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Introduction

This paper grew out of our joint work on the Brauer group. Our idea was to define the Brauer group in an equivariant situation, also a 'twisted' version incorporating anti-automorphisms, and give exact sequences for computing it. The theory related to representations of groups by automorphisms of various algebraic structures [4], [5], on one hand, and to the theory of quadratic and Hermitian forms (see particularly [17]) on the other.

In developing these ideas, we observed that many of the arguments could be developed in a purely abstract setting, and that this clarified the nature of the proofs. The purpose of this paper is to present this setting, together with such theorems as need no further structure.

To help motivate the reader, and fix ideas somewhat in tracing paths through the abstractions which follow, we list some of the examples to which the theory will be applied. These will be more fully developed, and their interrelations explored, in our next paper.

If R is any ring, we have the category $\mathscr{M}od_R$ of right R-modules. This has a natural product $\oplus$, which is coherently associative and commutative. We can (and usually will) restrict the modules to be finitely generated and/or projective. If R is commutative, we have a second product $\otimes$, also coherently associative and commutative, and distributive over $\oplus$. We are particularly interested in the subcategories $\mathscr{P}ic_R$ of invertible modules and $\mathscr{G}en_R$ of 'generators'. i.e. finitely generated faithfully projective modules. Finally, we have the category $\mathscr{A}z_R$ of Azumaya R-algebras, with product $\otimes$. Note the endomorphism functor End: $\mathscr{G}en_R \to \mathscr{A}z_R$ which preserves the product.

In the equivariant case, one can simply fix a group Γ, and consider each of the above with a group Γ of automorphisms. We generalise (and this is vital for including hermitian forms etc.) by fixing an action of a group Γ by automorphisms of the ring R. Now for (right) R-modules M, N, a morphism $M \to N$ of grade $\gamma \in \Gamma$ is a pair (f, γ), where $f: M \to N$ is a morphism of abelian groups with

$$f(mr) = f(m)^{\gamma} r \qquad \text{for all } m \in M,\ r \in R.$$

Thus we introduce our group by having graded categories throughout. The case when Γ acts trivially on R corresponds to the classical representation theory of Γ over R.

Another example is the 'Brauer category' $\mathscr{B}r(R)$ of a commutative ring R. Its objects are the same as those of $\mathscr{A}z(R)$, i.e. the R-Azumaya algebras. But a morphism $B \to A$ of such algebras in $\mathscr{B}r(R)$ is now given by an isomorphism class (M) of $A-B$-bimodules M, with A acting from the left, B from the right. More precisely a morphism of grade γ is a pair $((M), \gamma)$, where the two R-module structures an M, via A and via B are connected by the equations $mr = {}^{\gamma}r.m$ for all $r \in R$, $m \in M$, $\gamma \in \Gamma$.

We conclude this introduction with some general notation and conventions. Categories will be denoted by script letters $\mathscr{C}$, $\mathscr{D}$ etc. For any category $\mathscr{C}$, we regard $\mathscr{C}$ as denoting the class of morphisms, and write $\operatorname{ob}\mathscr{C}$ for the class of objects. Although we do not always identify $\operatorname{ob}\mathscr{C}$ with the identity morphisms in $\mathscr{C}$, we will usually denote the identity morphism of an object A by the same symbol A, though we sometimes use 1 for identity morphisms.

We write $\mathscr{S}et$ for the category of sets, and $\mathscr{C}at$ for the category of (small) categories. A category is called a *groupoid* if all morphisms are invertible, a *monoid* if it has only one object, a *group* if both; the corresponding full subcategories of $\mathscr{C}at$ are denoted by $\mathscr{G}pd$, $\mathscr{M}$, $\mathscr{G}$ respectively, and we use $\mathscr{A}\mathscr{M}$, $\mathscr{A}\mathscr{G}$ for abelian (i.e. commutative) monoids and groups. A *module* over a group Γ is a functor $\Gamma \to \mathscr{A}\mathscr{G}$; we write $\Gamma - \mathscr{A}\mathscr{G}$ for the category of these. Similarly we have the categories $\Gamma - \mathscr{A}\mathscr{M}$ of abelian Γ-monoids, $\Gamma - \mathscr{S}et$ of Γ-sets and $\Gamma - \mathscr{G}$ of Γ-groups.

For any category $\mathscr{C}$, write $k(\mathscr{C})$ for the set of isomorphism classes of objects of $\mathscr{C}$. We regard k as a functor $\mathscr{C}at \to \mathscr{S}et$.

An abelian monoid may be considered as a set with extra structure. We will study the notion of category with analogous extra structure: these will be called monoidal categories – they are those possessing a product which satisfies conditions shortly to be listed. In individual examples, the product may be more appropriately denoted by $+$, $\oplus$, $\times$, $\otimes$ or by $\perp$. We will adhere for the most part to the neutral symbol ∇.

The paper is divided into chapters, which can be grouped (by style) into movements:

1. Review of monoidal categories

The following problem was considered by MacLane [12]. We are given a category $\mathscr{C}$, and a covariant functor $\nabla: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, associative in that we are given a natural equivalence

$$a = a_{A,B,C}: A \nabla (B \nabla C) \to (A \nabla B) \nabla C.$$

Then we can compose objects $A_1, \cdots, A_n$ using ∇ in several ways, and a induces isomorphisms of these compositions. We wish the isomorphism to be unique. MacLane gives a precise formulation of this problem (see also section 5 below), and shows that the condition holds if and only if each diagram of the following form commutes:

$$(1.1)\qquad \begin{tikzcd} A \nabla (B \nabla (C \nabla D)) \arrow[r] \arrow[d] & (A \nabla B) \nabla (C \nabla D) \arrow[r] & ((A \nabla B) \nabla C) \nabla D \\ A \nabla ((B \nabla C) \nabla D) \arrow[rr] & & (A \nabla (B \nabla C)) \nabla D \arrow[u] \end{tikzcd}$$

Next, he considers the corresponding problem with commutativity. Here we have in addition an equivalence

$$c = c_{A,B}: A \nabla B \to B \nabla A.$$

It turns out that the desired set of conditions is

$$(1.2)\qquad c^2 = 1, \quad \text{i.e. } c_{B,A} \circ c_{A,B} = A \nabla B,$$

and that the following diagrams commute

$$(1.3)\qquad \begin{tikzcd} A \nabla (B \nabla C) \arrow[r, "a"] \arrow[d, "A \nabla c"] & (A \nabla B) \nabla C \arrow[r, "c"] & C \nabla (A \nabla B) \arrow[d, "a"] \\ A \nabla (C \nabla B) \arrow[r, "a"] & (A \nabla C) \nabla B \arrow[r, "c \nabla B"] & (C \nabla A) \nabla B. \end{tikzcd}$$

Finally, he supposes given also a 'unit object' E, and natural equivalence

$$e = e_A: E \nabla A \to A.$$

If we require further that all diagrams of instances of the equivalences a, c and e commute, it is only necessary to impose the further axioms

(1.4) $$c_{E,E} = E,$$

and that the following diagrams commute:

(1.5)
$$\begin{tikzcd} E \nabla (B \nabla C) \arrow[rr, "a"] \arrow[dr, "e"'] & & (E \nabla B) \nabla C \arrow[dl, "e \nabla C"] \\ & B \nabla C & \end{tikzcd}$$

(1.6)
$$\begin{tikzcd} A \nabla (E \nabla C) \arrow[r, "A \nabla e"] \arrow[d, "a"'] & A \nabla C \\ (A \nabla E) \nabla C \arrow[r, "c \nabla C"] & (E \nabla A) \nabla C. \arrow[u, "e \nabla C"'] \end{tikzcd}$$

Accordingly we define (following [14]) a *monoidal category* to be a category $\mathscr{C}$ with preferred object E and functor ∇, together with natural equivalences a, c, e satisfying (1.1)–(1.6). Moreover, it was shown by Kelly [8] that the other axioms follow from (1.1), (1.2), (1.3) and either (1.5) or (1.6).

Epstein [3] considered the situation of a functor $T:\mathscr{C} \to \mathscr{D}$ between monoidal categories. We wish T to preserve the product, so postulate a natural equivalence

$$p\colon TA \nabla TB \to T(A \nabla B)$$

of functors $\mathscr{C} \times \mathscr{C} \to \mathscr{D}$; and then again ask under what further conditions all 'naturally occurring diagrams' (the precise definition is given in [3]) will commute. Epstein considers only the natural equivalences a and c, and shows that it is enough to assume that the following diagrams commute:

(1.7)
$$\begin{tikzcd} TA \nabla (TB \nabla TC) \arrow[r, "TA \nabla p"] \arrow[d, "a"'] & TA \nabla T(B \nabla C) \arrow[r, "p"] & T(A \nabla (B \nabla C)) \arrow[d, "Ta"] \\ (TA \nabla TB) \nabla TC \arrow[r, "p \nabla TC"] & T(A \nabla B) \nabla TC \arrow[r, "p"] & T((A \nabla B) \nabla C), \end{tikzcd}$$

(1.8)
$$\begin{tikzcd} TA \nabla TB \arrow[r, "p"] \arrow[d, "c"'] & T(A \nabla B) \arrow[d, "Tc"] \\ TB \nabla TA \arrow[r, "p"] & T(B \nabla A). \end{tikzcd}$$

It is easy to see that to extend this result to include e it suffices to include an equivalence

$$q\colon TE_{\mathscr{C}} \to E_{\mathscr{D}}$$

(we shall drop the subscripts in future since no confusion can arise) and require commutativity of the diagrams

(1.9)
$$\begin{tikzcd} TE \nabla TA \arrow[r, "p"] \arrow[d, "q \nabla TA"] & T(E \nabla A) \arrow[d, "Te"] \\ E \nabla TA \arrow[r, "e"] & TA. \end{tikzcd}$$

We now define the category $\mathscr{M}\mathscr{C}$ of (small) monoidal categories. An object of $\mathscr{M}\mathscr{C}$ is a monoidal category – viz. a sextuple $(\mathscr{C}, \nabla, E, a, c, e)$ satisfying (1.1)–(1.6). We often simply write $\mathscr{C}$ for this. A morphism from $\mathscr{C}$ to $\mathscr{D}$ is a triple (T, p, q) as above satisfying (1.7)–(1.9). Composition of morphisms

$$\mathscr{C} \xrightarrow{(T,p,q)} \mathscr{D} \xrightarrow{(T',p',q')} \mathscr{E}$$

is defined to be (T'', p'', q''), where $T'' = T' \circ T$, $p'' = T'p \circ p'$ and $q'' = q' \circ T'q$. It is trivial to verify that (T'', p'', q'') does indeed satisfy (1.7)–(1.9), so we have a morphism; that composition of morphisms is associative, and that identity morphisms exist (take each of T, p, q to be the identity). Thus we have indeed constructed a category.

The above notion of morphism is, however, a little too rigid for most purposes, and we conclude by showing how to relax it. We define a *homotopy* between two morphisms (T, p, q), $(T', p', q'): \mathscr{C} \to \mathscr{D}$ to be a natural equivalence $X: T \to T'$ such that the following diagrams commute:

(1.10)
$$\begin{tikzcd} TA \nabla TB \arrow[r, "p"] \arrow[d, "XA \nabla XB"] & T(A \nabla B) \arrow[d, "X(A \nabla B)"] \\ T'A \nabla T'B \arrow[r, "p'"] & T'(A \nabla B), \end{tikzcd}$$

(1.11)
$$\begin{tikzcd} TE \arrow[rr, "XE"] \arrow[dr, "q"'] & & T'E \arrow[dl, "q'"] \\ & E & \end{tikzcd}$$

Homotopy is an equivalence relation among morphisms: for reflexivity, take X the identity, for symmetry use X^{-1} (which exists as X is an equivalence), and for transitivity we can use composites. It is also trivial to see that homotopy is compatible with composition in $\mathscr{M}\mathscr{C}$. We can therefore define the *homotopy category* $\mathscr{H}\mathscr{M}\mathscr{C}$ to be the quotient category with the same objects, but morphisms are homotopy classes of $\mathscr{M}\mathscr{C}$-morphisms.

LEMMA (1.12): *Suppose $(T, p, q): \mathscr{C} \to \mathscr{D}$ an $\mathscr{M}\mathscr{C}$-morphism. Then (T, p, q) is an equivalence in $\mathscr{H}\mathscr{M}\mathscr{C}$ if and only if T is an equivalence of categories.*

This 'homotopy' formulation of equivalence of categories is really nothing to do with monoidal categories, as is clear from the proof.

PROOF: If (T, p, q) is an equivalence in $\mathscr{HMC}$, it has an inverse, represented by an $\mathscr{MC}$-morphism (T_2, p_2, q_2) such that there are homotopies $\phi: TT_2 \to 1$, $\psi: T_2 T \to 1$. Hence any object D of $\mathscr{D}$ is isomorphic by ϕD to $TT_2 D$, so T is full on objects.

Now as ψ is natural, for any morphism $f: C_1 \to C_2$ in $\mathscr{C}$,

$$\psi C_2 \circ T_2 Tf = f \circ \psi C_1.$$

Hence the composite map

$$\mathrm{Hom}_{\mathscr{C}}(C_1, C_2) \xrightarrow{T} \mathrm{Hom}_{\mathscr{D}}(TC_1, TC_2) \xrightarrow{T_2} \mathrm{Hom}_{\mathscr{C}}(T_2 TC_1, T_2 TC_2)$$

sends f to $T_2 Tf = (\psi C_2)^{-1} \circ f \circ \psi C_1$, so is bijective. Thus T is injective. Similarly, T_2 is injective, so T is bijective. Thus T is full and faithful.

Conversely, if T is an equivalence we can choose, for each $D \in \mathrm{ob}\,\mathscr{D}$, $T_2 D \in \mathrm{ob}\,\mathscr{C}$ and an isomorphism $\phi D: TT_2\, D \to D$. We may choose in particular $T_2 E = E$ and $\phi E = q: TE \to E$. Then for any morphism $g: D_1 \to D_2$ of $\mathscr{D}$, define $T_2 g: T_2 D_1 \to T_2 D_2$ as the unique morphism (since T is an equivalence) with $TT_2 g = (\phi D_2)^{-1} \circ g \circ \phi D_1$. There is now a unique way to choose p_2 such that $(T_2, p_2, 1): \mathscr{D} \to \mathscr{C}$ is an $\mathscr{MC}$-morphism and $\phi: TT_2 \to 1$ a homotopy. Finally, for $C \in \mathrm{ob}\,\mathscr{C}$, define $\psi C: T_2 TC \to C$ as the morphism with $T\psi C = \phi(TC)$. Then $\psi: T_2 T \to 1$ is also a homotopy, so $(T_2, p_2, 1)$ is inverse to (T, p, q) in $\mathscr{HMC}$.

PROPOSITION (1.13): *There is a natural extension of k to a functor $k: \mathscr{HMC} \to \mathscr{AM}$.*

PROOF: Recall that $k(\mathscr{C})$ is the set of isomorphism classes of objects of $\mathscr{C}$. Thus $k(\mathscr{C} \times \mathscr{C}) = k(\mathscr{C}) \times k(\mathscr{C})$. Now suppose $\mathscr{C}$ a monoidal category. Then the functor $\nabla: \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ induces a map

$$k(\mathscr{C}) \times k(\mathscr{C}) = k(\mathscr{C} \times \mathscr{C}) \to k(\mathscr{C}).$$

The existence of natural equivalences a and c ensures that this product on $k(\mathscr{C})$ is associative and commutative. The class of E provides a unit, since e is an equivalence.

Now if (T, p, q) gives a morphism $\mathscr{C} \to \mathscr{D}$ of monoidal categories, the equivalence p shows that $k(T)$ preserves products, and q that it preserves the unit. Finally, it is clear that homotopic morphisms T, T' satisfy $k(T) = k(T')$.

2. The projective category

We will now describe a construction which generalises the passage from the category of finite dimensional vector spaces over a field k (with

$\nabla = \otimes$) to the corresponding category of projective spaces.

Let $\mathscr{C}$ be a monoidal category. Write $U = U(\mathscr{C}) = \operatorname{Aut}_{\mathscr{C}}(E)$ for the group of $\mathscr{C}$-automorphisms of the unit object: we call $U(\mathscr{C})$ the *unit group* of $\mathscr{C}$. Clearly U is a functor from $\mathscr{MC}$ to $\mathscr{G}$. For any object C of $\mathscr{C}$, $u \in U$, define $\theta_C(u)$ by the commutative diagram

(2.1)
$$\begin{tikzcd} E \nabla C \arrow[r, "u \nabla C"] \arrow[d, "e"] & E \nabla C \arrow[d, "e"] \\ C \arrow[r, "\theta_C(u)"] & C, \end{tikzcd}$$

i.e. $\theta_C(u) = e(u \nabla C)e^{-1}$.

THEOREM (2.2):

(i) *θ_C is a homomorphism $U \to \operatorname{Aut}_{\mathscr{C}}(C)$.*

(ii) *If $s: C_1 \to C_2$, then $s\theta_{C_1}(u) = \theta_{C_2}(u)s$; in particular, $\operatorname{Im}\theta_C$ is in the centre of $\operatorname{Aut}_{\mathscr{C}}(C)$.*

(iii) $\theta_{C_1 \nabla C_2}(u) = \theta_{C_1}(u) \nabla C_2 = C_1 \nabla \theta_{C_2}(u)$. $\theta_E(u) = u$.

(iv) *If there is an equivalence $f: C \nabla D \to E$ (i.e. if C is an invertible object), θ_C is an isomorphism.*

PROOF: (i) is immediate:

$$e(uv \nabla C)e^{-1} = e(u \nabla C)e^{-1} \cdot e(v \nabla C)e^{-1}.$$

(ii) follows since the diagram

$$\begin{tikzcd} C_1 \arrow[d, "s"] & E \nabla C_1 \arrow[l, "e"'] \arrow[r, "u \nabla C_1"] \arrow[d, "E \nabla s"] & E \nabla C_1 \arrow[r, "e"] \arrow[d, "E \nabla s"] & C_1 \arrow[d, "s"] \\ C_2 & E \nabla C_2 \arrow[l, "e"'] \arrow[r, "u \nabla C_2"] & E \nabla C_2 \arrow[r, "e"] & C_2 \end{tikzcd}$$

commutes, by naturality of ∇ and e.

For (iii), consider the diagram

$$\begin{tikzcd} & E \nabla (C_1 \nabla C_2) \arrow[r, "u \nabla (C_1 \nabla C_2)"] \arrow[dl, "e"'] \arrow[dd, "a"] & E \nabla (C_1 \nabla C_2) \arrow[dr, "e"] \arrow[dd, "a"] & \\ C_1 \nabla C_2 & & & C_1 \nabla C_2. \\ & (E \nabla C_1) \nabla C_2 \arrow[r, "(u \nabla C_1) \nabla C_2"] \arrow[ul, "e \nabla C_2"] & (E \nabla C_1) \nabla C_2 \arrow[ur, "e \nabla C_2"'] & \end{tikzcd}$$

The square commutes by naturality of a, the triangles by (1.5). Hence the upper composite, which is $\theta_{C_1 \nabla C_2}(u)$ equals the lower, $\theta_{C_1}(u) \nabla C_2$. Equality with $C_1 \nabla \theta_{C_2}(u)$ follows similarly using (1.6) from the diagram

$$
\begin{tikzcd}
& C_1 \nabla (E \nabla C_2) \arrow[dl, "C_1 \nabla e"'] \arrow[r, "C_1 \nabla (u \nabla C_2)"] \arrow[d, "a"] & C_1 \nabla (E \nabla C_2) \arrow[d, "a"] \arrow[dr, "C_1 \nabla e"] & \\
C_1 \nabla C_2 & (C_1 \nabla E) \nabla C_2 \arrow[r, "(C_1 \nabla u) \nabla C_2"] \arrow[d, "c \nabla C_2"] & (C_1 \nabla E) \nabla C_2 \arrow[d, "c \nabla C_2"] & C_1 \nabla C_2 . \\
& (E \nabla C_1) \nabla C_2 \arrow[ul, "e \nabla C_2"] \arrow[r, "(u \nabla C_1) \nabla C_2"] & (E \nabla C_1) \nabla C_2 \arrow[ur, "e \nabla C_2"'] &
\end{tikzcd}
$$

The last assertion follows from commutativity of

$$
\begin{tikzcd}
E \arrow[d, "u"] & E \nabla E \arrow[l, "e"'] \arrow[r, "c = 1"] \arrow[d, "E \nabla u"] & E \nabla E \arrow[d, "u \nabla E"] \arrow[ll, bend right, "e"'] \\
E & E \nabla E \arrow[l, "e"'] \arrow[r, "c = 1"] & E \nabla E . \arrow[ll, bend left, "e"]
\end{tikzcd}
$$

Finally for (iv) we define $\phi : \mathrm{Aut}_{\mathscr{C}}(C) \to U$ by

$$\phi(\alpha) = f(\alpha \nabla D) f^{-1}.$$

Then

$$
\begin{aligned}
\phi \theta_C(u) &= f(\theta_C(u) \nabla D) f^{-1} \\
&= f(\theta_{C \nabla D}(u)) f^{-1} && \text{by (iii)} \\
&= \theta_E(u) && \text{by (ii)} \\
&= u.
\end{aligned}
$$

In particular, ϕ is surjective. It will now suffice to prove ϕ injective. But if $\phi(\alpha) = 1$, then $\alpha \nabla D = 1$, so $(\alpha \nabla D) \nabla C : (C \nabla D) \nabla C \to (C \nabla D) \nabla C$ is the identity. Now write g for the composite isomorphism

$$(C \nabla D) \nabla C \xrightarrow{c} C \nabla (C \nabla D) \xrightarrow{C \nabla c} C \nabla (D \nabla C) \xrightarrow{a}$$
$$(C \nabla D) \nabla C \xrightarrow{f \nabla C} E \nabla C \xrightarrow{e} C :$$

it follows from obvious commutative diagrams that $g\{(\alpha \nabla D) \nabla C\} g^{-1} = \alpha$ is the identity also, whence the result.

COROLLARY (2.3): *$U(\mathscr{C})$ is an abelian group. Hence U defines a functor $\mathscr{HMC} \to \mathscr{AG}$.*

We now define an equivalence relation among morphisms of $\mathscr{C}$. If s_1, s_2 have domain C, write $s_1 \sim s_2$ if $s_2 = s_1 \circ \theta_C(u)$ for some $u \in U$.

It is immediate from (i) that this defines an equivalence relation, which respects also codomains. Now (ii) shows that the relation is compatible with composition in $\mathscr{C}$. Hence the equivalence classes of morphisms form a category $P-\mathscr{C}$, with the same objects as $\mathscr{C}$. Finally it follows from (iii) that the equivalence relation is also compatible with ∇, so $P-\mathscr{C}$ inherits a structure of monoidal category, such that the natural projection $\mathscr{C} \to P-\mathscr{C}$ defines a morphism in $\mathscr{M}\mathscr{C}$. This morphism is an equivalence precisely when $U(\mathscr{C})$ is trivial. Note that we always have

$$(2.4) \qquad k(P-\mathscr{C}) = k(\mathscr{C}), \qquad U(P-\mathscr{C}) = \{1\},$$

so $P-\mathscr{C} \to P-P-\mathscr{C}$ is an equivalence.

The hypothesis (iv) of (2.2) will become increasingly important as we continue. We now give a preliminary discussion of inverses, which will suffice for the first movement, though more precision will be needed in the second.

We first discuss morphisms. We are in general only interested in invertible morphisms. Given any category, there is always the maximal subgroupoid, consisting of invertible morphisms of the category. For our purposes, we need

REMARK (2.5): *If f, g are invertible morphisms in the monoidal category $\mathscr{C}$, so is $f \nabla g$.*

For let $f \circ f^{-1} = A$, $g \circ g^{-1} = B$. Then

$$(f \nabla g) \circ (f^{-1} \nabla g^{-1}) = (f \circ f^{-1}) \nabla (g \circ g^{-1}) = A \nabla B$$

since ∇ is a functor; and for the same reason, $A \nabla B$ is an identity. Thus $f \nabla g$ has a right inverse; similarly, it has a left inverse.

Thus the maximal subgroupoid of $\mathscr{C}$ is also monoidal. Next we consider objects. The monoid $k(\mathscr{C})$ has a maximal subgroup $C(k(\mathscr{C}))$; we denote by $\operatorname{Inv}\mathscr{C}$ the subcategory of $\mathscr{C}$ whose objects are the invertible objects of $\mathscr{C}$, and morphisms the invertible morphisms between them. In general, we call a monoidal category *group-like* if all objects and morphisms are invertible. Thus $\operatorname{Inv}\mathscr{C}$ is the maximal group-like subcategory of $\mathscr{C}$. The following observation is sometimes useful.

LEMMA (2.6): *Let $(\mathscr{C}, \nabla, E, a, c, e)$ be a group-like monoidal category. Then there exist a functor $\Phi: \mathscr{C} \to \mathscr{C}$ and a natural equivalence ϕ: $C \nabla \Phi(C) \to E$. This determines Φ up to equivalence.*

PROOF: Since $k(\mathscr{C})$ is a group, we can choose for each $C \in \operatorname{ob}\mathscr{C}$ another object $\Phi(C)$ of $\mathscr{C}$ and equivalence ϕ as above. We claim that there is now a unique way to define Φ on morphisms so that ϕ is natural.

If there is an isomorphism $s: C \to D$, then $\Phi(C)$ and $\Phi(D)$ define the same class in $k(\mathscr{C})$ (inverse to that of C), so we can find an isomorphism $\sigma: \Phi(C) \to \Phi(D)$. The other such isomorphisms are then of the form $\alpha \circ \sigma$, where α is an automorphism of $\Phi(C)$ and hence by (2.2 iv), of the form $\theta_{\Phi(C)}(u)$, for a unique $u \in U$. Now if $v^{-1} \in U$ is the composite $\phi_D(s \nabla \sigma)\phi_C^{-1}$: $E \to E$, we claim that

$$\begin{tikzcd} C \nabla \Phi(C) \arrow[rr, "s \nabla (\theta_{\Phi(C)}(u)\ \sigma)"] \arrow[dr, "\phi_C"'] & & D \nabla \Phi(D) \arrow[dl, "\phi_D"] \\ & E & \end{tikzcd}$$

commutes if and only if $u = v$. For

$$\begin{aligned} \phi_D(s \nabla (\theta_{\Phi(C)}(u) \circ \sigma))\phi_C^{-1} &= \phi_D(1 \nabla \theta_{\Phi(C)}(u))(s \nabla \sigma)\phi_C^{-1} \\ &= (\phi_D \theta_{C \nabla \Phi(C)}(u)\phi_D^{-1})(\phi_D(s \nabla \sigma)\phi_C^{-1}) && \text{by (2.2 iii)} \\ &= \theta_E(u)v^{-1} && \text{by (2.2 ii)} \\ &= uv^{-1}. && \text{by (2.2 iii)} \end{aligned}$$

This establishes all our claims. Uniqueness up to equivalence is immediate.

We will not investigate coherence of the inverse, as it is our contention that the inverse, when there is one, is sufficiently well determined by the structure already postulated.

Note for future reference (see Section 12) that if we denote by $C(M)$, $G(M)$ the universal groups for the monoid M, with homomorphisms $C(M) \to M \to G(M)$, then

$$(2.7) \qquad \begin{aligned} k(\mathrm{Inv}\,\mathscr{C}) &= C(k(\mathscr{C})) \\ U(\mathrm{Inv}\,\mathscr{C}) &= U(\mathscr{C}), \end{aligned}$$

since we have the same identity object. Also observe that we can express the algebraic K-theory of $\mathscr{C}$ by

$$(2.8) \qquad \begin{aligned} K_0(\mathscr{C}) &= G(k(\mathscr{C})) \\ K_1(\mathscr{C}) &= U(\mathscr{C}) \qquad \text{if } k(\mathscr{C}) \text{ is a group.} \end{aligned}$$

The former is clear; the latter also since if $k(\mathscr{C})$ is a group, the single object E is cofinal.

The following result is a first step to development of 'homotopy theory': it represents a sort of covering homotopy property.

PROPOSITION (2.9): *Let $\mathscr{C}$, $\mathscr{D}$ be monoidal categories with $\mathscr{C}$ group-like, and (T_1, p_1, q_1), (T_2, p_2, q_2) two morphisms $\mathscr{C} \to \mathscr{D}$ inducing the same maps $k(\mathscr{C}) \to k(\mathscr{D})$ and $U(\mathscr{C}) \to U(\mathscr{D})$. Then there exist a morphism (T_2, p'_2, q_2) and a homotopy $\omega: (T_1, p_1, q_1) \to (T_2, p'_2, q_2)$.*

PROOF: We first seek to construct a natural equivalence $\omega: T_1 \to T_2$. Begin by defining $\omega(E) = q_2^{-1} \circ q_1$. Since our two morphisms induce the same map $U(\mathscr{C}) \to U(\mathscr{D})$, this is natural for maps of E to itself. Next, for each object of $\mathscr{C}$ equivalent to E, choose an equivalence $e(C): E \to C$, and define $\omega(C) = T_2 e(C) \circ q_2^{-1} \circ q_1 \circ T_1 e(C)^{-1}$.

For C_1, C_2 two such objects, any morphism f: $C_1 \to C_2$ has the form $e(C_2) \circ u \circ e(C_1)^{-1}$ for some $u \in U(\mathscr{C})$. Then

$$
\begin{aligned}
\omega(C_2) \circ T_1(f) &= T_2 e(C_2) \circ q_2^{-1} \circ q_1 \circ T_1 e(C_2)^{-1} \circ T_1 e(C_2) \circ T_1(u) \circ T_1 e(C_1)^{-1} \\
&= T_2 e(C_2) \circ q_2^{-1} \circ q_1 \circ T_1(u) \circ T_1 e(C_1)^{-1} \\
&= T_2 e(C_2) \circ T_2(u) \circ q_2^{-1} \circ q_1 \circ T_1\ e(C_1)^{-1} \\
&= T_2(f) \circ \omega(C_1).
\end{aligned}
$$

Now for each other equivalence class of objects of $\mathscr{C}$ choose a representative C_0, and for each object in the class, a morphism $e(C): C_0 \to C$. Define $\omega(C_0)$ to be any (fixed) isomorphism $T_1(C_0) \to T_2(C_0)$, and $\omega(C) = T_2 e(C) \circ \omega(C_0) \circ T_1 e(C)^{-1}$. The same argument as above shows ω is natural provided we can check this on automorphisms of C_0. Now by (2.2 iv), any such automorphism has the form $\theta_{C_0}(u)$ for some $u \in U$. The desired naturality now follows from the commutative diagram

$$
\begin{tikzcd}
T_1 C_0 \arrow[ddd, bend right, "\omega C_0"'] & T_1(E \nabla C_0) \arrow[l, "T_1 e"'] \arrow[r, "T_1(u \nabla C_0)"] & T_1(E \nabla C_0) \arrow[r, "T_1 e"] & T_1 C_0 \arrow[ddd, bend left, "\omega C_0"] \\
E \nabla T_1 C_0 \arrow[u, "e"] \arrow[d, "E \nabla \omega C_0"] & T_1 E \nabla T_1 C_0 \arrow[l, "q_1 \nabla T_1 C_0"'] \arrow[u, "p_1"] \arrow[d, "\omega E \nabla \omega C_0"] \arrow[r, "T_1 u \nabla T_1 C_0"] & T_1 E \nabla T_1 C_0 \arrow[u, "p_1"] \arrow[d, "\omega E \nabla \omega C_0"] \arrow[r, "q_1 \nabla T_1 C_0"] & E \nabla T_1 C_0 \arrow[u, "e"] \arrow[d, "E \nabla \omega C_0"] \\
E \nabla T_2 C_0 \arrow[d, "e"] & T_2 E \nabla T_2 C_0 \arrow[l, "q_2 \nabla T_2 C_0"'] \arrow[d, "p_2"] \arrow[r, "T_2 u \nabla T_2 C_0"] & T_2 E \nabla T_2 C_0 \arrow[d, "p_2"] \arrow[r, "q_2 \nabla T_2 C_0"] & E \nabla T_2 C_0 \arrow[d, "e"] \\
T_2 C_0 & T_2(E \nabla C_0) \arrow[l, "T_2 e"'] \arrow[r, "T_2(u \nabla C_0)"] & T_2(E \nabla C_0) \arrow[r, "T_2 e"] & T_2 C_0
\end{tikzcd}
$$

where the upper and lower rows give the definitions of $T_i \theta_{C_0}(u)$, $i = 1, 2$.

We now have a natural equivalence satisfying (1.11). To complete the proof, all that is needed is to use the diagram (1.10) to define p'_2, i.e.

$$p'_2(A, B) = \omega(A \nabla B) \circ p_1(A, B) \circ (\omega A \nabla \omega B)^{-1}.$$

3. The graded case

We regard a group Γ as a category with one object and all morphisms invertible. A *Γ-grading* on a category $\mathscr{C}$ is a functor $g:\mathscr{C} \to \Gamma$. The grading is called *stable* if for all $C \in \operatorname{ob}\mathscr{C}$, $\gamma \in \Gamma$, there is an equivalence f in $\mathscr{C}$ with domain C and $g(f) = \gamma$. We refer to $g(f)$ as the *grade* of f. All the preceding theory admits immediate generalisation to the graded case. It is indeed this generalisation that we wish to study: we have described the ungraded case only for the sake of explaining each idea first in its simplest context.

If $(\mathscr{C}, g)$ is a Γ-graded category, we define $\operatorname{Ker}\mathscr{C}$ to be the subcategory consisting of all morphisms of grade 1. This is to be regarded as the underlying ungraded category of $(\mathscr{C}, g)$: in fact, we think of Γ acting in some sense on $\operatorname{Ker}\mathscr{C}$. Keeping this in mind, we generalise concepts to the graded case as follows.

For Γ-graded categories $(\mathscr{C}, g)$, $(\mathscr{C}', g')$, we write $\mathscr{C} \times_\Gamma \mathscr{C}'$ for the pullback of (g, g'): this has an obvious Γ-grading, which is stable if g and g' are. It is the product in the category of graded categories. A Γ-functor is one over the identity map of Γ, or equivalently, a functor preserving grades of morphisms. A natural transformation of Γ-functors will always be of grade 1 unless otherwise mentioned.

A *Γ-monoidal category* consists of: a stably Γ-graded category $(\mathscr{C}, g)$, a covariant Γ-functor $\nabla:\mathscr{C} \times_\Gamma \mathscr{C} \to \mathscr{C}$ – i.e. $h \nabla k$ is defined only when h, k have the same grade, and $g(h \nabla k) = g(h) = g(k)$, a covariant Γ-functor $E:\Gamma \to \mathscr{C}$ and natural equivalences (of grade 1) $a: A \nabla (B \nabla C) \to (A \nabla B) \nabla C$, $c: A \nabla B \to B \nabla A$, $e: E \nabla A \to A$ satisfying (1.1)–(1.6).

The category $\Gamma - \mathscr{M}\mathscr{C}$ has as objects the Γ-monoidal categories; a morphism $\mathscr{C} \to \mathscr{D}$ is a triple (T, p, q), where $T: \mathscr{C} \to \mathscr{D}$ is a Γ-functor, $p: TA \nabla TB \to T(A \nabla B)$ is a natural equivalence (of grade 1), $q: TE \to E$ is an isomorphism of grade 1, and (1.7)–(1.9) are satisfied. Exactly as in Section 1 we go on to define compositions, the notion of homotopy (of grade 1) and the category $\Gamma - \mathscr{H}\mathscr{M}\mathscr{C}$. Thus Ker defines a functor $\Gamma - \mathscr{M}\mathscr{C} \to \mathscr{M}\mathscr{C}$, which respects homotopies.

The proof of (1.12) carries over to the present situation. From (1.13) we obtain something new.

LEMMA (3.1): *k induces a functor $k_\Gamma: \Gamma - \mathscr{H}\mathscr{M}\mathscr{C} \to \Gamma - \mathscr{A}\mathscr{M}$.*

Here, a Γ-monoid is one on which Γ acts (on the left) as group of automorphisms.

PROOF: Combining the functors Ker (above) and k (from 1.13), we have a functor $\mathscr{C} \mapsto k(\operatorname{Ker}\mathscr{C})$ from $\mathscr{H}\mathscr{M}\mathscr{C}$ to the category of abelian monoids. It remains to define a natural Γ-action. We do this (as in [6]) by specifying that if $f: A \to B$ is an isomorphism of grade γ, then $\gamma\{A\} = \{B\}$. For f exists, by stability, and if $\{A\} = \{A'\}$, and $f': A' \to B'$ is an isomorphism of grade γ, there is an isomorphism $\phi: A \to A'$ in $\operatorname{Ker}\mathscr{C}$, i.e. of grade 1, and then also $f': \circ\, \phi \circ f^{-1}: B \to B'$ is an isomorphism of grade 1, thus $\{B\} = \{B'\}$. Thus $\gamma\{A\}$ is well-defined: we see easily that we have an action of Γ on $k(\operatorname{Ker}\mathscr{C})$, compatible with addition, and naturality is clear.

For any Γ-graded category $(\mathscr{C}, g)$, we write $\operatorname{Rep}(\mathscr{C}, g)$ for the category of Γ-functors $F: \Gamma \to \mathscr{C}$, and natural transformations (of grade 1). An object of $\operatorname{Rep}(\mathscr{C}, g)$ thus consists of an object C of $\mathscr{C}$ with a grade-preserving homomorphism $\Gamma \to \operatorname{Aut}_{\mathscr{C}}(C)$: a representation of Γ by automorphisms of C. The study of $\operatorname{Rep}(\mathscr{C}, g)$ is one of our major objectives. We have

LEMMA (3.2): Rep *induces a homotopy-preserving functor* $\Gamma - \mathscr{MC} \to \mathscr{MC}$. *If* $\mathscr{C}$ *is group-like, so is* $\operatorname{Rep}\mathscr{C}$.

Indeed, it is a trivial exercise to check that the structure (∇, a, c etc) on $\mathscr{C}$ carries over to $\operatorname{Rep}\mathscr{C}$. This, and the same assertion about $\operatorname{Ker}\mathscr{C}$, are special cases of more general remarks to be made later.

It is evident that if all morphisms in $\mathscr{C}$ are invertible, the same holds for $\operatorname{Rep}\mathscr{C}$. Finally, take an object of $\operatorname{Rep}\mathscr{C}$, defined by an object C of $\mathscr{C}$ and a group $f(\gamma)$ of automorphisms of it. By (2.6), we have a functor Φ: then $\Phi(C, f)$ define another object of $\operatorname{Rep}\mathscr{C}$, which we claim is inverse to (C, f). Indeed, the equivalence ϕ provided by (2.6) yields an isomorphism $(C, f) \nabla \Phi(C, f) \to E$ in $\operatorname{Rep}\mathscr{C}$, as required.

We now turn to the ideas of Section 2. Note that E is now given by an object, again called the identity object and by abuse of notation also denoted by E, together with a homomorphism $\Gamma \to \operatorname{Aut}_{\mathscr{C}}(E)$ which splits the grading map $g: \operatorname{Aut}_{\mathscr{C}}(E) \to \Gamma$. In other words, $\operatorname{Aut}_{\mathscr{C}}(E)$ is a split extension of the normal subgroup U of automorphisms of grade 1 by the subgroup $E(\Gamma)$ isomorphic to Γ. $U = U(\mathscr{C}) = U(\operatorname{Ker}\mathscr{C})$ is called the unit group of $\mathscr{C}$: by (2.3) it is abelian. The extension defines an action of Γ on U: we set

$$^{\gamma}u = E(\gamma) \circ u \circ E(\gamma^{-1}).$$

This is clearly natural, so U defines a functor $\Gamma - \mathscr{HMC} \to \Gamma - \mathscr{AG}$.

LEMMA (3.3): *Theorem* 2.2 *holds in the graded case. except that in* (ii), *if* $g(s) = \gamma$,

$$s \circ \theta_{C_1}(u) = \theta_{C_2}(^{\gamma}u) \circ s.$$

Note that we can only define $\theta_C(u)$ for $u \in U$, as if u does not have grade 1, $u \nabla C$ is undefined.

PROOF: The other assertions refer only to Ker$\mathscr{C}$: they hold there by (2.2). Now by naturality of ∇ and e, the diagram

$$\begin{tikzcd}
C_1 \arrow[d, "s"] & E \nabla C_1 \arrow[l, "e"'] \arrow[r, "\gamma_u \nabla C_1"] \arrow[d, "E(\gamma) \nabla s"] & E \nabla C_1 \arrow[r, "e"] \arrow[d, "E(\gamma) \nabla s"] & C_1 \arrow[d, "s"] \\
C_2 & E \nabla C_2 \arrow[l, "e"'] \arrow[r, "\gamma_u \nabla C_2"] & E \nabla C_2 \arrow[r, "e"] & C_2
\end{tikzcd}$$

commutes. The result follows.

It follows as before that we can define a projective category $P-\mathscr{C}$, with an induced Γ-grading, which inherits the structure of Γ-monoidal category.

The discussion of inverses, the definition of Inv$(\mathscr{C})$, and the definition of group-like are essentially the same as in the ungraded case. We now have

LEMMA (3.4): *Let $\mathscr{C}$ be a group-like Γ-monoidal category. Then* $k\,\mathrm{Rep}(P-\mathscr{C}) \cong H^0(\Gamma; k_\Gamma(\mathscr{C}))$, *the subgroup of Γ-invariant elements of* $k_\Gamma(\mathscr{C})$.

PROOF: If $(C, f) \in \mathrm{ob\,Rep}(P-\mathscr{C})$, then C has automorphisms of every grade in $P-\mathscr{C}$, hence in $\mathscr{C}$, so defines a Γ-invariant element of $k_\Gamma(\mathscr{C})$. Conversely if it does, it has automorphisms of every grade in $P-\mathscr{C}$. But in general if $f_1, f_2: C \to D$ in $\mathscr{C}$ have the same grade, $f_2^{-1} \circ f_1$ is an automorphism of C of grade 1, so by (2.2 iv) of the form $\theta_C(u)$; thus f_1, f_2 define the same morphism in $P-\mathscr{C}$. So our object has just one automorphism of each grade, hence defines a unique representation, or object of Rep$(P-\mathscr{C})$.

It is easy to see that

(3.5) $$U(\mathrm{Rep}\,\mathscr{C}) \cong H^0(\Gamma; U(\mathscr{C})).$$

No such simple result is available for $k(\mathrm{Rep}\,\mathscr{C})$: we next turn to this.

4. Exact sequences. Direct methods

Recall the interpretation of the cohomology set $H^1(\Gamma, V)$ of Γ with coefficients in a Γ-group V, in terms of splitting extensions. We start with a given extension of V by Γ

$$1 \to V \to H \to \Gamma \underset{f}{\overset{g}{\rightleftarrows}} 1$$

with a given splitting homomorphism f, defining on V the structure of a Γ-group. Among the splitting homomorphisms $f': \Gamma \to H$ with $g \circ f' = 1_\Gamma$

we introduce an equivalence relation $f' \sim f''$ meaning that $f' = i \circ f''$ where i is an inner automorphism of H induced by an element of the subgroup V. There is a bijection between the set of splitting homomorphisms f' and the set of 1-cocycles $c: \Gamma \to V$ given by $f'(\gamma) = c(\gamma) f(\gamma)$, and $f' \sim f''$ if and only if the corresponding cocycles are cohomologous.

Now let $(C, f) \in \operatorname{ob} \operatorname{Rep}(\mathscr{C})$. Here C stands for the underlying object of $\mathscr{C}$ and f is the given map $\Gamma \to \operatorname{Aut}_{\mathscr{C}}(C)$. Denote the latter group by H, and let g be the grading map. Then we are in the situation just discussed, with $V = \operatorname{Aut}_{\operatorname{Ker}\mathscr{C}}(C)$. The splitting homomorphisms f' correspond to the representations over the object C, and $f' \sim f''$ means exactly that the two representations are isomorphic in $\operatorname{Rep}(\mathscr{C})$. Let $k\operatorname{Rep}(C,f)$ denote in the sequel the set of isomorphism classes of objects (C, f') of $\operatorname{Rep}(\mathscr{C})$, or of '*forms of* (C,f)' as we shall say. Then we have established

Proposition (4.1): *There is a bijection*

$$k\operatorname{Rep}(C,f) \to H^1(\Gamma, \operatorname{Aut}_{\operatorname{Ker}(\mathscr{C})}(C)),$$

where $\operatorname{Aut}_{\operatorname{Ker}(\mathscr{C})}(C)$ *is a* Γ*-group via* f.

Example: Let L/K be a Galois extension of fields with Galois group Γ. Let $\mathscr{C}$ be the category whose objects are finite dimensional L-vector spaces, possibly with tensor elements attached (e.g. tensors defining symmetric or skew-symmetric bilinear forms, or algebras of some variety etc.). The morphisms of $\mathscr{C}$ are the L-semi-linear transformations of vector spaces preserving the given structure. $\operatorname{Rep}(\mathscr{C})$ is the category of K-vector spaces with the structure tensors defined over K. A form of (C, f) is then precisely a K-form in the well accepted sense [15].

Now let $\mathscr{C}$ be Γ-monoidal. In the particular case when $(C,f) = E$ with the standard action (also denoted by E), $\operatorname{Aut}_{\operatorname{Ker}\mathscr{C}}(E) = U(\mathscr{C})$ is an abelian group under $\circ$, and so $H^1(\Gamma, U(\mathscr{C}))$ is an abelian group. On the other hand, $k\operatorname{Rep}(E)$ is clearly a submonoid of $k\operatorname{Rep}(\mathscr{C})$ under ∇. Proposition (4.1) gives a bijection

$$k\operatorname{Rep}(E) \cong H^1(\Gamma, U(\mathscr{C})) \tag{4.2}$$

which we see, by (2.2), is an isomorphism of groups, so defines an injective homomorphism

$$\phi: H^1(\Gamma, U(\mathscr{C})) \to k\operatorname{Rep}(\mathscr{C}). \tag{4.3}$$

More generally, the bijections of (4.1) take the action of $H^1(\Gamma, U(\mathscr{C}))$ on $H^1(\Gamma, \operatorname{Aut}_{\operatorname{Ker}\mathscr{C}}(C))$ induced by θ_C to the action of $k\operatorname{Rep}(E)$ on $k\operatorname{Rep}(C,f)$ induced by $e: E \nabla C \to C$.

An object C of $\mathscr{C}$ is said to be *faithful* if θ_C is injective. If $(C,f) \in \operatorname{ob}\operatorname{Rep}(P - \mathscr{C})$, we have a diagram

(4.4)
$$\begin{tikzcd} & & \Gamma \arrow[d, "f"] & \\ U(\mathscr{C}) \arrow[r, "\theta_C"] & \mathrm{Aut}_{\mathscr{C}}(C) \arrow[r] & \mathrm{Aut}_{P-\mathscr{C}}(C) \arrow[r] & 1 \end{tikzcd}$$

with exact row, and if C is faithful this induces an extension of $U(\mathscr{C})$ by Γ. Note that by (3.3) the induced action of Γ on $U(\mathscr{C})$ is the standard one. Now extensions of the Γ-module $U(\mathscr{C})$ by Γ are classified by classes in $H^2(\Gamma, U(\mathscr{C}))$. This class depends only on the class of (C, f) in $k\,\mathrm{Rep}\,(P-\mathscr{C})$, thus if all objects of C are faithful it defines a map $\omega: k\,\mathrm{Rep}\,(P-\mathscr{C}) \to H^2(\Gamma, U(\mathscr{C}))$. By computing cocycles or by Baer addition, one sees that ω is a homomorphism.

THEOREM (4.5): *Let $\mathscr{C}$ be a Γ-monoidal category. Then the sequence of monoids*

$$1 \longrightarrow H^1(\Gamma, U(\mathscr{C})) \xrightarrow{\phi} k(\mathrm{Rep}\,(\mathscr{C})) \xrightarrow{\mu} k(\mathrm{Rep}\,(P-\mathscr{C})),$$

where μ is the obvious quotient map, is exact.

If all objects of $\mathscr{C}$ are faithful, $\mathrm{Im}\,\mu = \mathrm{Ker}\,\omega$. *If moreover either $k(\mathscr{C})$ is a group or there is an equivalence $J: \mathscr{C} \to \mathscr{C}$ of Γ-monoidal categories which acts by inversion on $U(\mathscr{C})$, then* $\mathrm{Im}\,\omega$ *is a group, so (μ, ω) is exact.*

PROOF: We have seen that ϕ is injective, and clearly $\mathrm{Im}\,\phi \subset \mathrm{Ker}\,\mu$. Suppose two objects (C_1, f_1), (C_2, f_2) of $\mathrm{Rep}\,\mathscr{C}$ define the same element of $\mathrm{Rep}\,(P-\mathscr{C})$. Then there is an isomorphism in $\mathrm{Rep}\,(P-\mathscr{C})$, which lifts to an isomorphism in $\mathscr{C}$, so by changing representatives, we may take $C_1 = C_2 = C$, say, and write $f_2(\gamma) = \theta_C(u(\gamma))f_1(\gamma)$ for suitable $u(\gamma)$. But then u is a 1-cocycle, defining a class in $H^1(\Gamma, U(\mathscr{C}))$. In view of our earlier remarks this suffices to establish exactness at $k(\mathrm{Rep}\,\mathscr{C})$.

Now if C is faithful and $(C, f) \in \mathrm{ob}\,\mathrm{Rep}\,(P-\mathscr{C})$, the corresponding class in $H^2(\Gamma, U(\mathscr{C}))$ is trivial if and only if the extension of $U(\mathscr{C})$ by Γ is trivial, i.e. in the diagram (4.4) f lifts to a homomorphism $\Gamma \to \mathrm{Aut}_{\mathscr{C}}(C)$. But this in turn is equivalent to (C, f) coming from an object of $\mathrm{Rep}\,\mathscr{C}$. Thus $\mathrm{Im}\,\mu = \mathrm{Ker}\,\omega$.

If $C \nabla D$ is equivalent to E, it is easy to see that $\omega(k\,\mathrm{Rep}\,P-D) = -\omega(k\,\mathrm{Rep}\,P-C)$, hence if $k(\mathscr{C})$ is a group so also is $\mathrm{Im}\,\omega$. Similarly if J is as described, then $k\,\mathrm{Rep}\,(P-\mathscr{C})$ has an automorphism J with $\omega \circ J(x) = -\omega(x)$, and again $\mathrm{Im}\,\omega$ is a group. For J acting by inversion means $J(u) = u^{-1}$ for $u \in U$, so J acts by -1 on the cohomology group in additive notation. Exactness now follows on taking note of Section 12 C. 2.

COROLLARY (4.6): *If $\mathscr{C}$ is a group-like Γ-monoidal category, there is an exact sequence (of groups)*

$$1 \to H^1(\Gamma; U(\mathscr{C})) \to k(\mathrm{Rep}(\mathscr{C})) \to H^0(\Gamma; k_\Gamma(\mathscr{C})) \to H^2(\Gamma; U(\mathscr{C})).$$

This follows at once by substituting from (3.4) in (4.5). The exact sequence of (4.5) is clearly natural for morphisms in $\Gamma - \mathscr{MC}$ and even, as homotopic morphisms induce the same map on each term, in $\Gamma - \mathscr{HMC}$. In particular, the inclusion $\mathrm{Inv}\,\mathscr{C} \subset \mathscr{C}$ induces a map of exact sequences, which is an isomorphism on the $H^i(\Gamma; U(\mathscr{C}))$.

Now consider an object (C,f) of $\mathrm{Rep}\,\mathscr{C}$. Then $k\,\mathrm{Rep}(C,f)$ is a set with base point (C,f). If $(C, P-f)$ is the image of (C,f) in $\mathrm{Rep}\,P-\mathscr{C}$ we shall write $kP-\mathrm{Rep}(C,f)$ (rather than $k\,\mathrm{Rep}(C, P-f)$) for the set of forms of $(C, P-f)$ (defined with respect to $P-\mathscr{C}$), and $\mathrm{Aut}_{\mathrm{Rep}(P-\mathscr{C})}(C,f)$ for the group of automorphisms of $(C, P-f)$.

THEOREM (4.7): *Let (C,f) be an object of $\mathrm{Rep}(\mathscr{C})$ with C faithful. Then we have an exact sequence of based sets*

$$1 \longrightarrow H^0(\Gamma, U(\mathscr{C})) \xrightarrow{\alpha_0} \mathrm{Aut}_{\mathrm{Rep}(\mathscr{C})}(C,f) \xrightarrow{\beta_0} \mathrm{Aut}_{\mathrm{Rep}(P-\mathscr{C})}(C,f)$$

$$\xrightarrow{\delta_0} H^1(\Gamma, U(\mathscr{C})) \xrightarrow{\alpha_1} k\,\mathrm{Rep}(C,f) \xrightarrow{\beta_1} kP-\mathrm{Rep}\,(C,f)$$

$$\xrightarrow{\delta_1} H^2(\Gamma, U(\mathscr{C}))$$

Up to $H^1(\Gamma, U(\mathscr{C}))$ this is also an exact sequence of Abelian groups. Moreover $H^1(\Gamma, U(\mathscr{C}))$ acts as a permutation group on $k\,\mathrm{Rep}(C,f)$, $\mathrm{Im}\,\alpha_1$ is the orbit of the base point, and the fibres of β_1 are the orbits of $H^1(\Gamma, U(\mathscr{C}))$.

PROOF: The exact sequence with central kernel

$$1 \to U(\mathscr{C}) \to \mathrm{Aut}_{\mathrm{Ker}\,\mathscr{C}}(C) \to \mathrm{Aut}_{\mathrm{Ker}\,P-\mathscr{C}}(C) \to 1$$

of Γ-groups with action coming from (C,f) gives rise to the usual seven-term exact sequence of non Abelian cohomology with all the properties mentioned in the Theorem. All that remains is to identify

$$H^0(\Gamma, \mathrm{Aut}_{\mathrm{Ker}\,\mathscr{C}}(C)) = \mathrm{Aut}_{\mathrm{Rep}(\mathscr{C})}(C)$$

(obvious) and $H^1(\Gamma, \mathrm{Aut}_{\mathrm{Ker}\,\mathscr{C}}(C)) = k\,\mathrm{Rep}(C,f)$ (Proposition 4.1), and analogously for $P-\mathscr{C}$ in place of $\mathscr{C}$.

5. Strict coherence and precise product

We return to the original discussion of monoidal categories, but now ask whether the natural equivalences a, c and e can be taken as the identity, if not in $\mathscr{C}$, at least in some equivalent object of $\mathscr{HMC}$. We call a monoidal category with a, c and e identities *precise*. We will show that this is related to a sharpening of the notion of coherence, which we will call strict coherence. There is an analogous problem for functors. As we now seek to extend, not merely review the earlier work, we recall the full definitions.

We have the category $\mathscr{C}$, and write $\mathscr{C}^n$ for the product $\mathscr{C} \times \cdots \times \mathscr{C}$ (n factors), and

$$\mathscr{O}\!\mathscr{p}(\mathscr{C}) = \bigcup_{m,n \geqq 0} \mathscr{O}\!\mathscr{p}_{m,n}(\mathscr{C}) = \bigcup_{m,n \geqq 0} \mathscr{H}\!om(\mathscr{C}^m, \mathscr{C}^n)$$

for the category of operators whose objects are functors $\mathscr{C}^m \to \mathscr{C}^n$ and morphisms are natural transformations. Here, $\mathscr{C}^0$ is the category with only one morphism. There are natural notions of sum and composition of operators

$$+: \mathscr{O}\!\mathscr{p}_{m_1,n_1}(\mathscr{C}) \times \mathscr{O}\!\mathscr{p}_{m_2,n_2}(\mathscr{C}) \to \mathscr{O}\!\mathscr{p}_{m_1+m_2,n_1+n_2}(\mathscr{C})$$

$$*: \mathscr{O}\!\mathscr{p}_{n,p}(\mathscr{C}) \times \mathscr{O}\!\mathscr{p}_{m,n}(\mathscr{C}) \to \mathscr{O}\!\mathscr{p}_{m,p}(\mathscr{C}).$$

Also, the symmetric group $\mathscr{T}_m$ acts by permutations on the category $\mathscr{C}^m$, hence on the categories $\mathscr{O}\!\mathscr{p}_{m,n}(\mathscr{C})$ and $\mathscr{O}\!\mathscr{p}_{l,m}(\mathscr{C})$.

Suppose now given a product $\nabla \in \operatorname{ob} \mathscr{O}\!\mathscr{p}_{2,1}(\mathscr{C})$ and unit $E \in \operatorname{ob} \mathscr{O}\!\mathscr{p}_{0,1}(\mathscr{C})$, with natural equivalences a, c, e as before: we do not yet make any assumption about coherence, as we wish to recall the definition. We define a subcategory $\mathscr{L}(\mathscr{C})$ of $\mathscr{O}\!\mathscr{p}(\mathscr{C})$. The set of objects is the least set containing ∇, E, $1_{\mathscr{C}} \in \operatorname{ob} \mathscr{O}\!\mathscr{p}_{1,1}(\mathscr{C})$ and projection on the first factor, $p \in \mathscr{O}\!\mathscr{p}_{2,1}(\mathscr{C})$, and closed under sum, composition and permutations. The set of morphisms is the least set containing the identity maps of these objects, $a \in \mathscr{O}\!\mathscr{p}_{3,1}(\mathscr{C})$, $c \in \mathscr{O}\!\mathscr{p}_{2,1}(\mathscr{C})$, $e \in \mathscr{O}\!\mathscr{p}_{1,1}(\mathscr{C})$ and closed under sum, $*$-composition, ordinary composition, permutations and inverses (each of the above is invertible).

We now define $(\mathscr{C}, \nabla, E, a, c, e)$ to be *coherent* if for any two objects $F_1, F_2 \in \mathscr{O}\!\mathscr{p}(\mathscr{C})$, there is at most one morphism $F_1 \to F_2$ in $\mathscr{L}(\mathscr{C})$, or equivalently, if the only morphisms $F \to F$ in $\mathscr{L}(\mathscr{C})$ are the identities. This is the rigorous definition to which we referred earlier. We now generalise it by allowing repetitions of the variables.— If we modify the above by including the diagonal functor $\Delta \in \mathscr{O}\!\mathscr{p}_{1,2}(\mathscr{C})$, we obtain a larger category $\mathscr{L}^+(\mathscr{C})$. If this has the property just described, we call $\mathscr{C}$ *strictly coherent.*

Some comments are in order here. First, note that $c\Delta$ defines a morphism in $\mathscr{L}^+(\mathscr{C})$ of $\nabla \circ \Delta \in \mathscr{O}\!\mathscr{p}_{1,1}(\mathscr{C})$ to itself, so a necessary condition for strict coherence is

(5.1) $\qquad c_{C,C} = 1 : C \nabla C \to C \nabla C \qquad$ for any $C \in \mathrm{ob}\,\mathscr{C}$.

This shows that, whereas categories with product arising naturally are nearly always coherent, the property of strict coherence is comparatively rare. For example, if M is a module, interchanging the copies of M in $M \oplus M$ is never the identity except when M is zero. On the other hand, if M is a rank one projective over a commutative ring R, the interchange in $M \otimes M$ is the identity. Our other main example will be the subcategory of $\mathscr{B}\imath(R)$ formed by Azumaya algebras.

THEOREM (5.2): *$\mathscr{C}$ is strictly coherent if and only if it is coherent and* (5.1) *holds.*

PROOF: Coherence is necessary since $\mathscr{L}(\mathscr{C}) \subset \mathscr{L}^+(\mathscr{C})$: we have already seen that (5.1) is needed.

Next we describe objects θ of $\mathscr{L}^+_{m,1}(\mathscr{C})$. $\theta : \mathscr{C}^m \to \mathscr{C}$ is a functor such that $\theta(C_1, \cdots, C_m)$ is the product (under ∇) of the objects C_i and E, with some multiplicities, and with some bracketing. We can thus write $\theta = F \circ \Omega$, where $\Omega : \mathscr{C}^m \to \mathscr{C}^p$ is an iterated diagonal, reproducing the C_i with the desired multiplicities, and F is an object of $\mathscr{L}_{p,1}(\mathscr{C})$. Further, it follows from our definition of morphisms in $\mathscr{L}^+(\mathscr{C})$ that any automorphism β of θ is of the form $\alpha\Omega$, where α has domain F.

Now assume $\mathscr{C}$ coherent and that (5.1) holds. We seek to show that any β as above is the identity. If α also has codomain F, then $\alpha = 1$ by coherence of $\mathscr{C}$, so $\beta = 1$. In general, since $\alpha\Omega$ is an automorphism, the codomain of α must be obtained from F by a permutation $\sigma \in \mathscr{T}_p$ which permutes (for each i) the entries where Ω substitutes C_i: we write it as $F\sigma$. Write $\sigma = \tau_1 \cdots \tau_k$ as a product of transpositions, each satisfying the same condition. Then for $0 \leqq j \leqq k$,

$$F_j = F\tau_1 \cdots \tau_j$$

is equivalent to F, so we have morphisms $\alpha_j : F_{j-1} \to F_j$ and can choose α_k so that $\alpha_1 \circ \cdots \circ \alpha_k = \alpha$. It thus suffices to show $\alpha_j\Omega = 1$, i.e. we can assume that σ is a transposition, say of r and s.

Choose a functor $G \in \mathscr{L}_{p,1}(\mathscr{C})$ of the form

$$G(A_1, \cdots, A_p) = (A_r \nabla A_s) \nabla H(A_1, \cdots, A_p)$$

and an isomorphism $\rho : F \to G$ in $\mathscr{L}$. Applying σ gives $\rho\sigma : F\sigma \to G\sigma$, and $G\sigma(A_1, \cdots) = (A_s \nabla A_r) \nabla H$. Thus

$$F \xrightarrow{\ \rho\ } G \xrightarrow{c \nabla 1} G\sigma \xrightarrow{(\rho\sigma)^{-1}} F\sigma$$

is an equivalence in $\mathscr{L}$; by coherence, it equals α. Thus

$$\beta = \alpha\Omega = (\rho\sigma\Omega)^{-1} \circ ((c \nabla 1)\Omega) \circ (\rho\Omega).$$

But $(c \nabla 1)\Omega = 1$ by (5.1), and $\rho\sigma\Omega = \rho\Omega$ by the condition on σ. Hence β is the identity.

Now since $\mathscr{L}^+(\mathscr{C})$ contains projections and diagonal maps, we see that the natural map

$$\mathscr{L}^+_{m,n}(\mathscr{C}) \to \mathscr{L}^+_{m,1}(\mathscr{C}) \times \cdots \times \mathscr{L}^+_{m,1}(\mathscr{C}) \qquad (n \text{ factors})$$

given by projections is an equivalence. Thus the result for arbitrary (m, n) follows from that for $(m, 1)$.

In respect of this last argument, we observe that the same does not hold for $\mathscr{L}$, but that any object of $\mathscr{L}_{m,n}$ can be uniquely expressed (after a suitable permutation in $\mathscr{T}_m$) as a sum of objects of $\mathscr{L}_{m_i,1}$ ($\sum m_i = m$), and that any morphism from it then respects the splitting, so here again coherence in general follows automatically from the case $n = 1$.

THEOREM (5.3): *Let $(\mathscr{C}, \nabla, E, a, c, e)$ be strictly coherent, N a set, $\omega: N \to \operatorname{ob}\mathscr{C}$ a map, X_N the free abelian monoid on N. Then there exist a precise monoidal category $\mathscr{C}'$, with $\operatorname{ob}\mathscr{C}' = X_N$, and a full and faithful functor $T: \mathscr{C}' \to \mathscr{C}$ forming an $\mathscr{M}\mathscr{C}$-morphism, such that ω is the composite*

$$N \subset X_N = \operatorname{ob}\mathscr{C}' \xrightarrow{T} \operatorname{ob}\mathscr{C}.$$

This determines $\mathscr{C}'$ and T up to equivalence. If the classes of $\omega(N)$ generate $k(\mathscr{C})$, T is an equivalence.

The corresponding result to get a, e identities has been independently obtained by D. W. Anderson, by essentially the same method: this will appear in [1].

PROOF: Choose an order for N. Then any $u \in X_N$ can be uniquely expressed as a product in increasing order

$$u = n_1 n_2 \cdots n_r;$$

we define $T(u)$ to be the left normalised product

$$T(u) = T(u, \omega) = (\cdots((\omega(n_1) \nabla \omega(n_2)) \nabla \omega(n_3)) \cdots \nabla \omega(n_r)):$$

the empty product (defining $T(1, \omega)$) is interpreted as E.

We identify $\mathscr{C}'(u, v)$ with $\mathscr{C}(T(u, \omega), T(v, \omega))$: then T is full and faithful, and the composition in $\mathscr{C}'$ is determined (uniquely) by that in $\mathscr{C}$. It remains to investigate products in $\mathscr{C}'$. Since any one formula will involve only a finite number of elements of N, it suffices for the rest of the proof to consider the case of N finite.

We will now allow $\omega: N \to \operatorname{ob}\mathscr{C}$ to vary or equivalently consider ω as a variable object of $\mathscr{C}^N$ (where we identify the set N with its cardinal). Then for u fixed, $T(u, \omega)$ defines a functor $\mathscr{C}^N \to \mathscr{C}$ belonging to $\mathscr{L}^+_{N,1}(\mathscr{C})$.

Note that $T(u, \omega) \nabla T(v, \omega)$ and $T(uv, \omega)$ coincide modulo equivalences coming from a, c and e. In other words, there is an equivalence

$$P = P(u, v, \omega)\colon T(u, \omega) \nabla T(v, \omega) \to T(uv, \omega)$$

in $\mathscr{L}^{+}_{N,1}(\mathscr{C})$. If we now define ∇ on objects of $\mathscr{C}'$ by the multiplication in X_N, we claim that there is a unique way to extend it to a functor yielding a precise monoidal category, and such that $(T, P, 1_E)$ is an $\mathscr{MC}$-morphism.

Indeed, if $f\colon u \to u'$ and $g\colon v \to v'$, commutativity of

$$\begin{tikzcd}
T(u) \nabla T(v) \arrow[r, "P"] \arrow[d, "T(f) \nabla T(g)"] & T(uv) \arrow[d, "T(f \nabla g)"] \\
T(u') \nabla T(v') \arrow[r, "P"] & T(u'v')
\end{tikzcd}$$

yields a unique value of $T(f \nabla g)$, hence of $f \nabla g$, such that P is natural. On account of the uniqueness, it is easy to see that this makes $\nabla\colon \mathscr{C}' \times \mathscr{C}' \to \mathscr{C}'$ into a functor. Now to prove this product precise, it is enough to see that the identity maps define equivalences a, c and e: i.e. that

$$f \nabla (g \nabla h) = (f \nabla g) \nabla h, \qquad f \nabla g = g \nabla f \quad \text{and} \quad E \nabla f = f.$$

As T is faithful, it is enough to prove these after after applying T.

We first consider the axioms (1.7)–(1.9). In each case, if we consider the diagram as functor of ω, we see that we have two natural transformations between the same pair of objects in $\mathscr{L}^{+}_{p,1}(\mathscr{C})$ ($p = 3$ resp. 2 resp. 1), which coincide by strict coherence.

Now to prove $f \nabla g = g \nabla f$, consider the diagram

$$\begin{tikzcd}
T(uv) \arrow[d, "T(f \nabla g)"] & T(u) \nabla T(v) \arrow[l, "P"'] \arrow[r, "c"] \arrow[d, "Tf \nabla Tg"] & T(v) \nabla T(u) \arrow[r, "P"] \arrow[d, "Tg \nabla Tf"] & T(vu) \arrow[d, "T(g \nabla f)"] \\
T(u'v') & T(u') \nabla T(v') \arrow[l, "P"'] \arrow[r, "c"] & T(v') \nabla T(u') \arrow[r, "P"] & T(v'u')
\end{tikzcd}$$

The outside squares commute by definition of $f \nabla g$, $g \nabla f$; the inner one by naturality of c. By (1.8), the composite map along each row is the identity. The desired conclusion follows. The other formulae follow similarly from (1.7) and (1.9) respectively.

The construction was forced on us uniquely at each stage, except the definition of T on objects. It is easy to see that any other suitable definition would give an equivalent result. The final assertion follows from a general category-theoretic argument, see e.g. [11, p. 52].

There is an analogous notion of precision for morphisms. We call an $\mathscr{MC}$-morphism (F, p, q), or simply the functor F, *precise* if p and q are identity transformations. This is rather simpler to deal with.

THEOREM (5.4): *Let $\mathscr{C}, \mathscr{D}$ be precise monoidal categories; $(F, p, q): \mathscr{C} \to \mathscr{D}$ an $\mathscr{M}\mathscr{C}$-morphism. Then there exist a precise monoidal category $\mathscr{C}'$ and precise functors $F': \mathscr{C}' \to \mathscr{D}$, $G: \mathscr{C}' \to \mathscr{C}$ with G an equivalence and $(F, p, q) \circ G$ homotopic to F'.*

PROOF: Define $\mathscr{C}'$ as follows. An object of $\mathscr{C}'$ is a triple (C, D, δ) with $C \in \operatorname{ob} \mathscr{C}$, $D \in \operatorname{ob} \mathscr{D}$ and $\delta: FC \to D$ an isomorphism. A morphism

$$(f, g): (C_1, D_1, \delta_1) \to (C_2, D_2, \delta_2)$$

consists of morphisms $f: C_1 \to C_2$, $g: D_1 \to D_2$ with

$$\delta_2 F(f) = g\delta_1.$$

It is trivial to check that we have a category.

We define a product by

$$(C_1, D_1, \delta_1) \nabla (C_2, D_2, \delta_2) = (C_1 \nabla C_2, D_1 \nabla D_2, (\delta_1 \nabla \delta_2) p_{C_1, C_2}^{-1}),$$
$$(f_1, g_1) \nabla (f_2, g_2) = (f_1 \nabla f_2, g_1 \nabla g_2),$$

with unit (E, E, q). Precise associativity, commutativity and unit follow from (1.7)–(1.9).

The forgetful functors

$$F'(C, D, \delta) = D \qquad F'(f, g) = g$$
$$G(C, D, \delta) = C \qquad G(f, g) = f$$

are now clearly precise, and δ yields the desired homotopy.

We now reformulate this result. Denote by $\mathscr{S}\mathscr{M}\mathscr{C}$ the full subcategory of $\mathscr{M}\mathscr{C}$ whose objects are strictly coherent, and by $\mathscr{H}\mathscr{S}\mathscr{M}\mathscr{C}$ its homotopy category. Denote by $\mathscr{F}\mathscr{M}\mathscr{C}$ the subcategory of $\mathscr{S}\mathscr{M}\mathscr{C}$ consisting of precise monoidal categories, and precise functors. We have homotopies here, too, and write $\mathscr{H}\mathscr{F}\mathscr{M}\mathscr{C}$ for the *category of fractions* of the homotopy category of $\mathscr{F}\mathscr{M}\mathscr{C}$, where equivalences are inverted [7].

THEOREM (5.5): *Inclusion induces an equivalence*

$$\mathscr{H}\mathscr{F}\mathscr{M}\mathscr{C} \to \mathscr{H}\mathscr{S}\mathscr{M}\mathscr{C}.$$

Thus to define a homotopy functor on $\mathscr{S}\mathscr{M}\mathscr{C}$, it suffices to define one on $\mathscr{F}\mathscr{M}\mathscr{C}$ which takes equivalences (of categories) to equivalences (in the target category).

PROOF: The inclusion $\mathscr{F}\mathscr{M}\mathscr{C} \subset \mathscr{S}\mathscr{M}\mathscr{C}$ respects homotopy, so induces a functor of homotopy categories. As any equivalence is invertible in $\mathscr{H}\mathscr{S}\mathscr{M}\mathscr{C}$ (a full subcategory of $\mathscr{H}\mathscr{M}\mathscr{C}$), we have a functor as asserted.

By Theorem (5.3), every object of $\mathscr{H}\mathscr{S}\mathscr{M}\mathscr{C}$ is equivalent in it to an object of $\mathscr{H}\mathscr{F}\mathscr{M}\mathscr{C}$. It thus suffices to consider two precise monoidal categories $\mathscr{C}_1, \mathscr{C}_2$. Given a functor $F:\mathscr{C}_1 \to \mathscr{C}_2$, by (5.4) we can find a precise monoidal category $\mathscr{C}_3$ and functors $F_1:\mathscr{C}_3 \to \mathscr{C}_1$, $F_2:\mathscr{C}_3 \to \mathscr{C}_2$ with F_1 an equivalence and $F \circ F_1 \simeq F_2$. Then $F_2 \circ F_1^{-1}$ defines a morphism in $\mathscr{H}\mathscr{F}\mathscr{M}\mathscr{C}$ whose image is the class of F. Thus our functor is surjective on morphisms.

Now observe that every morphism in $\mathscr{H}\mathscr{F}\mathscr{M}\mathscr{C}$ is of the form $F_2 \circ F_1^{-1}$ above. Indeed, this is clear provided the set of such morphisms is closed under composition. But given

$$\begin{tikzcd} & \mathscr{C}_4 \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & \mathscr{C}_5 \arrow[dl, "F_3"'] \arrow[dr, "F_4"] & \\ \mathscr{C}_1 & & \mathscr{C}_2 & & \mathscr{C}_3 \end{tikzcd}$$

as F_3 is an equivalence, there exists $F:\mathscr{C}_4 \to \mathscr{C}_5$ with $F_3 \circ F \simeq F_2$; then applying (5.4), we obtain the desired conclusion.

Finally, suppose that the precise functors

$$\begin{tikzcd} & \mathscr{C}_3 \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ \mathscr{C}_1 & & \mathscr{C}_2 \end{tikzcd} \qquad \begin{tikzcd} & \mathscr{C}_4 \arrow[dl, "F_3"'] \arrow[dr, "F_4"] & \\ \mathscr{C}_1 & & \mathscr{C}_2 \end{tikzcd}$$

with F_1, F_3 equivalences, define the same morphism in $\mathscr{H}\mathscr{S}\mathscr{M}\mathscr{C}$. This is equivalent to existence of a morphism $F:\mathscr{C}_3 \to \mathscr{C}_4$ in $\mathscr{S}\mathscr{M}\mathscr{C}$ with $F_3 \circ F \simeq F_1$ and $F_4 \circ F \simeq F_2$. By (5.4), we can find a precise monoidal category $\mathscr{C}_5$ and precise functors $F_5:\mathscr{C}_5 \to \mathscr{C}_3$, $F_6:\mathscr{C}_5 \to \mathscr{C}_4$ with F_5 an equivalence and $F \circ F_5 \simeq F_6$. Thus $F_2 \circ F_5 \simeq F_4 \circ F \circ F_5 \simeq F_4 \circ F_6$ and $F_1 \circ F_5 \simeq F_3 \circ F \circ F_5 \simeq F_3 \circ F_6$. Hence in $\mathscr{H}\mathscr{F}\mathscr{M}\mathscr{C}$,

$$\begin{aligned} F_2 \circ F_1^{-1} &= F_2 \circ F_5 \circ F_5^{-1} \circ F_1^{-1} && (F_5 \text{ an equivalence}) \\ &= F_4 \circ F_6 \circ (F_3 \circ F_6)^{-1} && (\text{homotopies}) \\ &= F_4 \circ F_3^{-1} && (F_6 \text{ an equivalence}). \end{aligned}$$

So our functor is injective on morphisms, which concludes the proof.

NOTE: This argument is close to constructing a calculus of fractions in the sense of Gabriel and Zisman [7]. However, it seems simpler to proceed as above than to verify their axioms.

In fact, even this result can be sharpened.

Proposition (5.6): *Any functor on $\mathcal{FMC}$ which preserves equivalences is a homotopy functor.*

Similar results hold in other categories of categories.

Proof: The idea of the proof is as follows. Consider homotopies in $\mathcal{FMC}, X: F_0 \simeq F_1: \mathcal{C} \to \mathcal{D}$, with $\mathcal{C}$ fixed. We show that there is a universal example

$$Y: i_0 \simeq i_1: \mathcal{C} \to \mathcal{C} \times I$$

with i_0, i_1 equivalences, and that there is a (precise) functor $p: \mathcal{C} \times I \to \mathcal{C}$ with $p \circ i_0 = 1 = p \circ i_1$. Thus any functor on $\mathcal{FMC}$ which preserves equivalences must take the same value on i_0 and i_1, hence on any two homotopic morphisms.

It is easy to see that there is a universal example, and that we can write $\mathrm{ob}(\mathcal{C} \times \mathrm{I}) = \mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{C}$ and $i_0(C) = C \times E$, $i_1(C) = E \times C$. Among the morphisms $(A, B) \to (C, D)$ in $\mathcal{C} \times I$ is, for any $f: A \nabla B \to C \nabla D$ in $\mathcal{C}$, $\langle f \rangle = (1_C \nabla Y_D) \circ (f \times 1_E) \circ (1_A \nabla Y_B)^{-1}$.

The set S of these morphisms clearly contains the Y_A, and is closed under composition. Since Y satisfies (1.10), S is closed under ∇. Further, S contains any morphism $f_1 \times f_2$ of $\mathcal{C} \times \mathcal{C}$: it suffices to consider the cases $f_1 = 1, f_2 = 1$ and add, and these cases are clear, taking $f = f_2$ resp. f_1. By universality, S contains all morphisms of $\mathcal{C} \times I$.

We define $p: \mathcal{C} \times I \to \mathcal{C}$ to be ∇ on $\mathcal{C} \times \mathcal{C}$ and with $pYA = A$. This is compatible with the universal relations (1.10), (1.11) which are all Y has to satisfy. Clearly $p \circ i_0 = p \circ i_1 = 1_{\mathcal{C}}$. Now p is surjective on objects; and the above description of morphisms shows it is full. Finally, it is faithful, for as $p\langle f \rangle = f$, if $p\langle f \rangle = p\langle g \rangle$ we must have $f = g$ and so $\langle f \rangle = \langle g \rangle$.

Corollary (5.7): *We can identify $\mathcal{HFMC}$, and hence also $\mathcal{HSMC}$ with the category of fractions of $\mathcal{FMC}$ in which equivalences are inverted. Thus any functor on $\mathcal{FMC}$ which preserves equivalences extends (essentially uniquely) to a homotopy functor on $\mathcal{SMC}$.*

6. Group categories

We next apply the ideas of the preceding section to the 'group-like' categories of Section 2: here somewhat stronger results hold. We then discuss the form taken by the results in the graded case.

We begin by analysing the notion of strict coherence.

Lemma (6.1): *Let $\mathcal{C}$ be a group-like monoidal category. Then the natural equivalences $c'_{A,B}: A \nabla B \to B \nabla A$ are given by*

$$c'_{A,B} = c_{A,B}\theta_{A \nabla B}(u(A, B))$$

for the various maps

$$u\colon k(\mathscr{C})\times k(\mathscr{C}) \to U(\mathscr{C}).$$

c' satisfies
(1.2) *if $u(B, A) = u(A, B)^{-1}$ for all A, B*
(1.3) *if (A, B) depends linearly on A*
(1.4) *if $u(E, E) = 1$*
(1.6) *if $u(A, E) = 1$ for all A;*
thus (a, c', e) is coherent if and only if u is skew-symmetric bilinear.

Proof: It follows from (2.2 iv) that any isomorphism $c'\colon A \nabla B \to B \nabla A$ is of the form $c_{A,B}\theta_{A\nabla B}(u)$ for some $u \in U$. As to naturality, if $s\colon A \to A'$ and $t\colon B \to B'$ are isomorphisms, then

$$\begin{tikzcd} A \nabla B \arrow[r, "c_{A,B}\theta_{A\nabla B}(u)"] \arrow[d, "s\nabla t"] & A \nabla B \arrow[d, "s\nabla t"] \\ A' \nabla B' \arrow[r, "c_{A',B'}\theta_{A'\nabla B}(v)"] & A' \nabla B' \end{tikzcd}$$

commutes if and only if $u = v$, as follows quickly from (2.2 ii). This proves our first assertion.

The next statements are easy verifications: for example,(1.3) for c' is obtained from the same diagram for c by composing at appropriate points with $\theta(A \nabla B, C)$ (upper row) $\theta(B, C)$ and $\theta(A, C)$ (lower row). In view of (2.2 ii) and (2.2 iii), we can move these θ's up to the front. The result now follows.

We now seek to choose u so that c' is strictly coherent.

Proposition (6.2): *For any $A \in \operatorname{ob}\mathscr{C}$, there is a unique $v = v(A)$ of order 2 in $U(\mathscr{C})$ with $c_{A,A} = \theta_{A\nabla A}(v)$. The map $A \to v(A)$ defines a homomorphism $k(\mathscr{C}) \to {}_2U(\mathscr{C})$. If c is replaced by c', v is changed to v' with $v'(A) = v(A)u(A, A)$. We can choose u to obtain any homomorphism $k(\mathscr{C}) \to {}_2U(\mathscr{C})$; in particular, we can make c' strictly coherent.*

Observe that we can also suppose c' not strictly coherent, unless $U(\mathscr{C})$ has no element of order 2, or $k(\mathscr{C})$ is divisible by 2. Observe also that the result depends crucially on $k(\mathscr{C})$ being a group: no amount of fiddling with natural equivalences will make the interchange in $R \oplus R$ the identity ($R \neq 0$), for example.

Proof: Existence and uniqueness of v follows again from (2.2 iv); $v^2 = 1$ from (1.2). The homomorphic property of v follows from (2.2) and the commutative diagram

$$\begin{tikzcd}
(A \nabla A) \nabla (B \nabla B) \arrow[r, "c_{A,A} \nabla c_{B,B}"] \arrow[d, "M"] & (A \nabla A) \nabla (B \nabla B) \arrow[d, "M"] \\
(A \nabla B) \nabla (A \nabla B) \arrow[r, "c_{A\nabla B, A\nabla B}"] & (A \nabla B) \nabla (A \nabla B),
\end{tikzcd}$$

where $M\{(A \nabla B) \nabla (C \nabla D)\} = (A \nabla C) \nabla (B \nabla D)$ and commutativity follows from coherence. Next,

$$c'_{A,A} = c_{A,A}\theta_{A \nabla A}(u(A, A)) = \theta_{A \nabla A}(v(A)u(A, A)).$$

It remains to show that any homomorphism $v: k(\mathscr{C}) \to {}_2U(\mathscr{C})$ can be obtained as diagonal of a skew-symmetric bilinear form u. Choose a base $\{u_\alpha\}$ of ${}_2U(\mathscr{C})$ as vector space over the field F_2 with two elements, and write

$$v(A) = \prod u_\alpha^{b_\alpha(A)}$$

where, for given A, all but finitely many $b_\alpha(A)$ vanish. We can now put

$$u(A, B) = \prod u_\alpha^{b_\alpha(A)\, b_\alpha(B)}.$$

Not only can we make a group-like category strictly coherent, and then precise, but we can get the inverse precise too.

THEOREM (6.3): *Let $\mathscr{C}$ be a strictly coherent group-like category. Then $\mathscr{C}$ is equivalent in $\mathscr{HMC}$ to a category $\mathscr{C}'$ with precise product for which the objects of $\mathscr{C}'$ form a group. Moreover, this group may be supposed free abelian.*

PROOF: Choose a set $\{x_\alpha\}_{\alpha \in I}$ of generators of the group $k(\mathscr{C})$. Choose objects C_α, D_α representing $x_\alpha, -x_\alpha$. Perform the construction of Theorem 5.3 with $N = \{A_\alpha, B_\alpha\}_{\alpha \in I}$. Thus we may suppose $\mathscr{C}$ a category with precise product, and ob $\mathscr{C}$ the free abelian monoid with generators A_α, B_α where we have isomorphisms $f_\alpha: A_\alpha \nabla B_\alpha \to E$. We wish to factor out by the f_α.

It will suffice to illustrate the argument in the case when I has one element. Then objects $X_{r,s}$ of $\mathscr{C}'$ are parametrised by pairs (r, s) of non-negative integers, with $X_{1,0} \to C$, $X_{0,1} \to D$ and $X_{r,s} \nabla X_{r',s'} = X_{r+r',s+s'}$. For each r, s we have

$$f \nabla X_{r,s}: X_{r+1,s+1} \to X_{r,s}$$

and composing these we obtain for each r, s, t a uniquely determined isomorphism from $X_{r+t,s+t}$ to $X_{r,s}$ and thus also to $X_{r-s,0}$ or $X_{0,s-r}$. This set of isomorphisms is, by definition, closed under composition. It is also closed under ∇, as we verify for

$$\operatorname{Hom}(X_{r+t,s+t}, X_{r,s}) \nabla \operatorname{Hom}(X_{r'+t',s'+t'}, X_{r',s'}) \to$$
$$\operatorname{Hom}(X_{r+r'+t+t',s+s'+t+t'}, X_{r+r',s+s'})$$

by induction on $t+t'$. Indeed, for $t=0$ there is nothing; for $t=1$, this reduces to the definition of our morphisms (recall that ∇ is precise), and the inductive step follows using naturality of ∇, e.g.

$$\begin{aligned} f \nabla f &= (E \circ f) \nabla (f \circ X_{1,1}) \\ &= (E \nabla f) \circ (f \nabla X_{1,1}) = f \circ (f \nabla X_{1,1}). \end{aligned}$$

Now form the quotient category identifying these isomorphisms to the identity. This inherits a product from ∇, still precise. But the set of objects is now the free abelian group on the image of $X_{1,0}$.

If $\mathscr{C}$ is a precise monoidal groupoid whose objects form a group, we will call $\mathscr{C}$ a *group category*. Group categories are particularly simple to work with, and we next give a partial analysis of their structure.

LEMMA (6.4):
(i) *Let $\mathscr{C}$ be a precise monoidal category. Then if $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathscr{C}$,*

$$g \nabla f = (g \circ f) \nabla B;$$

in particular, ∇ and $\circ$ coincide on $U(\mathscr{C})$.
(ii) *If $\mathscr{C}$ is a group category, the morphisms of $\mathscr{C}$ form an abelian group under ∇.*

PROOF: (i) We have

$$\begin{aligned} g \nabla f &= (g \circ B) \nabla (B \circ f) \\ &= (g \nabla B) \circ (B \nabla f) \\ &= (g \nabla B) \circ (f \nabla B) \\ &= (g \circ f) \nabla (B \circ B) = (g \circ f) \nabla B. \end{aligned}$$

(ii) The addition is well defined, and since a resp. c resp. e are identities, it is associative resp. commutative resp. E acts as zero. As to inverses, we have by (i) $f^{-1} \nabla f = A \nabla B$, so $f^{-1} \nabla (-A) \nabla (-B)$ is the desired inverse.

THEOREM (6.5): *Let $\mathscr{C}$ be a group category with* $\mathrm{ob}\,\mathscr{C}$ *free abelian. Then there exists a group category $\mathscr{C}_0$ with $U(\mathscr{C}_0) = 1$, and an isomorphism $\mathscr{C} \to U(\mathscr{C}) \times \mathscr{C}_0$.*

PROOF: We begin by analysing $\mathscr{C}$: consider first the group $(\mathscr{C}, +) = \mathscr{C}$ of (6.4 ii). There is an exact sequence (take domain and codomain)

$$0 \to U(\mathscr{C}) \to \mathscr{C} \to \mathrm{ob}\,\mathscr{C} \times \mathrm{ob}\,\mathscr{C} \to k(\mathscr{C}) \to 0.$$

Since identity maps form a subgroup, we also have

$$0 \to U(\mathscr{C}) \oplus \operatorname{ob}\mathscr{C} \to \mathscr{C} \to \operatorname{Ker}(\operatorname{ob}\mathscr{C} \to k(\mathscr{C})) \to 0.$$

As $\operatorname{ob}\mathscr{C}$, hence any subgroup, is free abelian, this sequence splits: let $(r, r'): \mathscr{C} \to U(\mathscr{C}) \oplus \operatorname{ob}\mathscr{C}$ be a retraction. Then $r: \mathscr{C} \to U(\mathscr{C})$ is a functor, for given f, g as in (6.4 i),

$$r(g \circ f) = r(g) \nabla r(f) \nabla r(-B) = r(g) \nabla r(f) = r(g) \circ r(f).$$

Clearly, r is a precise functor.

Now write $s = 1 - r$, i.e. $s(f) = f \nabla(-r(f))$. Since precise functors $\mathscr{C} \to \mathscr{C}$ form a group, s is also a precise functor. If f, g have the same domain and codomain, then $g = f \nabla u$ for some $u \in U$, so $r(g) = r(f) \nabla u$, and $s(g) = s(f)$. Thus s picks out a subcategory $\mathscr{C}_0$ of $\mathscr{C}$ with at most one morphism between any two objects, and $(r, s): \mathscr{C} \to U(\mathscr{C}) \times \mathscr{C}_0$ is an isomorphism, which we can regard as one of group categories.

Observe that $k(\mathscr{C}) = k(\mathscr{C}_0)$, and if we regard this as the (discrete) group category with all morphisms equivalences, then mapping each morphism of $\mathscr{C}_0$ to the class of its domain defines an equivalence $\mathscr{C}_0 \to k(\mathscr{C})$.

Corollary (6.6): *Two group-like categories are equivalent in $\mathscr{HSMC}$ if and only if their object and unit groups are isomorphic.*

For by (6.3), such a category is equivalent to a group category $\mathscr{C}$ with $\operatorname{ob}\mathscr{C}$ a group, and by the above $\mathscr{C}$ is equivalent to $U(\mathscr{C}) \times k(\mathscr{C})$.

This result does not quite trivialise the whole theory: the construction of the equivalence involved an essential choice. We do, however, obtain

Corollary (6.7): *Let $\mathscr{D}$ be a precise monoidal category, $\mathscr{C}$ a group category with* $\operatorname{ob}\mathscr{C}$ *free abelian. Then given two homomorphisms*

$$\phi: k(\mathscr{C}) \to k(\mathscr{D}), \qquad \psi: U(\mathscr{C}) \to U(\mathscr{D}),$$

there exists a precise functor $F: \mathscr{C} \to \mathscr{D}$ inducing them.

Proof: We can regard ψ itself as a precise functor which, using the decomposition $\mathscr{C} = U(\mathscr{C}) \times \mathscr{C}_0$, induces $F_1: \mathscr{C} \to \mathscr{D}$ which induces 0 on $k(\mathscr{C})$ and ψ on $U(\mathscr{C})$. As to ϕ, using freedom of $\operatorname{ob}\mathscr{C}$ and of $N\mathscr{C} = \operatorname{Ker}(\operatorname{ob}\mathscr{C} \to \mathrm{k}\mathscr{C})$, we lift

$$\operatorname{ob}\mathscr{C} \to k(\mathscr{C}) \to k(\mathscr{D})$$

to $l_1: \operatorname{ob}\mathscr{C} \to \operatorname{ob}\mathscr{D}$, and (identifying $N\mathscr{C}$ with the group of morphisms of $\mathscr{C}$ with domain E) lift $l_1|N\mathscr{C}$ to $l_2: N\mathscr{C} \to \mathscr{D}$ where any $l_2(x)$ has domain $E_{\mathscr{D}}$. (Note that the image lies in $\operatorname{Inv}\mathscr{D}$, which is a group category). Now for $f: C \to D$ in $\mathscr{C}$, set

$$F_2(f) = l_1(C) \nabla l_2(D - C):$$

this defines a precise functor which factors through $\mathscr{C}_0$, so induces 0 on $U(\mathscr{C})$ and ϕ on $k(\mathscr{C})$. We then take $F = F_1 \nabla F_2$.

However, two functors inducing the same map on k and on U need *not* be homotopic. Suppose $\mathscr{C}$, $\mathscr{D}$ group categories as above. Since we can subtract functors, it suffices to consider functors (or morphisms) inducing 0 on k and U. By (2.9), any such morphism is homotopic to one of the form $(0, p, 0)$. Here p is a map

$$p\colon \operatorname{ob}\mathscr{C} \times \operatorname{ob}\mathscr{C} \to U(\mathscr{D})$$

which, since it is natural, factors through $k(\mathscr{C}) \times k(\mathscr{C})$.Axiom (1.9) yields $p(E, A) = E$; (1.8) that p is symmetric, and (1.7) that p is a 2-cocycle. A homotopy changes p by a general coboundary. Hence the homotopy classes of morphisms are the 'symmetric' cohomology classes in $H^2(k(\mathscr{C}); U(\mathscr{D}))$. Since it is not our primary interest, we will not pursue the theory further here.

We now observe that the results of the Section 5 and (6.1)–(6.3) can be over to the graded case. Indeed, neither statements nor proofs of theorems need any change. Lemma (6.4) generalises to

LEMMA (6.8): *If $\mathscr{C}$ is a Γ-graded group category, the morphisms of $\mathscr{C}$ of a given grade γ form an abelian group.*

PROOF: As before, the addition is defined, is associative, and commutive, and $E(\gamma)$ acts as zero, and it remains to find inverses.

Let $f: X \to Y$ be an (iso)morphism of grade γ. Then $\{Y\} = \gamma\{X\}$ in $k_\Gamma(\mathscr{C})$, so $\{-Y\} = \gamma\{-X\}$, so there exists $f'\colon(-X) \to (-Y)$ of grade γ. Then $f \nabla f'$ is an automorphism of E. But any morphism $E \to E$ of grade γ is of the form $u \circ E(\gamma)$, $u \in U(\mathscr{C})$, and

$$\begin{aligned}(u \circ E(\gamma)) \nabla (u^{-1} \circ E(\gamma)) &= (u \nabla u^{-1}) \circ (E(\gamma) \nabla E(\gamma)) \\ &= 1 \circ E(\gamma) = E(\gamma),\end{aligned}$$

so $u^{-1} \circ E(\gamma)$ gives the desired inverse.

We conclude with a result which incorporates most of the work of the last two sections.

THEOREM (6.9): *Any functor on the category of Γ-group categories and precise functors, which preserves equivalences, extends canonically to a homotopy functor on the category of strictly coherent group-like Γ-monoidal categories.*

PROOF: In the same formal manner as (5.7) follows from (5.3), (5.4) and (5.6) this follows from (the Γ-graded version of) (5.3) and (6.3), (5.4) –

with the observation that if in the proof of (5.4), $\mathscr{C}$ and $\mathscr{D}$ are group categories, then so is $\mathscr{C}_1'$ as constructed – and a suitable analogue to (5.6). But again in the proof of (5.6), if $\mathscr{C}$ is a group category the same holds for $\mathscr{C} \times I$ (for $\mathrm{ob}(\mathscr{C} \times I) = \mathrm{ob}\,\mathscr{C} \times \mathrm{ob}\,\mathscr{C}$, and all morphisms are invertible).

The homotopy classification of Γ-group categories is less trivial than the ungraded case, and we will not undertake it here. See [16] for a different approach to this problem.

7. Cohomology of graded group categories

We shall suppose throughout this section that $\mathscr{C}$ is a Γ-graded group category, and will write $+$ instead of ∇ for the 'product'. We develop a cohomology theory, regarded as cohomology of Γ with coefficients in $\mathscr{C}$. To motivate the formulae, we first consider free Γ-graded groupoids. We shall call these, for short, 'Γ-groupoids' and similarly use the terms 'Γ-category', 'Γ-functor' etc. in the graded case.

Let X be a non-empty set. Consider pairs (c_1, c_2) of maps

$$c_1 : X \to \mathrm{ob}\,\mathscr{A}, \qquad c_2 : \Gamma \times X \to \mathscr{A},$$

for $\mathscr{A}$ a (stable) Γ-groupoid, such that $c_2(\gamma, x)$ has grade γ and domain $c_1(x)$. Clearly c_2 determines c_1. The universal object $(\mathscr{A}, c_1, c_2)$, the free Γ-groupoid on X, can be described as follows. Its objects are the symbols $\langle \gamma, x \rangle$ corresponding bijectively to $\Gamma \times X$. There exists a morphism $\langle \gamma, x \rangle \to \langle \delta, y \rangle$ only if $x = y$, and then it is unique of grade $\delta\gamma^{-1}$. Write (γ, x) for the morphism $\langle 1, x \rangle \to \langle \gamma, x \rangle$. Then the morphism $\langle \gamma, x \rangle \to \langle \delta, x \rangle$ is $(\delta, x) \circ (\gamma, x)^{-1}$.

Let $\mathscr{A}_n$ be the free Γ-groupoid on the n-fold Cartesian power Γ^n of Γ, $n \geqq 0$. This has objects

$$\langle \gamma_0, \gamma_1, \cdots, \gamma_n \rangle = \langle \gamma_0, (\gamma_1, \cdots, \gamma_n) \rangle$$

and morphisms

$$[\gamma_0, \gamma_1, \cdots, \gamma_n] = (\gamma_0, (\gamma_1, \cdots, \gamma_n))$$

and $[\delta, \gamma_1, \cdots, \gamma_n] \circ [\gamma_0, \gamma_1, \cdots, \gamma_n]^{-1}$. We define functors

$$\Delta_k = \Delta_k^n : \mathscr{A}_n \to \mathscr{A}_{n-1} \qquad (0 \leqq k \leqq n)$$

of Γ-graded groupoids by

$$\Delta_0 \langle 1, \gamma_1, \cdots, \gamma_n \rangle = \langle \gamma_1, \cdots, \gamma_n \rangle,$$

$$\Delta_i \langle 1, \gamma_1, \cdots, \gamma_n \rangle = \langle 1, \gamma_1, \cdots, \gamma_i \gamma_{i+1}, \cdots, \gamma_n \rangle \qquad (1 \leqq i < n)$$

$$\Delta_n \langle 1, \gamma_1, \cdots, \gamma_n \rangle = \langle 1, \gamma_1, \cdots, \gamma_{n-1} \rangle.$$

Since $\text{ob}\,\mathscr{A}_n$ is a free Γ-set, and $\mathscr{A}_{n-1}$ has at most one morphism with given source and target, this gives unique functors. Observe that

$$\Delta_i \Delta_j = \Delta_{j+1} \Delta_i \qquad (i \leqq j).$$

One can verify that

$$\Delta_0[\gamma_0, \cdots, \gamma_n] = [\gamma_0 \gamma_1, \gamma_2, \cdots, \gamma_n] \circ [\gamma_1, \gamma_2, \cdots, \gamma_n]^{-1}$$

$$\Delta_i[\gamma_0, \cdots, \gamma_n] = [\gamma_0, \cdots, \gamma_i \gamma_{i+1}, \cdots, \gamma_n] \qquad (1 \leqq i < n)$$

$$\Delta_n[\gamma_0, \cdots, \gamma_n] = [\gamma_0, \cdots, \gamma_{n-1}].$$

Now for $\mathscr{C}$ a Γ-category, define $C^{n+1}(\Gamma, \mathscr{C})$ to be the set of Γ-functors $\mathscr{A}_n \to \mathscr{C}$ $(n \geqq 0)$, or equivalently of maps $c_2: \Gamma^{n+1} \to \mathscr{C}$ as above. The Δ_i induce maps between these satisfying the usual simplicial identities. When $\mathscr{C}$ is a group category, each $C^{n+1}(\Gamma, \mathscr{C})$ is an abelian group, and we obtain a cochain complex in an obvious way.

Explicitly, for $n \geqq 0$, $C^{n+1}(\Gamma, \mathscr{C})$ is the group of maps $c: \Gamma^{n+1} \to \mathscr{C}$ such that

(i) the grade of $c(\gamma_0, \cdots, \gamma_n)$ is γ_0,

(ii) the domain of $c(\gamma_0, \cdots, \gamma_n)$ does not depend on γ_0.

The coboundary $\delta: C^n(\Gamma, \mathscr{C}) \to C^{n+1}(\Gamma, \mathscr{C})$ is given by

$$\delta c(\gamma_0, \cdots, \gamma_n) = c(\gamma_0 \gamma_1, \cdots, \gamma_n) \circ c(\gamma_1, \cdots, \gamma_n)^{-1} + \sum_{i=1}^{n-1} (-1)^i c(\gamma_0, \cdots, \gamma_i \gamma_{i+1}, \cdots, \gamma_n) + (-1)^n c(\gamma_0, \cdots, \gamma_{n-1}).$$

It is also convenient to introduce $C^0(\Gamma, \mathscr{C})$ as the group of morphisms c in $\mathscr{C}$ with domain E and grade 1. We define $\delta: C^0(\Gamma, \mathscr{C}) \to C^1(\Gamma, \mathscr{C})$ by

$$\delta c(\gamma) = c \circ E(\gamma) \circ c^{-1}.$$

It is easy to check that $\delta^2 = 0$.

We now obtain certain cohomology exact sequences. In the simplest case, when each isomorphism class of $\mathscr{C}$ contains but a single object, we obtain an exact sequence of cochain complexes

$$0 \longrightarrow C^*(\Gamma, U(\mathscr{C})) \xrightarrow{\alpha} C^*(\Gamma, \mathscr{C}) \xrightarrow{\beta} C^*(\Gamma, k_\Gamma(\mathscr{C})) \longrightarrow 0$$

with $\deg \alpha = 0, \deg \beta = -1$. Here the outside terms are the standard complexes of Γ with the indicated coefficient modules (see e.g. [13]).

For the general case, we must be a little more circumspect. Let $N(\mathscr{C}) = \text{Ker}\,(\text{ob}\,\mathscr{C} \to k_\Gamma(\mathscr{C}))$ be the isomorphism class of E in $\mathscr{C}$. Note that if the domain or codomain of a morphism of $\mathscr{C}$ (of any grade) is in $N(\mathscr{C})$, so is the other.

We define a filtration of $C^{n+1}(\Gamma, \mathscr{C})$ by

$c \in C_1^{n+1}$ if each $c(\gamma_0, \cdots, \gamma_n)$ has domain E and codomain E,
$c \in C_2^{n+1}$ if each $c(\gamma_0, \cdots, \gamma_n)$ has domain in $N(\mathscr{C})$ and codomain E,
$c \in C_3^{n+1}$ if each $c(\gamma_0, \cdots, \gamma_n)$ has domain in $N(\mathscr{C})$ and codomain in $N(\mathscr{C})$.

It is easy to see that $0 \subset C_1^* \subset C_2^* \subset C_3^* \subset C_4^* = C^*(\Gamma, \mathscr{C})$ are subcomplexes of $C^*(\Gamma, \mathscr{C})$.

THEOREM (7.1):
(i) *There is a natural isomorphism of degree* 0

$$\alpha: C^*(\Gamma;\, U(\mathscr{C})) \to C_1^*.$$

(ii) C_2^*/C_1^* *is contractible.*
(iii) C_3^*/C_2^* *is contractible.*
(iv) *There is a natural isomorphism of degree* -1

$$\beta: C_4^*/C_3^* \to C^*(\Gamma;\, k_\Gamma(\mathscr{C})).$$

PROOF: (i) C_1^n is the group of maps $c: \Gamma^n \to \mathscr{C}$ such that each $c(\gamma_0, \cdots, \gamma_{n-1}): E \to E$ has grade γ_0, while $C^n(\Gamma;\, U(\mathscr{C}))$ is the group of maps $a: \Gamma^n \to U(\mathscr{C})$. We establish an isomorphism $\alpha: C^n(\Gamma; U(\mathscr{C})) \to C_1^n$ by

$$\alpha(a)(\gamma_0, \cdots, \gamma_{n-1}) = a(\gamma_0, \cdots, \gamma_{n-1}) \circ E(\gamma_0).$$

For $n = 0$, α is the identity map on $C^0(\Gamma; U(\mathscr{C})) = U(\mathscr{C}) = C_1^0$.

It remains to verify that (at least up to sign) α preserves boundary maps. We have, for $n \geqq 1$

$$\delta(\alpha(a))(\gamma_0, \cdots, \gamma_n) = a(\gamma_0\gamma_1, \cdots, \gamma_n) \circ E(\gamma_0\gamma_1) \circ E(\gamma_1)^{-1} \circ a(\gamma_1, \cdots, \gamma_n)^{-1} +$$

$$+ \sum_{i=1}^{n-1} (-1)^i a(\gamma_0, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n) \circ E(\gamma_0) + (-1)^n a(\gamma_0, \cdots, \gamma_{n-1}) \circ E(\gamma_0),$$

$$\alpha(\delta(a))(\gamma_0, \cdots, \gamma_n) = \{{}^{\gamma_0}a(\gamma_1, \cdots, \gamma_n) - a(\gamma_0\gamma_1, \cdots, \gamma_n)$$
$$+ \sum_{i=1}^{n-1} (-1)^{i+1} a(\gamma_0, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n) + (-1)^{n+1} a(\gamma_0, \cdots, \gamma_{n-1})\} \circ E(\gamma_0).$$

Since ${}^{\gamma_0}a = E(\gamma_0) \circ a \circ E(\gamma_0^{-1})$ for all $a \in U(\mathscr{C})$, and by (6.4 i) addition and composition coincide on $U(\mathscr{C})$ we deduce that $\delta \circ \alpha = -\alpha \circ \delta$. For $n = 0$,

$$\delta(\alpha(a))(\gamma) = a \circ E(\gamma) \circ a^{-1}$$
$$= -\alpha(\delta(a))(\gamma).$$

(ii) Let D^n be the group of maps $\Gamma^n \to N(\mathscr{C})$. Since for $c \in C^{n+1}$ the domain of $c(\gamma_0, \cdots, \gamma_n)$ is independent of γ_0, we can define a map $\kappa: C_2^{n+1} \to D^n$ by

$$\kappa(c)(\gamma_1, \cdots, \gamma_n) = \text{domain } c(\gamma_0, \cdots, \gamma_n).$$

Clearly κ induces an isomorphism of C_2^{n+1}/C_1^{n+1} on D^n. Further, if we define a differential on D^* by

$$\delta(d)(\gamma_1, \cdots, \gamma_n) = \sum_{i=1}^{n-1} (-1)^{i-1} d(\gamma_1, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n) + (-1)^{n-1} d(\gamma_1, \cdots, \gamma_{n-1}),$$

we have $\delta \circ \kappa = -\kappa \circ \delta$, so it remains to check that D^* is contractible. But it is easily verified that

$$\sigma(d)(\gamma_1, \cdots, \gamma_{n-1}) = d(1, \gamma_1, \cdots, \gamma_{n-1})$$

gives a contracting homotopy (i.e. $\sigma \circ \delta + \delta \circ \sigma = 1$).

(iii) This case is like (ii) but simpler. Define $\lambda: C_3^{n+1} \to D^{n+1}$ by

$$\lambda(c)(\gamma_0, \cdots, \gamma_n) = \text{codomain } c(\gamma_0, \cdots, \gamma_n).$$

It is easily seen that λ induces a chain isomorphism of C_3^*/C_2^* onto D^*. Our assertion follows.

(iv) Analogously to (ii), define $\beta: C_4^{n+1} \to C^n(\Gamma; k_\Gamma(\mathscr{C}))$

$$\beta(c)(\gamma_1, \cdots, \gamma_n) = \text{class of domain } c(\gamma_0, \cdots, \gamma_n).$$

Note that the isomorphism class is just the class mod $N(\mathscr{C})$. The class of the codomain is the transform by γ_0 of this. Thus we have an isomorphism

$$\beta: C_4^{n+1}/C_3^{n+1} \to C^n(\Gamma; k_\Gamma(\mathscr{C})).$$

It remains to check compatibility with the boundary map: as before, this is routine.

Note that our introduction of $C^0(\Gamma, \mathscr{C})$ was used essentially in (i) and (iii) above.

COROLLARY (7.2): *There is an exact sequence*

$$\cdots H^n(\Gamma; U(\mathscr{C})) \to H^n(\Gamma; \mathscr{C}) \to H^{n-1}(\Gamma; k_\Gamma(\mathscr{C})) \to H^{n+1}(\Gamma; U(\mathscr{C})) \cdots.$$

This is simply the cohomology sequence [13] of the exact sequence of cochain complexes

$$0 \to C_1^* \to C_4^* \to C_4^*/C_1^* \to 0,$$

modified by noting that, in view of (ii) and (iii) of the theorem, $C_4^*/C_1^* \to C_4^*/C_3^*$ induces cohomology isomorphisms.

We now use some earlier results to extend this conclusion to group-like categories.

LEMMA (7.4): *A precise equivalence $h: \mathscr{C} \to \mathscr{C}'$ of Γ-group categories* *induces cohomology isomorphisms*

$$H^n(\Gamma; \mathscr{C}) \to H^n(\Gamma; \mathscr{C}').$$

PROOF: We observed earlier that h induces isomorphisms $U(\mathscr{C}) \to U(\mathscr{C}')$ and $k_\Gamma(\mathscr{C}) \to k_\Gamma(\mathscr{C}')$. Now h induces in a natural way a morphism of chain complexes $C^*(\Gamma; \mathscr{C}) \to C^*(\Gamma; \mathscr{C}')$ respecting filtrations. The conclusion follows on applying the Five Lemma [13] to the map of exact sequences (7.2).

Combining this with Theorem (6.6), we deduce

PROPOSITION (7.5): *The functors $H^n(\Gamma; \mathscr{C})$ extend (uniquely) to homotopy functors of strictly coherent group-like Γ-monoidal categories.*

We turn to the interpretation of the groups $H^n(\Gamma; \mathscr{C})$ for low values of n. By (7.2),

$$H^0(\Gamma; \mathscr{C}) = H^0(\Gamma; U(\mathscr{C}))$$

is the subgroup of Γ-invariant units in $U(\mathscr{C})$.

PROPOSITION (7.6): *There is a natural isomorphism $H^1(\Gamma; \mathscr{C}) \to k(\mathrm{Rep}\,\mathscr{C})$ such that the diagram*

$$\begin{array}{ccccccccc}
0 \to & H^1(\Gamma, U(\mathscr{C})) & \to & H^1(\Gamma, \mathscr{C}) & \to & H^0(\Gamma, k_\Gamma(\mathscr{C})) & \xrightarrow{\delta} & H^2(\Gamma, U(\mathscr{C})) \\
& \| & & \downarrow & & \| & & \| \\
0 \to & H^1(\Gamma, U(\mathscr{C})) & \to & k(\mathrm{Rep}\,\mathscr{C}) & \to & H^0(\Gamma, k_\Gamma(\mathscr{C})) & \xrightarrow{\omega} & H^2(\Gamma, U(\mathscr{C}),
\end{array}$$

where the upper sequence comes from (7.2) *and the lower from* (4.4), *commutes.*

Thus, in the special case of group-like categories, our present results include those of Section 4.

PROOF: It is sufficient (by (7.5)) to consider the case of group categories. An element of $C^1(\Gamma, \mathscr{C})$ is a function $c: \Gamma \to \mathscr{C}$ such that $\mathrm{dom}\, c(\gamma) = D$, say, is independent of γ, and $c(\gamma)$ has grade γ. We have

$$\delta c(\gamma_0, \gamma_1) = c(\gamma_0 \gamma_1) \circ c(\gamma_1^{-1}) - c(\gamma_0),$$

so c is a cocycle if and only if for all $\gamma_0, \gamma_1 \in \Gamma$ we have

$$c(\gamma_0 \gamma_1) \circ c(\gamma_1^{-1}) = c(\gamma_0), \qquad \text{i.e. } c(\gamma_0 \gamma_1) = c(\gamma_0) \circ c(\gamma_1).$$

This implies that each $c(\gamma)$ has codomain D also, and is equivalent to saying that c defines a representation of Γ on D.

Hence the group $Z^1(\Gamma, \mathscr{C})$ of cocycles is isomorphic to $\mathrm{ob}\,\mathrm{Rep}(\mathscr{C}, \Gamma)$.

We now show that the kernel of the map to $k\,\mathrm{Rep}(\mathscr{C}, \Gamma)$ corresponds to the subgroup of coboundaries. Indeed, for $c \in Z^1(\Gamma, \mathscr{C})$, an isomorphism of the corresponding representation with the standard one is an isomorphism $d: E \to D$ of grade 1 with $c(\gamma) = d \circ E(\gamma) \circ d^{-1}$. But this is precisely an expression of c as a coboundary.

Commutativity of the first two squares is immediate from the definitions. For the third, let D be an object of $\mathscr{C}$ whose class is Γ-invariant. Then we have an exact sequence (4.2)

$$\begin{tikzcd} 1 \arrow[r] & U(\mathscr{C}) \arrow[r, "\theta_\Gamma"] & \mathrm{Aut}_{\mathscr{C}}(D) \arrow[r] \arrow[dr, "g"] & \mathrm{Aut}_{P-\mathscr{C}}(D) \arrow[r] \arrow[d, "g'"] & 1, \\ & & & \Gamma & \end{tikzcd}$$

with g' an isomorphism (3.4). The extension determines a class in $H^2(\Gamma, U(\mathscr{C}))$ as follows. Choose a section $\lambda: \Gamma \to \mathrm{Aut}_{\mathscr{C}}(D)$ to g. Then $\lambda(\gamma_0\gamma_1)\lambda(\gamma_1)^{-1}\lambda(\gamma_0)^{-1}$ has grade 1 so there is a (unique) map $u: \Gamma \times \Gamma \to U(\mathscr{C})$ with

$$\theta_D u(\gamma_0, \gamma_1) = \lambda(\gamma_0\gamma_1)\lambda(\gamma_1)^{-1}\lambda(\gamma_0)^{-1}.$$

Then u is a 2-cocycle whose class is $\omega(D)$.

To compute $\delta(D)$, we first choose a cochain in $C^1(\Gamma, \mathscr{C}) = C_4^1$ whose image by β is the class of D. λ is a suitable choice.

I now assert that $\delta\lambda = \alpha(u)$, so that the class of the cocycle u also represents δD, which proves the desired commutativity. We have $\theta_D u(\gamma_0, \gamma_1) = u(\gamma_0, \gamma_1) + D$ by (2.1) and precision. Composing with $\lambda(\gamma_0) = E(\gamma_0) + \lambda(\gamma_0)$ gives

$$\theta_D u(\gamma_0, \gamma_1) \circ \lambda(\gamma_0) = (u(\gamma_0, \gamma_1) \circ E(\gamma_0)) + \lambda(\gamma_0)$$

and by definition of $u(\gamma_0, \gamma_1)$, the left hand side here simplifies to $\lambda(\gamma_0\gamma_1) \circ \lambda(\gamma_1)^{-1}$. Thus

$$\begin{aligned} \delta\lambda(\gamma_0, \gamma_1) &= \lambda(\gamma_0, \gamma_1) \circ \lambda(\gamma_1)^{-1} - \lambda(\gamma_0) \\ &= u(\gamma_0, \gamma_1) \circ E(\gamma_0) = \alpha(u)(\gamma_0, \gamma_1). \end{aligned}$$

PROPOSITION (7.7): *If Γ is finite of order N, then $N(H^n(\Gamma, \mathscr{C})) = 0$ for $n \geqq 2$.*

PROOF: For $n \geqq 1$ define

$$\rho: C^{n+1}(\Gamma, \mathscr{C}) \to C^n(\Gamma, \mathscr{C})$$

by

$$\rho c(\gamma_0, \cdots, \gamma_{n-1}) = \sum_{\gamma \in \Gamma} c(\gamma_0, \cdots, \gamma_{n-1}, \gamma).$$

One then verifies that $\delta\rho c - \rho\delta c = (-1)^n Nc$, and this implies the proposition.

We now consider briefly the relative case. Let $\mathscr{C}$, $\mathscr{D}$ be Γ-group categories and $F:\mathscr{C} \to \mathscr{D}$ a precise functor. Then F induces a chain-mapping $F^{\#}: C^*(\mathscr{C}) \to C^*(\mathscr{D})$, and we define $C^*(F)$ to be its mapping cone

$$C^{n+1}(F) = C^{n+1}(\mathscr{C}) \oplus C^n(\mathscr{D}),$$

with

$$\delta(c, d) = (\delta c, (-1)^{n+1} F^{\#}(c) + \delta d).$$

Then there is an exact sequence of cochain complexes

$$0 \to C^*(\mathscr{D}) \to C^*(F) \to C^*(\mathscr{C}) \to 0$$

which has an exact cohomology sequence whose boundary map coincides (up to sign) with that induced by F.

The above observations are formal, and have little interest unless one can say more about the relative cohomology groups. Comparing our construction with (7.1), we find with formal arguments and little effort

PROPOSITION (7.8): *If F is a precise functor as above, there is a diagram of exact sequences, commutative up to sign*

$$\begin{tikzcd}
H^n(U\mathscr{C}) \arrow[r, bend left] \arrow[dr] & H^n(U\mathscr{D}) \arrow[r, bend left] \arrow[dr] & H^n(\mathscr{D}) \arrow[r, bend left] \arrow[dr] & H^{n+1}(F) \arrow[r, bend left] \arrow[dr] & H^n(k_\Gamma F) \\
H^{n-2}(k_\Gamma \mathscr{P}) \arrow[ur] \arrow[dr] & H^n(\mathscr{C}) \arrow[ur] \arrow[dr] & H^{n+1}(U, F) \arrow[ur] \arrow[dr] & H^{n-1}(k_\Gamma \mathscr{D}) \arrow[ur] \arrow[dr] & H^{n+1}(\mathscr{C}) \\
H^n(F) \arrow[r, bend right] \arrow[ur] & H^{n-1}(k_\Gamma F) \arrow[r, bend right] \arrow[ur] & H^{n-1}(k_\Gamma \mathscr{C}) \arrow[r, bend right] \arrow[ur] & H^{n+1}(U\mathscr{C}) \arrow[r, bend right] \arrow[ur] & H^{n+1}(U\mathscr{D})
\end{tikzcd}$$

It follows that $H^0(F) = \operatorname{Ker}(U(\mathscr{C})^\Gamma \to U(\mathscr{D})^\Gamma)$, and for $H^1(F)$ we can employ the usual relativisation technique of algebraic K-theory. For any morphism $(F, p, q): \mathscr{C} \to \mathscr{D}$ of Γ-monoidal categories, define $\operatorname{Rep} F$ to be the category whose objects are triples (C, f, i) where (C, f) is an object of $\operatorname{Rep}\mathscr{C}$, and $i:(FC, Ff) \to E$ an isomorphism in $\operatorname{Rep}\mathscr{D}$. There are natural notions of morphism and product, and $\operatorname{Rep} F$ is a homotopy functor of F.

PROPOSITION (7.9): *If F is a precise functor of Γ-group categories, there is a natural isomorphism $H^1(F) \to k(\operatorname{Rep} F)$.*

We leave the necessary verifications to the reader.

8. Examples of group categories

The reader of Section 5 and Section 6 may feel that the concept of group category is austere, and that examples appear only in highly abstract situations. The following is perhaps the most illuminating example of a group category arising naturally.

Let R be a Dedeking ring, K its quotient field, Γ a group acting by automorphisms of R, hence of K. Define $\operatorname{ob}\mathscr{C}$ to be the set of fractional ideals I of R. These form a group under the natural product. A morphism of grade $\gamma \in \Gamma$ from I to J is determined by an element $a \in K$ with ${}^{\gamma}I \cdot a = J$. Composition of morphisms is defined by

$$(\gamma_1, a_1) \circ (\gamma_2, a_2) = (\gamma_1 \gamma_2, a_1{}^{\gamma_1}a_2)$$

and the product by

$$(\gamma, a_1) \nabla (\gamma, a_2) = (\gamma, a_1 a_2).$$

All our conditions are easily verified.

For this example, the class group $k_\Gamma(\mathscr{C})$ is just the class group of the ring R, and the unit group $U(\mathscr{C})$ is the group of units of R: this is one of the reasons for our terminology.

We now present an abstract generalisation of this example, and a careful analysis of its cohomology exact sequence. This will be of use in making calculations in specific examples, such as the above, and is also useful as a construction in its own right.

To see the relation of the example above to the abstract version to follow, use the notation

U = unit group of R
V = unit group of K
W = group of fractional ideals
P = group of principal ideals
C = ideal class group

Thus the natural homomorphism $V \to W$ has kernel U, image P and cokernel C.

Now let Γ be any group and $\psi\colon V \to W$ a homomorphism of (left) Γ-modules. We define a Γ-group category $\mathscr{C} = \mathscr{C}(\psi)$. Set $\operatorname{ob}\mathscr{C} = W$. The morphisms of $\mathscr{C}$ correspond bijectively to $\Gamma \times V \times W$; the morphism $\langle \gamma, v, w \rangle$ has domain w, codomain ${}^{\gamma}w + \psi(v)$ and grade γ. Composition is given by

$$\langle \gamma_1, v_1, w_1 \rangle \circ \langle \gamma_0, v_0, w_0 \rangle = \langle \gamma_1 \gamma_0, v_1 + {}^{\gamma_1}v_0, w_0 \rangle,$$

provided of course that $w_1 = {}^{\gamma_0}w_0 + \psi(v_0)$. This is associative, with identity morphisms $\langle 1, 0, w\rangle$, and inverses

$$\langle \gamma, v, w\rangle^{-1} = \langle \gamma^{-1}, -{}^{\gamma^{-1}}v, {}^{\gamma}w + \psi(v)\rangle,$$

so we have a (stable) Γ-graded groupoid.

We define a product – to be written as addition – by the given group structure on $\operatorname{ob}\mathscr{C} = W$, and

$$\langle \gamma, v_1, w_1\rangle + \langle \gamma, v_0, w_0\rangle = \langle \gamma, v_1 + v_0, w_1 + w_0\rangle.$$

Clearly, this defines a functor. Precise associativity and commutativity are immediate, and $E(\gamma) = \langle \gamma, 0, 0\rangle$ acts as unit. Hence we have a Γ-group category.

Objects w, w' are equivalent in $\operatorname{Ker}\mathscr{C}$ if and only if we can write $w' = w + \psi(v)$. Thus

$$k(\operatorname{Ker}\mathscr{C}) = \operatorname{Coker}\psi \colon (V \to W) = C, \quad \text{say}.$$

Since $\langle \gamma, 0, w\rangle$ has domain w and codomain ${}^{\gamma}w$, this isomorphism is one of Γ-modules, $k_\Gamma(\mathscr{C}) \cong C$. The unit group $U(\mathscr{C})$ consists of the morphisms $\langle 1, v, 0\rangle$ with codomain $\psi(v) = 0$, so $U(\mathscr{C}) \cong \operatorname{Ker}\psi$. Since

$$\langle \gamma, 0, 0\rangle \circ \langle 1, v, 0\rangle \circ \langle \gamma^{-1}, 0, 0\rangle = \langle 1, {}^{\gamma}v, 0\rangle,$$

this also is an isomorphism of Γ-modules.

We have thus shown most of

PROPOSITION (8.1): *$\psi \mapsto \mathscr{C}(\psi)$ is a functor from the category of homomorphisms of Γ-modules to the category of Γ-group categories, and there are natural equivalences*

$$k_\Gamma(\mathscr{C}(\psi)) \cong \operatorname{Coker}\psi, \qquad U(\mathscr{C}(\psi)) \cong \operatorname{Ker}\psi$$

of Γ-modules.

It remains to check naturality. But given a commutative square

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\phi_V"'] & W \arrow[d, "\phi_W"] \\ V' \arrow[r, "\psi'"] & W' \end{tikzcd}$$

defining a morphism $\psi \to \psi'$ of homomorphisms of Γ-modules, the obvious assignment

$$\langle \gamma, v, w\rangle \mapsto \langle \gamma, \phi_V(v), \phi_W(w)\rangle$$

defines a precise functor $\mathscr{C}(\psi) \to \mathscr{C}(\psi')$; and composition of such functors also behaves in the obvious way.

We now study the cohomology theory of the category $\mathscr{C}(\psi)$. Recall that $C^{n+1}(\Gamma, \mathscr{C})$ is the group of maps $c : \Gamma^{n+1} \to \mathscr{C}$ such that
(i) the grade of $c(\gamma_0, \cdots, \gamma_n)$ is γ_0
(ii) the domain of $c(\gamma_0, \cdots, \gamma_n)$ does not depend on γ_0.
Thus $c(\gamma_0, \cdots, \gamma_n) = \langle \gamma_0, v(\gamma_0, \cdots, \gamma_n), w(\gamma_1, \cdots, \gamma_n)\rangle$, and the maps v: $\Gamma^{n+1} \to V$, $w : \Gamma^n \to W$ are subject to no restrictions. It is thus natural to write the cochain c as $\langle v, w\rangle$, and to interpret

$$v \in C^{n+1}(\Gamma, V), \qquad w \in C^n(\Gamma, W).$$

LEMMA (8.2): $\delta\langle v, w\rangle = \langle -\delta v, \psi(v)+\delta(w)\rangle$.

PROOF: Applying the definition of $\delta c(\gamma_0, \cdots, \gamma_n)$ yields

$$\begin{aligned}&\langle \gamma_0\gamma_1, v(\gamma_0\gamma_1, \gamma_2, \cdots, \gamma_n), w(\gamma_2, \cdots, \gamma_n)\rangle \circ \langle \gamma_1, v(\gamma_1, \cdots, \gamma_n), w(\gamma_2, \cdots, \gamma_n)\rangle^{-1}\\ &\quad + \sum_{i=1}^{n-1} (-1)^i \langle \gamma_0, v(\gamma_0, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n), w(\gamma_1, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n)\rangle\\ &\quad + (-1)^n \langle \gamma_0, v(\gamma_0, \cdots, \gamma_{n-1}), w(\gamma_1, \cdots, \gamma_{n-1})\rangle.\end{aligned}$$

Using our definitions of addition and composition yields for the first component, γ_0 of course, for the second

$$\begin{aligned}&v(\gamma_0\gamma_1, \gamma_2, \cdots, \gamma_n) - {}^{\gamma_0}v(\gamma_1, \cdots, \gamma_n)\\ &\quad + \sum_{i=1}^{n-1} (-1)^i v(\gamma_0, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n) + (-1)^n v(\gamma_0, \cdots, \gamma_{n-1})\\ &\quad = -\delta v(\gamma_0, \cdots, \gamma_n)\end{aligned}$$

and for the third component

$$\begin{aligned}&\psi(v(\gamma_1, \cdots, \gamma_n)) + {}^{\gamma_1}w(\gamma_2, \cdots, \gamma_n) + \sum_{i=1}^{n-1} (-1)^i w(\gamma_1, \cdots, \gamma_i\gamma_{i+1}, \cdots, \gamma_n)\\ &\qquad + (-1)^n w(\gamma_1, \cdots, \gamma_{n-1}) = \psi(v)(\gamma_1, \cdots, \gamma_n) + \delta w(\gamma_1, \cdots, \gamma_n).\end{aligned}$$

COROLLARY (8.3): *We have (up to signs) an exact sequence of cochain complexes*

$$0 \to C^*(\Gamma, W) \to C^*(\Gamma, \mathscr{C}) \to C^*(\Gamma, V) \to 0$$

with exact cohomology sequence

$$\cdots H^{n-1}(\Gamma, W) \xrightarrow{\pi^*} H^n(\Gamma, \mathscr{C}) \xrightarrow{\nu^*} H^n(\Gamma, V) \xrightarrow{\psi^*} H^n(\Gamma, W) \longrightarrow \cdots$$

Proof: We can define $\pi(w) = \langle 0, w\rangle$ and $\nu\langle v, w\rangle = v$. The lemma shows that these are chain maps, and it is clear that we have a short exact sequence. It also follows from the lemma that the 'coboundary' maps coincides (up to sign) with ψ^*.

Next we relate this to the exact sequence (7.2). Write $U = \operatorname{Ker}\psi$, $P = \operatorname{Im}\psi$, $C = \operatorname{Cok}\psi$. Then (8.1) gives natural isomorphisms $C \cong k_\Gamma(\mathscr{C}(\psi))$, $U \cong U(\mathscr{C}(\psi))$, so the exact sequence (7.2) can be written as

$$\cdots H^n(\Gamma, U) \xrightarrow{\tau^*} H^n(\Gamma, \mathscr{C}) \xrightarrow{\mu^*} H^{n-1}(\Gamma, C) \xrightarrow{\omega^*} H^{n+1}(\Gamma, U) \cdots,$$

where our notation for the maps derives from Section 4. Also, the short exact sequences of Γ-modules

$$0 \longrightarrow U \xrightarrow{\phi} V \xrightarrow{\lambda} P \longrightarrow 0$$

$$0 \longrightarrow P \xrightarrow{\sigma} W \xrightarrow{\chi} C \longrightarrow 0$$

have cohomology exact sequences. We now economise notation by omitting the symbol Γ.

Theorem (8.4): *The diagram formed from these four exact sequences is commutative* (*up to sign*).

$$\begin{tikzcd}
H^{n-1}(W) \arrow[rr, bend left, "\chi^*"] \arrow[dr, "\pi^*"] & & H^{n-1}(C) \arrow[rr, bend left, "\omega^*"] \arrow[dr, "\xi^*"] & & H^{n+1}(U) \arrow[rr, bend left, "\phi^*"] \arrow[dr, "\tau^*"] & & H^{n+1}(V) \\
& H^n(\mathscr{C}) \arrow[ur, "\mu^*"] \arrow[dr, "\nu^*"] & & H^n(P) \arrow[ur, "\rho^*"] \arrow[dr, "\sigma^*"] & & H^{n+1}(\mathscr{C}) \arrow[ur, "\nu^*"] \arrow[dr, "\mu^*"] & \\
H^n(U) \arrow[ur, "\tau^*"] \arrow[rr, bend right, "\phi^*"'] & & H^n(V) \arrow[ur, "\lambda^*"] \arrow[rr, bend right, "\psi^*"'] & & H^n(W) \arrow[ur, "\pi^*"] \arrow[rr, bend right, "\chi^*"'] & & H^n(C)
\end{tikzcd}$$

Proof: We will verify each case by direct calculation. Observe that $\pi^*, \nu^*, \phi^*, \lambda^*, \sigma^*, \chi^*, \psi^*$ are induced by the correspondingly named chain-mappings, and that – comparing (7.2) with our present notation – τ^* is induced by the chain map $\tau(c) = \langle \phi c, 0\rangle$ and μ^* by the chain map $\mu\langle v, w\rangle = \chi(w)$. The boundary maps ρ (resp. ξ) are described by: if $c \in C^n(\Gamma, V)$ (resp. W) has $\delta\lambda c = 0$ (resp. $\delta\chi c = 0$), then $\phi^{-1}\delta c$ (resp. $\sigma^{-1}\delta c$) represents $\rho^*\{\lambda c\}$ (resp. $\xi^*(\chi c)$).

Now $\mu \circ \pi = \chi$, $\nu \circ \tau = \phi$, and $\sigma \circ \lambda = \psi$, so three of the triangles certainly commute. We next consider the two diamonds. If $\langle v, w\rangle \in C^n(\Gamma, \mathscr{C})$

is a cocycle with class x say, then by (8.2) $\psi v+\delta w = 0$. Now μ^*x is represented by $\chi(w)$ and $\xi^*\mu^*x$ by $\sigma^{-1}\delta w = -\sigma^{-1}\psi v = -\lambda v$. On the other hand, v represents v^*x, so λv represents λ^*v^*x. Thus $\lambda^*\circ v^* = -\xi^*\circ\mu^*$.

Next let $c\in C^n(\Gamma, P)$ be a cocyle with class y. Choose $d\in C^n(\Gamma, V)$ with $c = \lambda d$. Then $\phi^{-1}\delta d$ represents ρ^*y, so $\tau^*\rho^*y$ is represented by $\langle\delta d, 0\rangle$. Now $\pi^*\sigma^*y$ is represented by $\langle 0, \sigma c\rangle = \langle 0, \psi d\rangle$; since by (8.2) $\delta\langle d, 0\rangle = \langle -\delta d, \psi d\rangle$ these differ by a coboundary, so $\pi^*\circ\sigma^* = \tau^*\circ\rho^*$.

It remains to show $\omega^* = \rho^*\circ\xi^*$. Let $c\in C^{n-1}(\Gamma, C)$ be a cocycle with class z. Choose $b\in C^{n-1}(\Gamma, W)$ with $\chi b = c$: then $\delta b = \sigma a$ for a unique cocycle a representing ξ^*z. Now choose d with $\lambda d = a$. Then $\phi^{-1}\delta d$ represents $\rho^*\xi^*z$. Consider the cochain $\langle -d, b\rangle\in C^n(\Gamma, \mathscr{C})$: we have $\mu\langle -d, b\rangle = \chi b = c$. Also,

$$\delta\langle -d, b\rangle = \langle\delta d, -\psi d+\delta b\rangle = \langle\delta d, 0\rangle$$

since $\delta b = \sigma\lambda d = \psi d$. But this is the image by τ of the above cocycle representing $\rho^*\xi^*z$. Hence this same cocycle represents ω^*z, by the usual construction for a boundary map (here, the contractible subquotient complex C_3^*/C_1^* can be ignored). This completes the proof.

9. Change of groups

Let $\mathscr{C} = (\mathscr{C}, \nabla, E, a, c, e)$ be a Γ-monoidal category. Then we have defined the monoidal categories $\operatorname{Ker}\mathscr{C}$, $\operatorname{Rep}\mathscr{C}$ in Section 3. We now wish to study the dependence of $\operatorname{Rep}\mathscr{C}$ and particularly $k(\operatorname{Rep}\mathscr{C})$ on Γ. To do this, we model ourselves on the axiomatic system of Dress [2].

Let $\hat{\Gamma}$ denote the category of finite Γ-sets and equivariant maps, where we will now assume that Γ acts on the right. For any Γ-set X, we define the Γ-graded category X_Γ to have object set X and morphism set $X\times\Gamma$, where (x, γ) has grade γ, domain x and codomain $x\gamma$. This determines composition: $(x, \gamma)\circ(y, \delta)$ is defined if $y = x\gamma$ and then equals (x, δ). The case when $X = Y\times\Gamma$ is a free Γ-set was already mentioned in Section 7, except that there we considered left action.

We now define, for X a finite Γ-set,

$$\mathscr{H}(X) = \mathscr{H}_{\mathscr{C}}(X) = \mathscr{H}om_\Gamma(X_\Gamma, \mathscr{C}),$$

where $\mathscr{H}om_\Gamma$ signifies morphisms of Γ-graded categories. Before continuing, notice that this is really nothing new. If Γ acts transitively on X, we can identify X with the coset space $\Delta\backslash\Gamma$ of a subgroup Δ, and it is easy to see that $\mathscr{H}(X)$ is essentially the same as $\operatorname{Rep}(\mathscr{C}, \Delta)$: the representation category of the restriction of $\mathscr{C}$ to an Δ-graded category. Thus for $X = \{1\}$, $\Delta = \Gamma$, we have $\operatorname{Rep}\mathscr{C}$ and for $X = \Gamma$, $\Delta = \{1\}$, we have $\operatorname{Ker}\mathscr{C}$. In general, we decompose X into Γ-orbits, and observe that for a disjoint union $X = X_1 \cup X_2$ we have $\mathscr{H}(X) = \mathscr{H}(X_1) \times \mathscr{H}(X_2)$.

Next observe that $\mathscr{H}(X)$ can itself be considered as a monoidal category. We can define a sum ∇ using the sum in $\mathscr{C}$; the identity object E comes from the identity object of $\mathscr{C}$, and the equivalences a, c and e in $\mathscr{C}$ induce ones for $\mathscr{H}(X)$. Further, a map $\phi\colon Y \to X$ of Γ-sets induces in an obvious way a Γ-functor $\phi_\Gamma\colon Y_\Gamma \to X_\Gamma$. Composing with ϕ_Γ gives a functor $\phi^*\colon \mathscr{H}(X) \to \mathscr{H}(Y)$. Since the products in $\mathscr{H}(X)$ and in $\mathscr{H}(Y)$ are both induced from $\mathscr{C}$, ϕ^* is a precise $\mathscr{MC}$-morphism.

Proposition (9.1): *The above construction defines a contravariant functor $\mathscr{H}_{\mathscr{C}}$ from the category $\hat{\Gamma}$ of finite Γ-sets to $\mathscr{MC}$.*

Indeed, it remains only to observe that $(\phi\psi)^* = \psi^*\phi^*$.

We now show that $\mathscr{H}$ can be regarded as a bifunctor in the sense of [2]. It will be convenient first to make a slight modification, and let $\hat{\Gamma}$ be the category of *ordered* finite Γ-sets (morphisms as before, they need not respect the order). Let $\phi\colon Y \to X$ be a morphism in $\hat{\Gamma}$, let $\rho \in \operatorname{ob} \mathscr{H}(Y)$ be a functor $\rho\colon Y_\Gamma \to \mathscr{C}$. We now define $\phi_* \rho\colon X_\Gamma \to \mathscr{C}$.

On objects, we set

$$\phi_* \rho(x) = \nabla\{\rho(y)\colon \phi(y) = x\},$$

which we make explicit as follows. We have ordered Y, and hence the set $\{y \in Y\colon \phi(y) = x\}$, say as $y_1, \cdots, y_n$. The right hand side is defined as

$$(\cdots((\rho(y_1) \nabla \rho(y_2)) \cdots \nabla \rho(y_n)).$$

To define $\phi_* \rho(x, \gamma)\colon \phi_* \rho(x) \to \phi_* \rho(x\gamma)$, we have indeed

$$\{y \in Y\colon \phi(y) = x\gamma\} = \{y\gamma \in Y\colon \phi(y) = x\} = \{y_1\gamma, \cdots, y_n\gamma\},$$

but this may not be the preferred order. However, we do have morphisms $\rho(y_i, \gamma)\colon \rho(y_i) \to \rho(y_i\gamma)$, hence a morphism of the above to

$$(\cdots((\rho(y_1\gamma) \nabla \rho(y_2\gamma)) \nabla \rho(y_3\gamma)) \cdots \nabla \rho(y_n\gamma)).$$

Now using coherence of a and c we have a natural unique morphism of this to the corresponding sum in the preferred order, $\phi_* \rho(x\gamma)$. This completes the definition of $\phi_* \rho$: it is evident that we have a Γ-functor $\phi_0 \rho\colon X_\Gamma \to \mathscr{C}$. Thus we have defined $\phi_*\colon \mathscr{H}(Y) \to \mathscr{H}(X)$ on objects.

Now if $T\colon \rho_1 \to \rho_2$ is a morphism in $\mathscr{H}(Y)$, we define $\phi_* T$ in the obvious way (with no need for reordering): with the above notation $\phi_* T(x)$ is the left normalised sum of the $T(y_i)$. This completes the construction of the functor $\phi_*\colon \mathscr{H}(Y) \to \mathscr{H}(X)$.

Lemma (9.2): $1_* = 1$, $(\phi\psi)_* \simeq \phi_* \psi_*$.

PROOF: The first assertion is obvious. For the second, let $\psi: Z \to Y$, let $\rho: Z_\Gamma \to \mathscr{C}$, and let $x \in X$ with $\phi^{-1}(x) = \{y_1, \cdots, y_n\}$ and $\psi^{-1}(y_i) = \{z_{i1}, \cdots, z_{in_i}\}$. Then

$$(\phi\psi)^{-1}(x) = \{z_{ij}: 1 \leqq i \leqq n, 1 \leqq j \leqq n_i\}$$

in some order, and each of $(\phi\psi)_* \rho(x)$ and $\phi_*(\psi_* \rho)(x)$ is the sum (in some order and with some bracketing) of the $\rho(z_{ij})$. Since a and c are coherent, they provide a (unique) isomorphism $(\phi\psi)_* \rho(x) \to \phi_*(\psi_* \rho)(x)$. Since this is natural and unique, it is easy to see that we have a natural equivalence $(\phi\psi)_* \rho \to \phi_*(\psi_* \rho)$. Again, it follows easily that this is also natural in ρ.

We thus have a bifunctor (in the terminology of [2]) $\mathscr{H}: \hat{\Gamma} \to \mathscr{H}\mathscr{M}\mathscr{C}$.

THEOREM (9.3): *$\mathscr{H}$ is a Mackey functor.*

PROOF: We have already observed that $\mathscr{H}$ carries finite sums (disjoint unions) into products. It remains to show that for each pullback diagram in $\hat{\Gamma}$,

$$\begin{array}{ccc} X = X_1 \underset{Y}{\times} X_2 & \xrightarrow{\Psi} & X_1 \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \phi} \\ X_2 & \xrightarrow{\psi} & Y, \end{array}$$

we have $\Psi_* \circ \Phi^* = \phi^* \circ \psi_*: \mathscr{H}(X_2) \to \mathscr{H}(X_1)$.

In the following calculation, we will use the imprecise notation ∇ for a set of objects: the justification using coherence is as above.

Let $\rho \in \mathscr{H}(X_2)$. Then $\Phi^*\rho$ is the composite

$$X_\Gamma \xrightarrow{\Phi_\Gamma} (X_2)_\Gamma \xrightarrow{\rho} \mathscr{C}$$

and for $x_1 \in X_1$,

$$\Psi_* \Phi^* \rho(x_1) = \nabla\{\rho(\Phi(y)): \Psi(y) = x_1\};$$

similarly for (x_1, γ). Now we can write $y \in X$ as a pair $(x_1, x_2) \in X_1 \times X_2$, and such a pair is in X if and only if $\phi(x_1) = \psi(x_2)$. Since also $\Psi(x_1, x_2) = x_1$, y has the form (x_1, x_2) for some x_2, so the above is

$$\begin{aligned} &\nabla\{\rho(\Phi(x_1, x_2)): \phi(x_1) = \psi(x_2)\} \\ &\qquad = \nabla\{\rho(x_2): \psi(x_2) = \phi(x_1)\} \qquad (\text{as } \Phi(x_1, x_2) = x_2) \\ &\qquad = \psi_* \rho(\phi(x_1)) = \phi^* \psi_* \rho(x_1)). \end{aligned}$$

Since the same holds for (x_1, γ), this completes the proof.

COROLLARY (9.4): *Composing with any covariant functor on $\mathcal{HMC}$ – e.g. k, U or K_i: $\mathcal{HMC} \to \mathcal{Ab}$ – yields further Mackey-functors.*

We now discuss a different, but not unrelated construction. First suppose the group Δ acts on the left of X, and commutes with the right action of Γ. Then we can enrich $\mathcal{H}(X)$ to an Δ-graded category as follows. Define a morphism T of grade $\delta \in \Delta$ between $\phi_1, \phi_2 : X_\Gamma \to \mathcal{C}$ to be a pair (T, δ), where T is a set of morphisms (of grade 1)

$$T(x) \colon \phi_1(x) \to \phi_2(\delta^{-1}x) \qquad \forall x \in X$$

such that for all $x \in X$, $\gamma \in \Gamma$ the following diagram commutes:

$$\begin{tikzcd} \phi_1(x) \arrow[r, "T(x)"] \arrow[d, "\phi_1(x,\gamma)"] & \phi_2(\delta^{-1}x) \arrow[d, "\phi_2(\delta^{-1}x,\gamma)"] \\ \phi_1(x\gamma) \arrow[r, "T(x\gamma)"] & \phi_2(\delta^{-1}x\gamma). \end{tikzcd}$$

Observe that this construction generalises the enrichment described in [6] of Rep$(\Delta, \mathcal{C})$ to a Γ/Δ-graded category, when $\Delta \lhd \Gamma$, taking $X = \Gamma/\Delta$. This, too is functorial in the same sense as described above. Further, we now have an associativity law. A $(\Delta - \Gamma)$ set X defines a functor

$$\mathcal{MC}_\Gamma \xrightarrow{\mathcal{F}(X)} \mathcal{MC}_\Delta .$$

PROPOSITION (9.5): *Let Y be a $(K - \Delta)$ set and X a $(\Delta - \Gamma)$ set. Then*

$$\mathcal{F}(Y \underset{\Delta}{\times} X) \simeq \mathcal{F}(Y) \circ \mathcal{F}(X).$$

PROOF: Evaluate both sides on the Γ-monoidal category $\mathcal{C}$. An object of $\mathcal{F}(X)\mathcal{C}$ is a functor $X_\Gamma \to \mathcal{C}$. An object of $\mathcal{F}(Y)(\mathcal{F}(X)(\mathcal{C}))$ is a functor $Y_\Delta \to \mathcal{F}(X)(\mathcal{C}) = \mathcal{H}om(X, \mathcal{C})$. It thus assigns to each $y \in Y$ and $x \in X$ an object $F(y, x)$ of $\mathcal{C}$; to (y, x, γ) a morphism of grade γ, $F(y, x, \gamma) \colon F(y, x) \to F(y, x\gamma)$, and to (y, δ, x) a morphism of grade 1 $F(y, \delta, x) \colon F(y, x) \to F(y\delta, \delta^{-1}x)$: these to satisfy certain identities. Observe that when each $F(y, \delta, x)$ is the identity, we have exactly a functor $(Y \times_\Delta X)_\Gamma \to \mathcal{C}$.

A morphism of grade $k \in K$ between F_1, F_2 is a set of morphisms (of grade 1) $T(y) \colon F_1(y, x) \mapsto F_2(k^{-1}y, x)$, again satisfying the obvious identities. We can thus identify $\mathcal{F}(Y \times_\Delta X)(\mathcal{C})$ with a subcategory of $\mathcal{F}(Y)\mathcal{F}(X)(\mathcal{C})$; it is easy to see that this is equivalent to the whole category.

Let $\mathcal{SG}$ be the category whose objects are groups, morphisms from Γ to Δ are isomorphism classes of finite $(\Delta - \Gamma)$ sets ${}_\Delta X_\Gamma$, and composition of ${}_K Y_\Delta$ and ${}_\Delta X_\Gamma$ is defined as $Y \times_\Delta X$.

Then this proposition shows that we have a functor from $\mathcal{SG}$ to the category with objects the categories $\mathscr{HMC}_\Gamma$ (Γ finite) and morphisms the functors between them.

We observe that if $\Delta \subset \Gamma$ and we take $X = {}_\Delta\Gamma_\Gamma$ with actions by translation, the corresponding functor is the forgetful functor taking a Γ-graded category to the subcategory of morphisms with grade in Δ; similarly ${}_\Gamma\Gamma_\Delta$ corresponds to induction from Δ up to Γ. More interesting, if $\Delta \lhd \Gamma$, with quotient $Q = \Gamma/\Delta$, take $X = {}_Q Q_\Gamma$ (again with action by translation): this yields a Q-grading on $\mathrm{Rep}(\mathscr{C}, \Delta)$. Composing with ${}_1\{1\}_Q$, (9.5) yields an equivalence

$$\mathrm{Rep}(\mathrm{Rep}(\mathscr{C}, \Delta), \Gamma/\Delta) \cong \mathrm{Rep}(\mathscr{C}, \Gamma) \tag{9.6}$$

THEOREM (9.7): *Let $\mathscr{C}$ be a Γ-monoidal category, $\Delta \lhd \Gamma$. Then there is an exact sequence*

$$0 \to H^1(\Gamma/\Delta; H^0(\Delta; U(\mathscr{C}))) \to k\,\mathrm{Rep}(\mathscr{C}, \Gamma) \to H^0(\Gamma/\Delta, k\,\mathrm{Rep}(\mathscr{C}, \Delta)) \to$$
$$\to H^2(\Gamma/\Delta; H^0(\Delta; U(\mathscr{C}))).$$

PROOF: This is the exact sequence of (4.6) for the Γ/Δ-graded category $\mathrm{Rep}(\mathscr{C}, \Delta)$, interpreting the terms using the isomorphisms (9.6) and (3.5):

$$U(\mathrm{Rep}(\mathscr{C}, \Delta)) = H^0(\Delta, U(\mathscr{C})).$$

One would expect for a Γ-group category to obtain a spectral sequence, but the right formulation of this is not yet clear.

10. Twisting

We begin by analysing the structure imposed on $\mathrm{Ker}\,\mathscr{C}$ for a Γ-graded category $\mathscr{C}$ by virtue of the rest of $\mathscr{C}$, in the case when Γ is cyclic. This allows formal construction of $\boldsymbol{Z}/n\boldsymbol{Z}$-graded categories, and also of $\Gamma \times \boldsymbol{Z}/n\boldsymbol{Z}$-gradings. A special case which is important for applications gives a 'twisting' construction allowing us from one Γ-graded (or monoidal) category to form another, using the above with $n = 2$. We have an exact sequence relating the equivariant unit and class groups in this case.

Let Γ be a cyclic group of order n, with generator T. Let $\mathscr{C}$ be a stable Γ-graded category. For each object $C \in \mathrm{ob}\,\mathscr{C}$, choose an invertible morphism t_C with domain C and grade T. Define a functor $D\colon \mathrm{Ker}\,\mathscr{C} \to \mathrm{Ker}\,\mathscr{C}$ by

$$D(C) = \text{codomain } t_C$$
$$D(f\colon C \to C') = t_{C'} \circ f \circ t_C^{-1}.$$

Observe that

$$\psi(C) = t_{D^{n-1}(C)} \circ \cdots \circ t_{D^2(C)} \circ t_{D(C)} \circ t_C \colon C \to D^n(C)$$

has grade $T^n = 1$, and defines a natural transformation $\psi : I \to D^n$ satisfying $D(\psi(C)) = \psi(D(C))$ for all $C \in \mathrm{ob}\,\mathscr{C}$.

Conversely, we have

PROPOSITION (10.1): *Given a category $\mathscr{A}$, a functor $D : \mathscr{A} \to \mathscr{A}$ and a natural equivalence $\psi : I \to D^n$ with $D(\psi(A)) = \psi(D(A))$ for all $A \in \mathrm{ob}\,\mathscr{A}$, there exists a stably $\mathbf{Z}/n\mathbf{Z}$-graded category $\mathscr{C}$ as above.*

PROOF: We construct $\mathscr{C}$ explicitly. The objects (and morphisms of grade 1) are those of $\mathscr{A}$. The morphisms $A \to A'$ of grade r $(0 \leqq r < n)$ are pairs (f, r) for all morphisms $f \colon D^r(A) \to A'$ of $\mathscr{A}$. Composition is defined by

$$A \xrightarrow{(f,r)} A' \xrightarrow{(f',r')} A''$$

$$(f', r') \circ (f, r) = \begin{cases} (f' \circ D^{r'} f, r+r') & 0 \leqq r+r' < n \\ (f' \circ D^{r'} f \circ \psi(D^{r+r'-n} A), r+r'-n) & n \leqq r+r', \end{cases}$$

and is easily seen to be associative. We arrived at these formulae by setting

$$(f, r) = f \circ t_{D^{r-1}(A)} \circ \cdots \circ t_{D(A)} \circ t_A$$

in the situation above.

We can extend these considerations to monoidal categories. If $\mathscr{A}$ is an object of $\mathscr{M}\mathscr{C}$ and $(D, \Phi, 1) \colon \mathscr{A} \to \mathscr{A}$ a morphism, with a homotopy $\psi : I \to D^n$ satisfying $D(\psi(A)) = \psi(D(A))$, then in the category $\mathscr{C}$ constructed above we can define a product extending that on $\mathscr{A}$ by setting, if

$$A \xrightarrow{(f,r)} A' \qquad B \xrightarrow{(g,r)} B',$$

$$(f, r) \nabla (g, r) = (\Phi_r(A, B)^{-1} \circ (f \nabla g), r),$$

where $(D, \Phi, 1)^r = (D^r, \Phi_r, 1)$ in $\mathscr{M}\mathscr{C}$, i.e.

$$\Phi_r(A, B) = \Phi_{D^{r-1}A, D^{r-1}B} \circ D\Phi_{D^{r-2}A, D^{r-2}B} \circ \cdots D^{r-1}\Phi_{A,B}.$$

Again, it is easily verified that this is a product: observe, for example, that (1.10) for ψ implies $\psi_{(A \nabla B)} = \Phi_n(A, B) \circ (\psi_{(A)} \nabla \psi_{(B)})$. Further, if we identify $\mathscr{A} = \mathrm{Ker}\,\mathscr{C}$, the same natural equivalences a, c and e will do for $\mathscr{C}$ as for $\mathscr{A}$.

The construction remains valid if $\mathscr{A}$ is already Γ-graded (Γ again an arbitrary group). We shall need only the simplest case: when D preserves degrees, and ψ has grade 1. We receive a $\Gamma \times (\mathbf{Z}/n\mathbf{Z})$-graded category $\mathscr{C}$: details are just as before. Since $\mathrm{Ker}\,\mathscr{C} = \mathrm{Ker}\,\mathscr{A}$, $\mathscr{C}$ and $\mathscr{A}$ have the same unit and class groups U and k: the operation of Γ is the same in each case, and the operation on the generator T of $\boldsymbol{Z}/n\boldsymbol{Z}$ is induced by the functor D.

Finally, we note the corresponding result for morphisms. Suppose given morphisms

$$(D, \Phi, 1)\ \ \mathscr{A} \to \mathscr{A}, \qquad (D', \Phi', 1)\colon \mathscr{A}' \to \mathscr{A}'$$

with ψ, ψ' as above. Then if $(T, p, q)\colon \mathscr{A} \to \mathscr{A}'$ is a morphism, and

$$\Omega\colon(D', \Phi', 1) \circ (T, p, q) \to (T, p, q) \circ (D, \Phi, 1)$$

a homotopy, we can extend (T, p, q) to a morphism $\mathscr{C} \to \mathscr{C}'$ by setting, for $f\colon DA \to A'$,

$$T(f, 1) = (Tf \circ \Omega A, 1)$$

(the formula for $T(f, r)$ can be deduced from this).

We now specialise to the case of particular interest. Let $(\mathscr{C}, \nabla, E)$ be Γ-monoidal; $(D, \Phi, 1)$ a self-morphism, $\Psi\colon I \to D^2$ a homotopy with $D\Psi(A) = \Psi(DA)$ for $A \in \operatorname{ob}\mathscr{C}$. Construct a $\Gamma \times \boldsymbol{Z}/2\boldsymbol{Z}$-graded category $\mathscr{C}^e$ as above. Let $w\colon \Gamma \to \boldsymbol{Z}/2\boldsymbol{Z}$ be a homomorphism, grading Γ into even and odd elements: its graph Γ^w is a subgroup of $\Gamma \times \boldsymbol{Z}/2\boldsymbol{Z}$. Restricting the grading to this subgroup and then identifying $\Gamma^w = \Gamma$ yields a new Γ-monoidal category, $\tilde{\mathscr{C}}$ say. Thus objects of $\tilde{\mathscr{C}}$, and morphisms of even grade, are as in $\mathscr{C}$; those of odd grade γ, $C_1 \to C_2$ are the $\tilde{f} = (f, 1)$ with $f\colon DC_1 \to C_2$ of grade γ in $\mathscr{C}$. Composition is given by

$$k \circ \tilde{h} = \widetilde{D(k) \circ h}, \qquad \tilde{k} \circ \tilde{h} = \psi(C_3) \circ D(k) \circ h.$$

In our next paper, we will apply this to the following example. R is a commutative ring; $\mathscr{C}$ the category $\mathscr{P}(R)$ of finitely generated projective R-modules, and isomorphisms. $D(M) = \operatorname{Hom}_R(M, R) = M^*$ is the dual module, $D(f) = f^{*-1}$; $\psi(M)\colon M \to M^{**}$ is the natural map. In $\tilde{\mathscr{C}}$, morphisms of odd grade are given by isomorphisms $M^* \cong N$, i.e. by nonsingular bilinear pairings $M \times N \to R$. Another important example will be the category of R-algebras, where we twist by having anti-automorphisms. The important functor relating these types of example is the endomorphism algebra $M \mapsto \operatorname{End}_R(M)$.

We now construct an exact sequence which, in a special case relates the original category and the twisted one: we consider only the case of group categories. Let $\mathscr{C}$ be a Γ-group category, $w\colon \Gamma \to \boldsymbol{Z}/2\boldsymbol{Z}$ a surjection with kernel Δ. Take $D(f) = -f$ for all morphisms f of $\mathscr{C}$ (which gives a precise functor).

Proposition (10.2): *There is an exact sequence*

$$\cdots H^i(\Gamma, \mathscr{C}) \to H^i(\Delta, \mathscr{C}) \to H^i(\Gamma, \tilde{\mathscr{C}}) \to H^{i+1}(\Gamma, \mathscr{C}) \cdots$$

Proof: It will be notationally convenient to denote the given copy of Γ sometimes by Γ_1, and the copy acting on the twisted category by Γ_2.

We begin with some homological remarks. Define Λ, $\boldsymbol{Z}^t$ to be the $\boldsymbol{Z}/2\boldsymbol{Z}$-modules with additive groups $\boldsymbol{Z}+\boldsymbol{Z}$ resp. $\boldsymbol{Z}$ and action given by

$$T(m, n) = (n, m) \qquad \text{resp. } T(n) = -n.$$

Thus if we define $a(n) = (n, n)$, $b(m, n) = m-n$, we have a short exact sequence of $\boldsymbol{Z}/2\boldsymbol{Z}$-modules

$$0 \to \boldsymbol{Z} \to \Lambda \to \boldsymbol{Z}^t \to 0. \tag{10.3}$$

We can also regard these as Γ-modules via the epimorphism $w: \Gamma \to \boldsymbol{Z}/2\boldsymbol{Z}$. Since the sequence is additively split, we can tensor by any Γ-module M to obtain another exact sequence of Γ-modules

$$0 \to M \to \Lambda \underset{\boldsymbol{Z}}{\otimes} M \to \boldsymbol{Z}^t \underset{\boldsymbol{Z}}{\otimes} M \to 0.$$

But the induced action of Γ on $\boldsymbol{Z}^t \otimes_{\boldsymbol{Z}} M$ corresponds, under the obvious isomorphism $\boldsymbol{Z}^t \otimes_{\boldsymbol{Z}} M \cong M$, to that of Γ_2 on M. Thus $H^i(\Gamma; \boldsymbol{Z}^t \otimes_{\boldsymbol{Z}} M) \cong H^i(\Gamma_2; M)$ for any i. And $\Lambda \otimes_{\boldsymbol{Z}} M$ is isomorphic, as Γ-module, to the module induced from the Δ-module M, so

$$H^i(\Gamma; \Lambda \underset{\boldsymbol{Z}}{\otimes} M) \cong H^i(\Delta; M), \qquad \text{all } i,$$

Thus the exact sequence of Γ-modules has exact cohomology sequence

$$\cdots H^i(\Gamma_1, M) \to H^i(\Delta, M) \to H^i(\Gamma_2, M) \to H^{i+1}(\Gamma_1, M) \cdots$$

Now tensor (10.3) by the cochain complex $C^*(\Gamma, \mathscr{C})$. We receive a short exact sequence of cochain complexes, which thus has an exact cohomology sequence. Now since D simply acts by -1, it is easy to identify

$$\boldsymbol{Z}^t \underset{\boldsymbol{Z}}{\otimes} C^*(\Gamma, \mathscr{C}) \cong C^*(\Gamma_2, \tilde{\mathscr{C}}).$$

Now the natural splitting $\Lambda = \boldsymbol{Z} \oplus \boldsymbol{Z}$ is Δ-invariant so, as chain complex over Δ,

$$\Lambda \underset{\boldsymbol{Z}}{\otimes} C^*(\Gamma, \mathscr{C}) \cong C^*(\Gamma, \mathscr{C}) \oplus C^*(\Gamma, \mathscr{C}).$$

Projecting to the first summand and restricting n-chains to Δ^n thus defines a chain-map

$$\Lambda \underset{\mathbf{Z}}{\otimes} C^*(\Gamma, \mathscr{C}) \to \quad C^*(\Delta, \mathscr{C}).$$

Now if we construct such a map with $\mathscr{C}$ replaced by a Γ-module, by Shapiro's lemma, it induces cohomology isomorphisms. But applying (7.2) to $C^*(\Delta, \mathscr{C})$ and the same technique to $\Lambda \otimes_{\mathbf{Z}} C^*(\Gamma, \mathscr{C})$, we see that the above chain map defines a map of exact sequences in which (by Shapiro's lemma) two out of every three maps are isomorphisms. Hence by the 5 lemma, so are the others: the above chain map also induces cohomology isomorphisms.

Hence the cohomology sequence of our exact sequence of cochain complexes is as asserted in the Proposition.

COROLLARY (10.4): *In the same situation, we have an exact sequence*

$$0 \to H^0(\Gamma_1, U(\mathscr{C})) \to H^0(\Delta, U(\mathscr{C})) \to H^0(\Gamma_2, U(\mathscr{C})) \to k\operatorname{Rep}(\mathscr{C}, \Gamma)$$
$$\to k\operatorname{Rep}(\mathscr{C}, \Delta) \to k\operatorname{Rep}(\tilde{\mathscr{C}}, \Gamma).$$

This follows from the Proposition on substituting the isomorphisms (7.6). Here, the map $H^0(\Gamma_2, U(\mathscr{C})) \to k\operatorname{Rep}(\mathscr{C}, \Gamma)$ is induced as in (4.5), the next map is the obvious restriction, and the last map is a sort of 'norm': given a representation of Δ on C, and a morphism $C \to C'$ of grade in $\Gamma - \Delta$, we construct a representation on $C \nabla D(C')$. Exactness up to one term short of this can also be proved directly under much weaker hypotheses.

11. Ring-like categories

The category $\mathscr{M}od_R$ of R-modules, for R a commutative ring, admits two structures of monoidal category, given by direct sum and tensor product, which are related by the distributive isomorphisms

$$d_{A,B,C}: A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C).$$

This situation was formalised by Laplaza [9], [10] who gives a list of axioms for coherence in this situation. The list is long (another paper like [8] is needed!) but if we ignore the relations concerning 'zero' and 'unit' objects, and those for separate coherence of $\oplus$ and $\otimes$, only four more are needed, viz. commutativity of

(11.1)
$$\begin{tikzcd} A \otimes (B \oplus C) \arrow[r, "d"] \arrow[d, "A \otimes c"] & (A \otimes B) \oplus (A \otimes C) \arrow[d, "c"] \\ A \otimes (C \oplus B) \arrow[r, "d"] & (A \otimes C) \oplus (A \otimes B) \end{tikzcd}$$

$$\begin{tikzcd}
A\otimes(B\oplus(C\oplus D)) \arrow[r, "d"] \arrow[d, "A\otimes a"] & A\otimes B\oplus A\otimes(C\oplus D) \arrow[r, "A\otimes B\oplus d"] & A\otimes B\oplus(A\otimes C\oplus A\otimes D) \arrow[d, "a"] \\
A\otimes((B\oplus C)\oplus D) \arrow[r, "d"] & A\otimes(B\oplus C)\oplus A\otimes D \arrow[r, "d\oplus A\otimes D"] & (A\otimes B\oplus A\otimes C)\oplus A\otimes D,
\end{tikzcd} \tag{11.2}$$

$$\begin{tikzcd}
A\otimes(B\otimes(C\oplus D)) \arrow[r, "a"] \arrow[dr, "A\otimes d"] & (A\otimes B)\otimes(C\oplus D) \arrow[r, "d"] & (A\otimes B)\otimes C\oplus(A\otimes B)\otimes D \\
& A\otimes(B\otimes C\oplus B\otimes D) \arrow[r, "d"] & A\otimes(B\otimes C)\oplus A\otimes(B\otimes D), \arrow[u, "a\oplus a"]
\end{tikzcd} \tag{11.3}$$

and

$$\begin{tikzcd}
(A\oplus B)\otimes(C\oplus D) \arrow[r, "d"] \arrow[d, "c"] & (A\oplus B)\otimes C\oplus(A\oplus B)\otimes D \arrow[d, "c\oplus c"] \\
(C\oplus D)\otimes(A\oplus B) \arrow[d, "d"] & C\otimes(A\oplus B)\oplus D\otimes(A\oplus B) \arrow[d, "d\oplus d"] \\
(C\oplus D)\otimes A\oplus(C\oplus D)\otimes B \arrow[d, "c\oplus c"] & (C\otimes A\oplus C\otimes B)\oplus(D\otimes A\oplus D\otimes B) \arrow[d, "(c\oplus c)\oplus(c\oplus c)"] \\
A\otimes(C\oplus D)\oplus B\otimes(C\oplus D) \arrow[d, "d\oplus d"] & (A\otimes C\oplus B\otimes C)\oplus(A\otimes D\oplus B\otimes D) \arrow[dl, "M"] \\
(A\otimes C\oplus A\otimes D)\oplus(B\otimes C\oplus B\otimes D) &
\end{tikzcd} \tag{11.4}$$

where M, the 'middle four interchange', is defined by

$$\begin{tikzcd}
(P\oplus Q)\oplus(R\oplus S) \arrow[r, "a"] \arrow[d, "M"] & ((P\oplus Q)\oplus R)\oplus S & (P\oplus(Q\oplus R))\oplus S \arrow[l, "a\oplus S"'] \arrow[d, "(P\oplus c)\oplus S"] \\
(P\oplus R)\oplus(Q\oplus S) \arrow[r, "a"] & ((P\oplus R)\oplus Q)\oplus S & (P\oplus(R\oplus Q))\oplus S. \arrow[l, "a\oplus S"']
\end{tikzcd}$$

Clearly, the definition can be taken over verbatim to the Γ-graded case. The structure can be viewed in (at least) two ways.

First, we can study induction theorems, following Dress [2]. Recall the Mackey-functor $\mathscr{H}$ of (9.3). If $\mathscr{C}$ is ring-like, there is a natural product on $K_0\circ\mathscr{H}$, making it a ring-valued functor, and as in [2], one sees that this is a Green functor. With this, the formal apparatus of induction theory is at our disposal. Indeed, the higher K-groups $K_i\circ H$ are Green-modules over this functor, so the theory applies there also.

On the other hand, we can use the additive structure to study the multiplicative. Let C be an object of $\mathscr{C}$. Each structure, $\oplus$ and $\otimes$, provides an 'induced' object of Rep $(\mathscr{C}, \Gamma)$: consider the object (Σ, f) induced using $\oplus$.

Proposition (11.1): *Consider* (Σ, f) *as object of* $\mathrm{Rep}\,\mathscr{C}$ *with* $\otimes$ *product: assume it faithful. Then the map in* (4.7), $\delta_1 : kP-\mathrm{Rep}(\Sigma, f) \to H^2(\Gamma, U(\mathscr{C}))$ *is surjective, so* $\omega : k\,\mathrm{Rep}(P-\mathscr{C}) \to H^2(\Gamma, U(\mathscr{C}))$ *is also onto.*

Proof: Consider Σ as the sum of objects $C(\gamma)$ indexed by Γ. Now $U(\mathscr{C})$ can act on each of these, so we have an action of $U(\mathscr{C})^\Gamma$ on the sum. As Γ acts by permuting the summands, we have an action of the wreath product $U(\mathscr{C}) \wr \Gamma$. Moreover, θ_Σ embeds $U(\mathscr{C})$ as the *diagonal* subgroup of $U(\mathscr{C})^\Gamma$ (it is here that we use the ring-like property of $\mathscr{C}$).

Represent $U(\mathscr{C}) \wr \Gamma$ as the monomial group acting on the basis $\{e_\gamma : \gamma \in \Gamma\}$, where the γ copy of $U(\mathscr{C})$ acts by multiplication on e_γ, and Γ acts by ${}^\gamma e_\delta = e_{\gamma\delta}$. For any 2-cocycle $c : \Gamma \times \Gamma \to U(\mathscr{C})$, so

$$ {}^{\gamma_0}c(\gamma_1, \gamma_2)c(\gamma_0, \gamma_1\gamma_2) = c(\gamma_0, \gamma_1)c(\gamma_0\gamma_1, \gamma_2), $$

define $f_c(\gamma)(e_\delta) = c(\gamma, \delta)e_{\gamma\delta}$. Then

$$ \begin{aligned} f_c(\gamma_0)f_c(\gamma_1)(e_{\gamma_2}) &= c(\gamma_1, \gamma_2)^{\gamma_0}c(\gamma_0, \gamma_1\gamma_2)e_{\gamma_0\gamma_1\gamma_2} \\ &= c(\gamma_0, \gamma_1)f_c(\gamma_0\gamma_1)e_{\gamma_2}. \end{aligned} $$

Thus f_c defines a projective representation corresponding to the given cocycle, which proves the result.

It is perhaps worth noting that for the important example $\mathscr{C} = \mathscr{M}od_R$, if we take $\nabla = \oplus$ then $U(\mathscr{C})$ is trivial, while for $\nabla = \otimes$, $U(\mathscr{C})$ is the multiplicative group of units of R – another reason why we are more interested in the latter case.

12. Exactness for abelian monoids

In the first section, we study kernels and cokernels; in the second, we classify morphisms; and in a final section, we study relationships between monoids and groups.

A. Kernels and Cokernels

A.1. *Every morphism* $\phi : X \to Y$ *has a kernel viz., the inclusion of the submonoid* $\phi^{-1}\{0\}$.
This is immediate from the definition of kernel.

A.2. *Every morphism* $\phi : X \to Y$ *has a cokernel viz., the projection of* Y *on its quotient by the equivalence relation*

$$ y_1 \sim y_2 \Leftrightarrow \exists\, x_1, x_2 \in X, \qquad y_1 + \phi(x_1) = y_2 + \phi(x_2). $$

First we check that this is an equivalence relation. Symmetry and reflexivity are clear, and if (x_1, x_2) resp. (x_2', x_3) give equivalences $y_1 \sim y_2 \sim y_3$, then

$$\begin{aligned} y_1+\phi(x_1+x_2') &= (y_1+\phi(x_1))+\phi(x_2') = y_2+\phi(x_2)+\phi(x_2') \\ &= (y_3+\phi(x_3))+\phi(x_2) = y_3+\phi(x_2+x_3). \end{aligned}$$

Further, each element of $\phi(X)$ is equivalent to 0.

Suppose conversely $\psi: Y \to Z$ such that $\psi \circ \phi = 0$. It will suffice to show that $y_1 \sim y_2$ implies $\psi(y_1) = \psi(y_2)$. But if $y_1 \sim y_2$, then

$$\begin{aligned} \psi(y_1) &= \psi(y_1)+\psi(\phi(x_1)) = \psi(y_1)+\phi(x_1)) \\ &= \psi(y_2+\phi(x_2)) = \psi(y_2). \end{aligned}$$

For any abelian monoid X, write $\operatorname{Sub} X$ for the set of submonoids of X. We can also define this as the set of equivalence classes of injective homomorphisms $\phi: Y \to X$, where $\phi_1 \sim \phi_2$ if for some isomorphism ψ, $\phi_1 = \phi_2 \circ \psi$. If for any ψ(necessarily injective) $\phi_1 = \phi_2 \circ \psi$, we write $Y_1 \leqq Y_2$: of course this is equivalent to $\phi_1(Y_1) \subseteq \phi_2(Y_2)$. However, the definition as given dualises at once to give a partly ordered set $\operatorname{Quot} X$ of surjective homomorphisms with domain X. Observe that if (in either case) $Y_1 \leqq Y_2 \leqq Y_1$, then $Y_1 \sim Y_2$.

A.3. Ker, Cok *define a Galois connexion between* $\operatorname{Sub} X$ *and* $\operatorname{Quot} X$.

For any $\phi: X \to Y$, $\operatorname{Ker} \phi$ is an injective homomorphism $K \to X$, defined precisely up to equivalence in the above sense. It is also clear that if ϕ_1, ϕ_2 are equivalent surjections, then $\operatorname{Ker} \phi_1 = \operatorname{Ker} \phi_2$. Thus

$$\operatorname{Ker}: \operatorname{Quot} X \to \operatorname{Sub} X$$

is well-defined; dually, so is

$$\operatorname{Cok}: \operatorname{Sub} X \to \operatorname{Quot} X.$$

Now let $\phi_1 \leqq \phi_2$ in $\operatorname{Sub} X$, say $\phi_1 = \phi_2 \circ \psi$. Then

$$\operatorname{Cok} \phi_2 \circ \phi_1 = \operatorname{Cok} \phi_2 \circ \phi_2 \circ \psi = 0 \circ \psi = 0,$$

so by universality, $\operatorname{Cok} \phi_2$ factors through $\operatorname{Cok} \phi_1$, i.e.

$$\operatorname{Cok} \phi_2 \leqq \operatorname{Cok} \ \phi_1.$$

By the same argument, if $\psi_1 \leqq \psi_2$ in $\operatorname{Quot} X$, then

$$\operatorname{Ker} \psi_1 \geqq \operatorname{Ker} \psi_2.$$

Finally, for any $\phi \in \operatorname{Sub} X$, $\operatorname{Cok} \phi \circ \phi = 0$, so by universality ϕ factors through $\operatorname{Ker} \operatorname{Cok} \phi$. Thus

$$\phi \leqq \operatorname{Ker}\operatorname{Cok}\phi.$$

The Galois connexion defines a closure operation Ker Cok (resp. Cok Ker) on Sub X (resp. Quot X) in which the closed objects are the kernels (resp. cokernels). Hence, in particular,

A.4. *$\phi: X \to Y$ is a kernel iff $\phi = \operatorname{Ker}\operatorname{Cok}\phi$, or equivalently, iff $x \in X$, $y \in Y$, $\phi(x) + y \in \phi(X)$ implies $y \in \phi(X)$.*
For the first assertion note that either condition implies ϕ injective. The second follows using the characterisations of Ker, Coker in 1.1, 1.2.

A.5. *$\phi: X \to Y$ is a cokernel iff $\phi = \operatorname{Cok}\operatorname{Ker}\phi$, or equivalently, iff $\phi(x_1) = \phi(x_2)$ implies that for some k_1, $k_2 \in X$ with $\phi(k_1) = \phi(k_2) = 0$, $x_1 + k_1 = x_2 + k_2$.*
The same comments apply in this case also.

Any map $\phi: X \to Y$ has a unique (up to equivalence) factorisation $\phi = \alpha \circ \beta$ with β surjective and α injective. We write $\alpha = \operatorname{Im}\phi$, $\beta = \operatorname{Coim}\phi$ Further, clearly $\operatorname{Ker}\phi = \operatorname{Ker}(\operatorname{Coim}\phi)$ and $\operatorname{Cok}\phi = \operatorname{Cok}(\operatorname{Im}\phi)$. Since A.3 emphasises the importance of kernels and cokernels, and the dual role they play, we make the

DEFINITION (A.6): The sequence $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ is *exact if*

$$\operatorname{Im}\phi = \operatorname{Ker}\psi = \operatorname{Ker}(\operatorname{Coim}\psi)$$

and

$$\operatorname{Coim}\psi = \operatorname{Coker}\phi = \operatorname{Cok}(\operatorname{Im}\phi).$$

B. Classification of Morphisms

We first list the properties of a morphism $\phi: X \to Y$ whose interrelations we will study: these occur in dual pairs.

(a) ϕ is *injective* resp. *surjective* as map of sets.

(b) ϕ is a *kernel* resp. *cokernel*, in the sense studied above.

(c) ϕ is an *equaliser* resp. *coequaliser*. Given two morphisms ψ_1, ψ_2: $Y \to Z$, ϕ is an equaliser if $\psi_1 \circ \phi = \psi_2 \circ \phi$, and whenever $\psi_1 \circ \theta = \psi_2 \circ \theta$, $\theta = \phi \circ \chi$ for a unique χ. The notion of coequaliser is dual.

(d) ϕ is a *monomorphism* resp, *epimorphism*. The former means that $\phi \circ a = \phi \circ b$ implies $a = b$.

(e) ϕ is a *weak monomorphism* resp. *weak epimorphism*. Here, the condition is that $\phi \circ a = 0$ (resp. $a \circ \phi = 0$) implies $a = 0$, or equivalently that $\operatorname{Ker}\phi = 0$ resp. $\operatorname{Cok}\phi = 0$.

(f) ϕ is *cofinal*, i.e. for all $y \in Y$ we can find $x \in X$, $y' \in Y$ with $\phi(x) = y + y'$. It is easily seen that this is equivalent to having $\operatorname{Cok}\phi$ a group. The dual notion is thus that $\operatorname{Ker}\phi$ is a group; call this *coinitial*.

As a preliminary analysis, we recall the canonical factorisation $\phi = \operatorname{Im}\phi \circ \operatorname{Coim}\phi$ with $\operatorname{Coim}\phi$ surjective and $\operatorname{Im}\phi$ injective. Our conditions reduce as follows:

(a) is clear.

(b), (c) Any kernel or equaliser is injective; any cokernel or coequaliser is surjective.

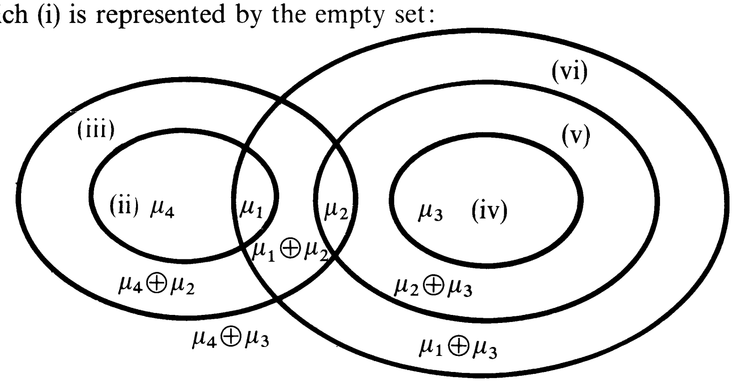
(d), (e), (f) ϕ is a monomorphism, weak monomorphism or coinitial if and only if $\operatorname{Coim}\phi$ is; it is an epimorphism, weak epimorphism or cofinal if and only if $\operatorname{Im}\phi$ is. All these assertions are trivial consequences of the definitions.

It thus suffices to analyse injective maps with respect to the properties: (i) isomorphism, (ii) kernel, (iii) equaliser, (iv) epimorphism, (v) weak epimorphism, (vi) cofinal, and surjective maps with respect to the dual properties.

B.1. (iii) $\Leftarrow$ (ii) $\Leftarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) *in either case.*
These implications are immediate consequences of the definitions.

B.2. (iii) & (iv) $\Rightarrow$ (i), (ii) & (v) $\Rightarrow$ (i).
If ϕ equalises a and b, $a \circ \phi = b \circ \phi$. If ϕ is an epimorphism, it follows that $a = b$. But the equaliser of a with itself is the class of the identity map. The second case is similar, but with $b = 0$. The arguments are just the same in the dual situations.

B.3. *For injective maps, there are no further relations among the classes.* Those obtained so far may be indicated as follows on a Venn diagram in which (i) is represented by the empty set:



We now consider some examples. Throughout $\boldsymbol{Z}_+$ is the monoid of non negative integers. X is the monoid of elements x_n ($n \in \boldsymbol{Z}_+$, $n \neq 1$), y_1 and z_1, addition being defined so that $y_1 + z_1 = x_2$ and so that each of the maps $\phi, \psi : \boldsymbol{Z}_+ \to X$ is a monoid homomorphism, where $\phi(n) = \chi(n) = x_n$ ($n \neq 1$), $\phi(1) = y_1$, $\psi(1) = z_1$. Now we get:

$\mu_1 : 2\boldsymbol{Z}/4\boldsymbol{Z} \subset \boldsymbol{Z}/4\boldsymbol{Z}$ is a kernel (of ε_1 below) and cofinal.

$\mu_2: \boldsymbol{Z}_+ - \{1\} \subset \boldsymbol{Z}_+$ is an equaliser, of ϕ and ψ above and a weak epimorphism.

$\mu_3: \boldsymbol{Z}_+ \subset \boldsymbol{Z}$ is an epimorphism.

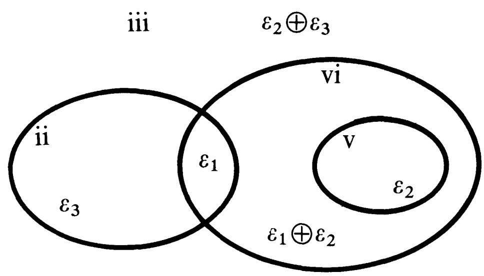
$\mu_4: \{0\} \subset \boldsymbol{Z}_+$ is a kernel but not cofinal.

Since each of properties (i)–(vi) holds for a direct sum if and only if it holds for each factor, the chart above indicates how each possibility is realised.

Here, however, duality breaks down at last.

B.4. *Every surjective map is a coequaliser.*
Let $\phi\colon\ X \to Y$ be surjective. Let C be the coproduct (= restricted direct product) of copies of $\boldsymbol{Z}_+$ (representing the forgetfunctor from abelian monoids to sets), indexed by pairs $(x, x') \in X \times X$ with $\phi(x) = \phi(x')$. Let α resp. $\alpha'\colon C \to X$ be the morphism taking the generator 1 of the copy of $\boldsymbol{Z}_+$ indexed by (x, x') to x resp. x'. It is easy to see that ϕ is the coequaliser of α and α'.

B.5. *For surjective maps, there are no further relations among the classes.* The classes are: (i) isomorphism, (ii) cokernel, (iii) coequaliser, (iv) monomorphism, (v) weak monomorphism, (vi) coinitial. The Venn diagram (excluding isomorphisms) showing relations now obtained is



as every monomorphism is an isomorphism.

It now suffices to exhibit the examples

$\varepsilon_1: \boldsymbol{Z}/4\boldsymbol{Z} \to \boldsymbol{Z}/2\boldsymbol{Z}$ is a cokernel (of μ_1) and coinitial

$\varepsilon_2: \boldsymbol{Z}_+ \to \{0, 1\}$ with $\varepsilon_2(n) = 1$ for $n \geqq 1$ is a weak monomorphism, and

$\varepsilon_3: \boldsymbol{Z}_+ \to \{0\}$ is a cokernel, not coinitial.

C. Exact sequences

We defined $X \xrightarrow{f} Y \xrightarrow{g} Z$ to be exact if $\operatorname{Im} f = \operatorname{Ker} g$ and $\operatorname{Cok} f = \operatorname{Coim} g$. Thus if $\phi: A \to B$ is a surjection which is not a cokernel,

$$\operatorname{Ker} \phi \longrightarrow A \xrightarrow{\ \phi\ } B$$

is not exact since the second condition fails: dually, if ϕ is an injection which is not a kernel,

$$A \longrightarrow B \xrightarrow{\phi} \operatorname{Cok} \phi$$

satisfies the second condition but not the first.

We can further analyse our condition into (i) $\operatorname{Im} f \leqq \operatorname{Ker} g$, (ii) $\operatorname{Im} f \geqq \operatorname{Ker} g$, (iii) $\operatorname{Cok} f \leqq \operatorname{Coim} g$, (iv) $\operatorname{Cok} f \geqq \operatorname{Coim} g$. Using the construction of kernels and cokernels in A.1, A.2 these give

(i)$\Leftrightarrow$(iv)$\Leftrightarrow g \circ f = 0$

(ii): $g(y) = 0 \Rightarrow y = f(x)$, some x

(iii): $g(y_1) = g(y_2) \Rightarrow y_1 + f(x_1) = y_2 + f(x_2)$, some x_1, x_2.

We have just seen that these conditions are in general independent.

C.1. *If $f(X)$ is a group,* (iii)$\Rightarrow$(ii). For we can take $y_2 = 0$ and write $f(x) = f(x_2) - f(x_1)$.

C.2. *If $g(Y)$ is a group, which happens in particular when f is cofinal* (for then $\operatorname{Cok} f$ is a group, and hence also $\operatorname{Coim} g \leqq \operatorname{Cok} f$), (ii)$\Rightarrow$(iii). For given $g(y_1) = g(y_2)$, choose y_3 with $g(y_3) = -g(y_1)$. Applying (ii) to $y_1 + y_3$, $y_2 + y_3$ we receive x_2, x_1 with

$$y_1 + y_3 = f(x_2), \qquad y_2 + y_3 = f(x_1).$$

But then $y_1 + f(x_1) = y_1 + y_2 + y_3 = y_2 + f(x_2)$.

Another special case of C.2 is when there is a homomorphism $D: Y \to Y$ with $g \circ D + g = 0$. For then $g(D(y))$ is inverse to $g(y)$, so again $g(Y)$ is a group.

Exactness is a useful term to convey information, but is much less useful as a tool for monoids than for groups, partly due to the (above noted) failure of exactness of the kernel-cokernel sequence. We do not know any generalisation of the 5 lemma or the snake lemma, which can be regarded as test cases: but as one can try such a variety of hypotheses (see preceding chapter), there may well be one.

If $A' \to A \to A''$ is exact with A', A'' groups, then A need not be, as we see from the example $0 \to \boldsymbol{Z}^+ \to \boldsymbol{Z}$. However,

C.3. *If $A' \xrightarrow{p} A \xrightarrow{q} A'' \to 0$ is exact with A', A'' groups, then so is A.* For given $a \in A$, choose $b \in A''$ with $q(a) + b = 0$. By exactness, q is surjective, so we can find $a_1 \in A$ with $q(a_1) = b$. Then $a + a_1 \in \operatorname{Ker} q = \operatorname{Im} p$, say $a + a_1 = p(c)$. Then $a_1 + p(-c)$ is an inverse to a in A. Observe the asymmetry of this result: a quotient monoid of a group is a group, but a submonoid need not be.

C.4. *The category of $\mathscr{A}\mathscr{G}$ is contained in $\mathscr{A}\mathscr{M}$ as a reflective and coreflective subcategory.*

For X a monoid, we have

$$C(X) = \{x \in X : \exists x' \in X, x+x' = 0\}$$

$$G(X) = X \times X/(a, b) \sim (c, d) \qquad \text{if } \exists e, a+d+e = b+c+e.$$

There are then natural transformations $C(X) \to X \to G(X)$. The construction of G is usually attributed to Grothendieck, but in fact goes back much further. It is amusing to note that $C(X) \subset X$ is a kernel and $X \times X \to G(X)$ a cokernel: in fact, the cokernel of the diagonal map. However in general the functors C and G are not good for exactness properties. C preserves injective maps and G surjective ones. Here is a kernel μ with $G(\mu) = 0$. A any group. $B = A \vee \boldsymbol{Z}^+$ with $a+m = m$ for $m > 0$. Then $A \subset B$ is a kernel, but $G(A) = A \to \boldsymbol{Z} = G(B)$ is zero.

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