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INVERSE LIMITS OF SIMPLICIAL COMPLEXES

M. D. Alder

Summary

Eilenberg and Steenrod have shown that any compact space may be expressed as an inverse limit in $\mathcal{Top}$, the category of topological spaces, of a diagram of simplicial complexes. [1]

We show that any paracompact space may be expressed as a limit of a diagram of nerves; further that $[-, K] \circ \mathcal{H} : \mathcal{Top} \rightarrow \mathcal{Ens}$ preserves such limits when K is a complex, where $[-, K]$ denotes the contravariant hom-functor from $\mathcal{Htp}$ (the category of spaces and homotopy classes of maps) to $\mathcal{Ens}$ the category of sets, and $\mathcal{H} : \mathcal{Top} \rightarrow \mathcal{Htp}$ is the canonical quotient functor.

1. Introduction

Let Δ denote a small category having the property that for A, B objects of Δ , there exists a C in Δ and maps $f: C \rightarrow A, g: C \rightarrow B$ respectively.

1.1 DEFINITION: A diagram in a category $\mathcal{M}$ with scheme Δ will be said to be a semi inverse diagram.

1.2. DEFINITION: $\Gamma: \mathcal{M} \rightarrow \mathcal{B}$ is a *semi-directly continuous* contravariant functor if it takes the limit of every semi inverse diagram $(\Delta, \Phi: \Delta \rightarrow \mathcal{M})$ in $\mathcal{M}$ to the colimit of $(\Delta, \Gamma \circ \Phi: \Delta \rightarrow \mathcal{B})$ in $\mathcal{B}$.

1.3 REMARK: We assume familiarity with the diagram of nerves of a space arising from locally finite partitions of unity, and with the natural map which we shall denote by $\pi_j: X \rightarrow N_j$ from X a space admitting a locally finite partition of unity having nerve N_j . We recall that if one locally finite partition of unity with nerve N_j refines another with nerve N_k , then there is a map $\alpha_{jk}: N_j \rightarrow N_k$ which homotopy commutes with the canonical maps from X .

1.4 REMARK: Our first proposition shows that when X is paracompact we may choose maps between the nerves such that X is the limit of the resulting semi-inverse diagram. This is the content of §2. In §3 we show that if K is a complex, $[-, K] \circ \mathcal{H} : \mathcal{Top} \rightarrow \mathcal{Ens}$ preserves limits of this type. Finally in §4 we conclude with remarks on some applications of these results.

2.

2.1 DEFINITION: Let $\{f_u\}_U$ and $\{g_v\}_V$ denote locally finite partitions of unity on a space X , with corresponding nerves N and M . The *product partition* has maps the non-zero functions $\{f_u \cdot g_v\}_{U \times V}$, multiplying values in $[0, 1]$ pointwise, and the *product nerve* $N \times M$ is the nerve of this cover. It is trivial to verify that the product partition is a locally finite partition of unity.

2.2 PROPOSITION: *There are projections $p_n: N \times M \rightarrow N$, $p_m: N \times M \rightarrow M$ which commute in $\mathcal{T}op$ with the canonical maps from X , $\pi_{N \times M}$, π_N and π_M .*

PROOF: Let $\{f_u\}_U$ and $\{g_v\}_V$ be the partitions of unity on X giving rise to N , M respectively. Let $\{i_u\}_U$, $\{j_v\}_V$ be the sets of vertices of N , M respectively, corresponding to f_u , g_v , and let (i_u, j_v) be the vertex of $N \times M$ corresponding to the map $f_u \cdot g_v: X \rightarrow [0, 1]$, the vertex existing only when the map is not the zero map. Define $p_N: N \times M \rightarrow N$ on vertices by $(i_u, j_v) \mapsto i_u$. Then it is clear that when any finite collection

$$(i_1, j_1), (i_2, j_2) \cdots (i_r, j_r)$$

determines a simplex of $N \times M$, then the set $i_1, i_2, \dots, i_r$ determines a simplex of N , and we may extend the vertex map linearly. Hence p_N is well defined. We show it commutes with $\pi_{N \times M}$ and π_N . Take x in X . If $\pi_N(x)$ has non-zero co-ordinates the set $f_1(x), f_2(x), \dots, f_t(x)$ in N , with respect to vertices i_1 to i_t and if $\pi_M(x)$ has non-zero co-ordinates the set of maps $g_1(x), g_2(x), \dots, g_s(x)$ in M , with respect to vertices j_1 to j_s , then $\pi_{N \times M}(x)$ in $N \times M$ is in the simplex with vertices the (i_a, j_b) for $1 \leq a \leq t, 1 \leq b \leq s$, some of which may be missing, and co-ordinates the $f_a \cdot g_b$ respectively. Now we have by linearity:

$$\begin{aligned} p_N(\pi_{N \times M}(x)) &= p_N \sum_{\substack{1 \leq a \leq t \\ 1 \leq b \leq s}} (f_a \cdot g_b)(x) \cdot (i_a, j_b) \\ &= \sum_{1 \leq a \leq t} p_N \sum_{1 \leq b \leq s} f_a(x) \cdot g_b(x) \cdot (i_a, j_b) \\ &= \sum_{1 \leq a \leq t} f_a(x) \cdot p_N \sum_{1 \leq b \leq s} g_b(x) \cdot (i_a, j_b) \\ &= \sum_{1 \leq a \leq t} f_a(x) \cdot \sum_{1 \leq b \leq s} g_b(x) \cdot i_a \\ &= \sum_{1 \leq a \leq t} f_a(x) \cdot i_a \quad \text{Since } \left(\sum_{1 \leq b \leq s} g_b(x) = 1 \right) \\ &= \pi_N(x). \end{aligned}$$

2.3 DEFINITION: Let N be a nerve arising from a partition of unity on X , and M a subdivision of N . Then π_N regarded as a map to M defines a

partition of unity on X which has as its nerve the subset of M containing simplices intersecting $\pi_N(X)$, and π_M composed with the inclusion is π_N . We shall call such a refinement a *proper refinement*, and refer to the nerve M as a *proper refinement* of N .

2.4 DEFINITION: For any space X , the *proper diagram of nerves* is the diagram containing all nerves of locally finite partitions of unity on X , and having maps either the inclusions of 2.3 or else the projections of 2.2.

2.5 REMARK: It is clear that the diagram is a semi-inverse diagram, but in general for N a proper refinement of M , and $N \times M$ the product, $i: N \rightarrow M$ the inclusion, it is not the case that $i \cdot p_N$ is equal to p_M . Hence the diagram is not an inverse system. However, we do have $i \cdot p_N$ homotopic to p_M , as is well known.

2.6 PROPOSITION: *A paracompact space X is the limit in $\mathcal{T}op$ of its proper diagram of nerves.*

PROOF: The family of natural maps $\pi_N: X \rightarrow N$ is a compatible family of maps into the diagram, and hence determines a unique map $\pi: X \rightarrow L$, where L is the limit of the diagram. We show that π is a homeomorphism.

It is immediate that π is continuous, and evident that it is injective, since any two points of X may be sent to 0 and 1 in a nerve which is a copy of the unit interval. This only requires X to be completely regular, which of course it is.

We show that π is surjective: suppose there is a y in $L - \pi X$. Since L is a completely regular space (all the nerves are, and L is a subset of their product) we take all the closed neighbourhoods of y in L and observe that their intersection is y .

The family of complements is an open cover of πX , and hence determines an open cover of X . Since X is paracompact, there is a subordinates partition of unity, and a nerve arising from it which we shall call N . We consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & L \\ & \searrow \pi_N & \downarrow m \\ & & N \end{array}$$

where m is the canonical map from the limit into the element of the diagram. By construction the diagram commutes.

Now if $y \in L$, my is in some simplex of N , say $(v_1, v_2, \dots, v_s)$, and by the continuity of m , there is some neighbourhood U of y in L such that $mU \subset \text{st } v_1$, where st denotes the open star on the vertex. If V_1 is the open

set on X corresponding to the vertex v_1 , then it follows that $\pi_N^{-1}(mU) \subset V_1$, and by commutativity that $(m \cdot \pi)^{-1}(mU) \subset V_1$, that is to say we have that

$$\pi^{-1}(m^{-1}(mU)) \subset V_1 \subset X$$

whence

$$\pi^{-1}U \subset V_1 \subset X,$$

and

$$\pi \cdot \pi^{-1}U \subset \pi V_1 \subset L.$$

By hypothesis, πV_1 is disjoint from y ; moreover, by our construction of the cover of X , we can find a W open in L and containing y such that $W \subset \pi, x^{-1}U = \emptyset$. Without loss of generality, we may take $W \subset U$. Now it follows that W is disjoint from πX in L , for if $w \in W \cap \pi X$, $\exists w' \in X : \pi w' = w$, and $w \in W \cap \pi \cdot \pi^{-1}U$, contra. We have shown therefore that if $y \in L - \pi X$, $y \in L - \pi \bar{X}$ also, hence that πX is closed in L . Now an open set W on y in L contains on y the intersection of some open sets with L

$$L \cap m_1^{-1}(W_1) \cap m_2^{-1}(W_2) \cap \cdots, m_t^{-1}(W_t)$$

for some open sets $W_1, W_2, \cdots, W_t$ in nerves $N_1, N_2, \cdots, N_t$, where the maps $m_1, m_2, \cdots, m_t$ are the canonical maps from the limit.

Form $M = N_1 \times N_2 \times \cdots \times N_t$, the nerve product. Let $p_1, p_2, \cdots, p_t$ denote the projections from M to $N_1, N_2, \cdots, N_t$ respectively. Then $p_1^{-1}(W_1) \cap p_2^{-1}(W_2) \cap \cdots \cap p_t^{-1}(W_t)$ is an open set in M , containing $m_M(y)$, and is clearly disjoint from $\pi_M X$. But in this event, we can take a subdivision of M excluding $m_M(y)$, contradicting y being an element of L . Hence there are no elements in $L - \pi X$, i.e. π is surjective.

Finally to prove that π is open, take A open in X and $a \in A$. If $A = X$ there is nothing to prove, so take $A \neq X$. Then there is a map $\alpha : X \rightarrow [0, 1]$ such that $\alpha(a) = 1$ and $\alpha^{-1}(0, 1] \subset A$. This yields a nerve I of a covering with α as canonical map, and if m_I is the map from L to this nerve, then $B = m_I^{-1}(0, 1]$ is open in L . Clearly $\pi x \in B$, and since π is onto, $B = \pi \cdot \pi^{-1}B$, but $\pi^{-1}B = (\pi \cdot m_I)^{-1}(0, 1] \subset A$, hence $B \subset \pi A$ and the result follows.

3.

3.1 PROPOSITION : *With the notation of 2.6, if K is a simplicial complex, and a paracompact space X is a limit of its proper diagram of nerves, Δ , then*

$$[- , K] \cdot \mathcal{H}(X) = \varinjlim [- , K] \cdot \mathcal{H} \Delta.$$

PROOF: For all N_δ, N_ϵ objects of Δ , and all $h_{\delta\epsilon}: N_\delta \rightarrow N_\epsilon$ the corresponding maps of Δ , we have the set

$$\{[-, K] \circ \mathcal{H}(N_\delta): N_\delta \in \Delta\}$$

i.e. the set of homotopy classes of maps, $\{[N_\delta, K]: N_\delta \in \Delta\}$. On this set we have the relation $\sim$ given by

$$[g_\delta] \sim [g_\epsilon] \text{ iff } [g_\delta] = [g_\epsilon \circ h_{\delta\epsilon}]$$

i.e. iff

$$([- , K] \circ \mathcal{H}(h_{\delta\epsilon}))[g_\epsilon] = [g_\delta]$$

Then $t < \epsilon$ set $\lim_{\leftarrow} [-, K] \circ \mathcal{H} \Delta$ is the set $\{[-, K] \circ \mathcal{H}(N_\delta): N_\delta \in \Delta\}$ factored by the smallest equivalence containing the relation $\sim$.

We write $\{g_\epsilon\}$ for the class containing $[g_\epsilon] \in [N_\epsilon, K]$. Since the canonical maps $\pi_\delta: X \rightarrow N_\delta$ commute with the maps of the diagram Δ we may define

$$\ell: \varinjlim [-, K] \cdot \mathcal{H} \Delta \rightarrow [-, K] \cdot \mathcal{H} X \quad \text{by} \quad \{g_\epsilon\} \mapsto [g_\epsilon \cdot \pi_\epsilon]$$

and this is well defined, i.e. it is independent of the choice of representative in $\{g_\epsilon\}$.

ℓ is surjective. For take $[g] \in [X, K]$. Choosing a representative $g \in [g]$, $g: X \rightarrow K$ in $\mathcal{T}op$, we note that g factors through the canonical map to a nerve N (pull back the star open cover of K ; we get not only an open cover of X but also a natural partition of unity. Then N is a subcomplex of K , viz. that complex having simplices of K intersecting gX) and we write $g = \pi_N \cdot j$ where $j: N \rightarrow K$ is the inclusion. Then j certainly defines a class $\{j\}$ in the limit, and $\ell(\{j\}) = [g]$. Choosing a different representative of $[g]$ might conceivably give us a different $\{j\}$, but any one suffices. In fact this does not happen, because: ℓ is injective. Suppose $\ell(\{g_\epsilon\}) = \ell(\{f_\delta\})$, that is to say, we have $[g_\epsilon]$ and $[f_\delta]$ such that $[g_\epsilon \cdot \pi_\epsilon] = [f_\delta \cdot \pi_\delta]$. This assures us of a homotopy $H: X \times I \rightarrow K$ between $g_\epsilon \circ \pi_\epsilon$ and $f_\delta \circ \pi_\delta$, which determines a partition of unity on $X \times I$, arising from the star open cover of K . We consider the cover defined by the portion of unity. For each x in X and each t in I , there are some finite number of sets of the cover, say $V_t^1, V_t^2, \dots, V_t^n$ on x, t . Let V_t be their intersection. Cover $x \times I$ by such sets. Since it is compact, there is a finite subcover, say $V_1, V_2, \dots, V_s$. Project each set down to X and take V_x to be the intersection.

Doing this for each x in X we obtain a cover of X , χ such that $\forall t \in [0, 1]$, H_t^{-1} st.o.c. K is refined by χ , where st.o.c. K is of course the star open cover of K . Let N be a nerve arising from a partition of unity subordinate to χ ; then $H_t: X \rightarrow K$ factors through N for all t , i.e. H factors through

$N \times I$. Since we have refined the partition of unity corresponding to N_δ (and N_ε) in the process, there is a map from N to each of them, say α, β respectively, with $g_\varepsilon \circ \beta$ homotopic to $f_\delta \circ \alpha$ by the map from $N \times I$ to K . It is to be remembered that α, β will not in general be in Δ , but we may take then a nerve product $N \times N_\varepsilon \times N_\delta$ in place of N , and the result follows.

3.2 REMARK: The two preceding propositions admit the following partial converse: if P is paracompact and $[-, P] \cdot \mathcal{H}$ preserves limits of proper diagrams of nerves then P has the homotopy type of a complex. This follows immediately from Milnor [2] where it is shown that a space dominated by a CW complex has the homotopy type of one, and the observation that the identity map on P must factor through a nerve.

4.

4.1 DEFINITION: Let $\mathcal{C}$ denote the full subcategory of simplicial complexes; $\mathcal{HC}$ we take to be either the image of $\mathcal{C}$ under $\mathcal{H}$ in $\mathcal{H}tp$ or the full subcategories of $\mathcal{H}tp$ having as objects those spaces having the homotopy type of a simplicial complex (this includes the CW complexes, by [2]).

4.2 PROPOSITION: *Let $\Gamma: \mathcal{T}op \rightarrow \mathcal{E}ns$ be a contravariant functor preserving limits of proper diagrams of simplicial complexes, and factoring through $\mathcal{H}tp$ to give $[\Gamma]$, with $[\Gamma]|_{\mathcal{HC}}$ representable by a complex K . Then the representation extends to spaces having the homotopy type of a paracompact space.*

PROOF: We have $\sigma: [-, K] \rightarrow [\Gamma]$ a natural transformation arising from some $s \in \Gamma K$ by the Yoneda lemma, and $\sigma|_{\mathcal{HC}}$ is an equivalence.

If P is a paracompact space we express it as the limit of the appropriate diagram of nerves in $\mathcal{T}op$. Now

$[P, K] \approx [\lim \Delta, K] \approx \text{colim } [\Delta, K]$ by continuity of $[-, K] \cdot \mathcal{H}$
and

$$\begin{aligned} \text{colim } [\Delta, K] &\approx \text{colim } [\Gamma](\Delta) \approx \text{colim } \Gamma \Delta \\ \text{colim } \Gamma \Delta &\approx \Gamma \lim \Delta \quad \text{by continuity of } \Gamma \\ &\approx \Gamma P \end{aligned}$$

where the $\approx$ signs denote various natural equivalences, and some minor abuses of language have taken place.

4.3 REMARK: It follows by the same kind of argument that the Čech cohomology theory is, as is well known, representable on paracompacta.

More generally, if a suitable functor is given on $\mathcal{HC}$ and extended by the generalized Čech process, then we can again expect to obtain an extension of representability.

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