\\ F\neq 0}} \frac{\displaystyle\int_D \|\mathrm{d}F\|_h^2\,\mathrm{d}v_h}{\displaystyle\int_D F^2\,\mathrm{d}v_h}$$

where $\|dF\|_h$ is the norm induced by h and dv_h the induced Riemannian measure. Set

$$H = \{f: [0, d/2] \to \mathbb{R} \in C^1 : f'(0) = f(d/2) = 0\}$$

and $\mathscr{H}_0$ to be the set of functions $F: D \to \mathbb{R}$ in $\mathscr{H}$ of the form $F(s, \Theta) = f(s)$ where $f \in H$. Then the rotational symmetry of the metric h implies that the eigenspace of σ_1 contains a function $U \in \mathscr{H}_0$ of the form $U(s, \Theta) = u(s)$, which implies

$$\sigma_1 = \inf_{\substack{F \in \mathscr{H}_0 \\ f \neq 0}} \frac{\int_D \|dF\|^2 \, dv_h}{\int_D F^2 \, dv_h}$$

$$= \inf_{\substack{f \in H \\ f \neq 0}} \frac{\int_0^{d/2} (f'(s))^2 \Phi^+(s) \, ds}{\int_0^{d/2} (f(s))^2 \Phi^+(s) \, ds}$$

$$= \lambda_+ .$$

Thus u satisfies (11) and hence (9) which implies Theorem 2.

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The Graduate School
and University Center
of the Cith University of New York,
Ph.D. Program in Mathematics,
Graduate Center: 33 West 42 Street,
New York, N.Y. 10036, U.S.A.
The City College of New York,
C.U.N.Y.
Hunter College C.U.N.Y.