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A CONDITION EQUIVALENT TO COVERING DIMENSION FOR NORMAL SPACES

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In this paper a concept called *boundary covering dimension* is defined. Boundary covering dimension is proven to be equivalent to covering dimension for normal spaces. Also included is a definition of complete boundary covering dimension. Complete boundary covering dimension is proven to be equivalent to complete covering dimension for paracompact T_2-spaces (complete covering dimension is equivalent to covering dimension for paracompact T_2-spaces).

NOTATIONS: If X is a space and $V \subset X$, then $B(V)$ denotes the boundary of V. If X is a space, $M \subset X$, and $H \subset M$, then $B(M, H)$ denotes the boundary in the subspace M of H.

DEFINITIONS: The collection G of subsets of the space X is *discrete* means every point of X is contained in an open set that intersects at most one element of G.

Covering dimension is denoted by *dim*. $\dim X \leqq n$ means if G is a *finite* open cover of X, then there exists an open cover R of X such that R refines G and $\operatorname{ord} R \leqq n+1$.

Boundary covering dimension is denoted by *bcd*. For $n \geqq 1$, $\operatorname{bcd} X \leqq n$ means if H is a closed set, W is an open set, $H \subset W$, and G is a *finite* open cover of X, then there are an open set V and discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H \subset V \subset W$, $\bigcup_{j=1}^{n} G_j$ refines G, and $B(V) = \bigcup(\bigcup_{j=1}^{n} G_j)$. Now $\operatorname{bcd} X = n$ means $\operatorname{bcd} X \leqq n$ and $\operatorname{bcd} X \nleqq n-1$.

Complete covering dimension is denoted by *complete dim*. Complete $\dim X \leqq n$ means if G is an open cover of X, then there exists an open cover R of X such that R refines G and $\operatorname{ord} R \leqq n+1$.

Complete boundary covering dimension is denoted by *complete bcd*. For $n \geqq 1$, complete $\operatorname{bcd} X \leqq n$ means if H is a closed set, W is an open set, $H \subset W$, and G is an open cover of X, then there exist an open set V and discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H \subset V \subset W$, $\bigcup_{j=1}^{n} G_j$ refines G, and $B(V) = \bigcup(\bigcup_{j=1}^{n} G_j)$.

[1] The work for that paper was done while the author was on a Cottrell College Science Grant for Research Corporation.

Remark: What is meant by bcd $X \leqq 0$? Let us note that the definition for bcd can be written another way. bcd $X \leqq n$ means if H is a closed set, W is an open set, $H \subset W$, and G is a finite open cover of X, then there are an open set V and a collection T of at most n elements such that $H \subset V \subset W$, each element of T is a discrete collection of closed sets, $\bigcup T$ refines G, and $B(V) = \bigcup(\bigcup T)$. So when $n = 0$, $T = \phi$ and $B(V) = \phi$. Thus, bcd $X \leqq 0$ means Ind $X \leqq 0$. In our proofs we will not be considering the case where $n = 0$ since it will be evident what the proof would be for $n = 0$.

Lemma 1: *If X is a topological space,* bcd $X \leqq n$, *and M is a closed subset of X, then* bcd $M \leqq n$. (*The proof is straight-forward and will not be given.*)

Lemma 2: *If X is a topological space and* bcd $X \leqq n$, *then if H is a closed set, W is an open set, $H \subset W$, and G is a finite open cover of X, then there are an open set V and finite discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H \subset V \subset W$, $\bigcup_{i=1}^{n} G_i$ refines G, and $B(V) = \bigcup(\bigcup_{i=1}^{n} G_i)$.* (*The proof is straight-forward and will not be given.*)

Lemma 3: *If each of $G_1, G_2, \cdots, G_n$ is a finite open cover of the topological space X, then there is a finite open cover G of X such that for every $i \in \{1, \cdots, n\}$, G refines G_i.*

Proof: For every $p \in X$, let $T(p) = \{g \mid \exists i \in \{1, \cdots, n\}$ such that $g \in G_i$ and $p \in g\}$. Let $G = \{\bigcap T(p) \mid p \in X\}$. G is a finite open cover of X such that for every $i \in \{1, \cdots, n\}$. G refines G_i.

Lemma 4: *If X is a paracompact T_2-space, $M \subset X$, M is closed, n is a positive integer, G is a collection of open sets of X covering M, and no point of M belongs to $n+1$ elements of G, then there exist discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $\bigcup_{j=1}^{n} G_j$ refines G and $\bigcup(\bigcup_{j=1}^{n} G_j) = M$.*

Proof: Since every paracompact T_2-space is collectionwise normal, Theorem 2 of [1] can be applied to prove the Lemma.

Lemma 5: *If X is a normal topological space, $M \subset X$, M is closed, n is a positive integer, G is a finite collection of open sets of X covering M, and no point of M belongs to $n+1$ elements of , then there exist discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $\bigcup_{j=1}^{n} G_j$ refines G and $\bigcup(\bigcup_{j=1}^{n} G_j) = M$.*

Proof: The proof is similar to the proofs of Theorem 1 and Theorem 2 of [1]. Only normality is needed instead of collectionwise normality since the open cover G is finite.

THEOREM 1: *If X is a normal topological space, then* $\operatorname{bcd} X = \dim X$.

PROOF:

Part I: Show $\dim X \leqq \operatorname{bcd} X$. Assume n is a positive integer and $\operatorname{bcd} X \leqq n$. Assume G is a finite open cover of X. Let $G = \{g_1, \cdots, g_m\}$. Let $H_1 = g_1 - (\bigcup_{j=2}^m g_j) = X - \bigcup_{j=2}^m g_j$. Now g_1 is an open set containing the closed set H_1. Since $\operatorname{bcd} X \leqq n$, by Lemma 2, there exist an open set V_1, and finite discrete collections $L_1, L_2, \cdots, L_n$ of closed sets such that $H_1 \subset V_1 \subset g_1$, $\bigcup_{j=1}^n L_j$ refines G, and $B(V_1) = \bigcup(\bigcup_{j=1}^n L_j)$. For every $j \in \{1, \cdots, n\}$, let $S(1, j) = L_j$. Let $X_1 = X$.

Assume k is a positive integer such that $1 \leqq k \leqq m$ and for every $i \in \{1, \cdots, k\}$,

(a) $X_i = X - \bigcup_{j=1}^{i-1} V_j = X_{i-1} - V_{i-1}$

(b) $H_i = X_i - \bigcup_{j=i+1}^m g_j$

(c) $H_i \subset V_i \subset g_i$, $V_i \subset X_i$, V_i open in X_i (Hence X_i and H_i are closed in X)

(d) $\forall j \in \{1, \cdots, n\}$, $S(i, j)$ is a finite discrete collection of closed sets and $S(i, j)$ refines G, and

(e) $\bigcup(\bigcup\limits_{j=1}^{n} S(i, j)) = \bigcup\limits_{j=1}^{i} B(X_j, V_j)$.

Now let $X_{k+1} = X - \bigcup_{j=1}^k V_k = X_k - V_k$ and let $H_{k+1} = X_{k+1} - \bigcup_{j=k+2}^m g_j$

Now $H_{k+1} \subset g_{k+1}$. For every $j \in \{1, \cdots, n\}$, let $E_j = \{e(j, w) | w \in S(k, j)\}$ and $F_j = \{f(j, w) | w \in S(k, j)\}$ be finite discrete collections of open sets such that F_j refines G, $\forall w \in S(k, j)$ $w \subset e(j, w) \subset \overline{e(j, w)} \subset f(j, w)$ and $f(j, w)$ intersects only one element of $S(k, j)$, and let $T_j = \{f(j, w) \cap X_{k+1} | w \in S(k, j)\} \cup \{[g - (\overline{\bigcup E_j})] \cap X_{k+1} | g \in G\}$. By Lemma 3, there is a finite cover T of X_{k+1} such that each element of T is open in X_{k+1}, and for every $j \in \{1, \cdots, n\}$, T refines T_j. By Lemma 1, $\operatorname{bcd} X_{k+1} \leqq n$ so by Lemma 2 there exist a set V_{k+1}, open in X_{k+1}, and finite discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H_{k+1} \subset V_{k+1} \subset g_{k+1}$, $\bigcup_{j=1}^n G_j$ refines T, and $B(X_{k+1}, V_{k+1}) = \bigcup(\bigcup_{j=1}^n G_j)$. $\forall j \in \{1, \cdots, n\}$, $\forall w \in S(k, j)$, let $b(j, w) = \{w\} \cup \{h | h \in G_j$ and $h \subset f(j, w)\}$. $\forall j \in \{1, \cdots, n\}$, let $M_j = \{h | h \in G_j$ and $\forall w \in S(k, j)$, $h \notin b(j, w)\}$ and let $S(k+1, j) = \{\bigcup b(j, w) | w \in S(k, j)\} \cup M_j$. $\forall j \in \{1, \cdots, n\}$, $S(k+1, j)$ is a finite collection of closed sets and $S(k+1, j)$ refines G.

Assume $j \in \{1, \cdots, n\}$. It will now be shown that $S(k+1, j)$ is discrete. Since $S(k+1, j)$ is finite, we need only to show that no two elements of $S(k+1, j)$ intersect. It should be clear that no two elements of M_j intersect and no two elements of $\{\bigcup b(j, w) | w \in S(k, j)\}$ intersect. Assume $\exists w_0 \in S(k, j)$ and $h_0 \in M_j$ such that $\bigcup b(j, w_0)$ intersects h_0.

Case 1: $\exists h_1 \in G_j$ such that $h_1 \subset f(j, w_0)$ and h_1 intersects h_0. Since no two elements of G_j intersect, $h_0 = h_1$. $\forall w \in S(k, j)$, $h_0 \notin b(j, w)$ since

$h_0 \in M_j$. But h_0, which is h_1, is an element of $b(j, w_0)$. Contradiction.

Case 2: *h_0 intersects w_0*. Since G_j refines T which refines T_j, there is an element g_0 of T_j such that $h_0 \subset g_0$. Thus g_0 intersects w_0, and $w_0 \subset \overline{\bigcup E_j}$. No element of $\{[g-(\overline{\bigcup E_j})] \cap X_{k+1} | g \in G\}$ intersects $\overline{\bigcup E_j}$ so $g_0 \in \{f(j, w) \cap X_{k+1} | w \in S(k, j)\}$. Thus $\exists w_1 \in S(k, j)$ such that $g_0 = f(j, w_1) \cap X_{k+1}$. This means $h_0 \subset f(j, w_1)$. So $h_0 \in b(j, w_1)$. Since $h_0 \in M_j$, we know that $\forall w \in S(k, j)$, $h_0 \notin b(j, w)$. This means $h_0 \notin b(j, w_1)$, but $h_0 \in b(j, w_1)$. Contradiction. Therefore, no two elements of $S(k+1, j)$ intersect.

It follows that $\bigcup(\bigcup_{j=1}^{n} S(k+1, j)) = \bigcup_{j=1}^{k+1} B(X_j, V_j)$. We have now completed our inductive definition. Thus each of $S(m, 1), S(m, 2), \cdots, S(m, n)$ is a finite discrete collection of closed sets that refines G. $\forall j \in \{1, \cdots, n\}$, let Z_j be a finite discrete collection of open sets such that $S(m, j)$ refines Z_j and Z_j refines G. $\forall i \in \{1, \cdots, m\}$, let $V_i' = V_i - [\bigcup(\bigcup_{j=1}^{n} Z_j)]$. Now $\{V_1', V_2', \cdots, V_m'\}$ is a finite collection of mutually exclusive closed sets such that $\forall i \in \{1, \cdots, m\}$, $V_i' \subset g_i$. Let $Z_{n+1} = \{a_1, \cdots, a_m\}$ be a finite discrete collection of open sets such that $\forall i \in \{1, \cdots, m\}$, $V_i' \subset a_i \subset g_i$. Let $Z = \bigcup_{j=1}^{n+1} Z_j$. Z is an open cover of X such that Z refines G and ord $Z \leqq n+1$. Thus $\dim X \leqq n$.

Part II: *Show* $\operatorname{bcd} X \leqq \dim X$. Assume n is a positive integer and $\dim X \leqq n$. Assume H is a closed set, W is an open set $H \subset W$, and G is a finite open cover of X. Let F be a finite open cover of X such that F refines G and every element of F that intersects H is a subset of W. Let $T = \{t_i | i = 1, \cdots, k\}$ be a finite open cover of X such that T refines F, ord $T \leqq n+1$, and if $i \neq j$, then $t_i \neq t_j$. Let $R = \{r_i | i = 1, \cdots, k\}$ be an open cover of X such that $\forall i \in \{1, \cdots, k\}$, $\overline{r_i} \subset t_i$. Let $V = \bigcup \{r_i | i \in \{1, \cdots, k\}$ and r_i intersects $H\}$. Assume $p \in B(V)$ and $n+1$ elements of R contain p. There exist positive integers $j_1 < j_2 < \cdots < j_{n+1} \leqq k$ such that $\forall i \in \{1, \cdots, n+1\}$, $p \in r_{j_i}$. Since R is finite, $\exists j_{n+2} \in \{1, \cdots, k\}$ such that $p \in B(r_{j_{n+2}})$. $\forall i \in \{1, \cdots, n+2\}$, $p \in t_{j_i}$ since $\overline{r_{j_i}} \subset t_{j_i}$. Thus, $n+2$ elements of T contain p, which is a contradiction. Therefore no point of $B(V)$ is contained by $n+1$ elements of R. By Lemma 5, there exist discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $\bigcup_{j=1}^{n} G_j$ refines G and $B(V) = \bigcup(\bigcup_{j=1}^{n} G_j)$. So $\operatorname{bcd} X \leqq n$.

THEOREM 2: *If X is a paracompact T_2-space, then* $\operatorname{bcd} X = \dim X =$ complete $\operatorname{bcd} X =$ complete $\dim X$.

PROOF: Assume X is a paracompact T_2-*space*. Theorem II.6 page 22 of [2] makes it clear $\dim X =$ complete $\dim X$, Theorem 1 gives us $\operatorname{bcd} X = \dim X$. It is trivial that $\operatorname{bcd} X \leqq$ complete $\operatorname{bcd} X$. It will now be shown that complete $\operatorname{bcd} X \leqq \operatorname{bcd} X$. Assume n is positive integer and $\operatorname{bcd} X \leqq n$. Thus $\dim X \leqq n$, and hence complete $\dim X \leqq n$.

Assume H is a closed set, W is an open set, $H \subset W$, and G is an open cover of X. Let F be an open cover of X such that F refines G and every element of F that intersects H is a subset of W. Let $T = \{t_b | b \in B\}$ be a *locally finite* open cover of X such that T refines F, $\operatorname{ord} T \leqq n+1$, and if $b_1, b_2 \in B$ and $b_1 \neq b_2$ then $t_{b_1} \neq t_{b_2}$ (Theorem 3 of [1] assures the existence of such a T). Let $R = \{r_b | b \in B\}$ be an open cover of X such that $\forall b \in B$, $\overline{r_b} \subset t_b$. Let $V = \bigcup \{r_b | b \in B$ and r_b intersects $H\}$. Assume $p \in B(V)$ and $n+1$ elements of R contain p. There exist $n+1$ elements $b_1, b_2, \cdots, b_{n+1}$ of B such that $\forall i \in \{1, \cdots, n+1\}$, $p \in r_{b_i}$. Since R is locally finite, there exists $b_{n+2} \in B$ such that $p \in B(r_{b_{n+2}})$. $\forall i \in \{1, \cdots, n+2\}$, $p \in t_{b_i}$, since $\overline{r_{b_i}} \subset t_{b_i}$. Thus $n+2$ elements of R contain p, which is a contradiction. Therefore, no point of $B(V)$ is contained by $n+1$ elements of R. By Lemma 4, there exist discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $\bigcup_{j=1}^{n} G_j$ refines G and $B(V) = \bigcup (\bigcup_{j=1}^{n} G_j)$. Thus complete bcd $X \leqq n$. Therefore bcd $X = \dim X =$ complete bcd $X =$ complete dim X.

COROLLARY: *Assume X is a normal topological space. Then* $\dim X \leqq n$ *if and only if for all mutually exclusive closed sets H and K, for every finite (the word 'finite' can be deleted for X a paracompact T_2-space) open cover G of X, there exist mutually exclusive open sets D_H and D_K and a collection T of at most n elements such that $H \subset D_H$, $K \subset D_K$, every element of T is a discrete collection of closed sets, $\bigcup T$ refines G, and $X - (D_H \cup D_K) = \bigcup (\bigcup T)$.*

PROOF: The proof follows from Theorem 1 (If X is T_2-paracompact and the open cover G is not necessarily finite, then the proof follows from Theorem 2).

REMARK: Note the similarity between the above Corollary and the following familiar theorem on large inductive dimension (denoted Ind): For X normal, Ind $X \leqq n$ if and only if for all mutually exclusive open sets H and K, there exist mutually exclusive open sets D_H and D_K and a closed set T such that $H \subset D_H$, $K \subset D_K$, Ind $T \leqq n-1$, and $X - (D_H \cup D_K) = T$. The similarity of the Corollary and this theorem on Ind enable one to pattern some dim proofs after some Ind proofs.

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