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A CONDITION EQUIVALENT TO COVERING DIMENSION FOR NORMAL SPACES

James Austin French

In this paper a concept called boundary covering dimension is defined. Boundary covering dimension is proven to be equivalent to covering dimension for normal spaces. Also included is a definition of complete boundary covering dimension. Complete boundary covering dimension is proven to be equivalent to complete covering dimension for paracompact $T_2$-spaces (complete covering dimension is equivalent to covering dimension for paracompact $T_2$-spaces).

**Notations:** If $X$ is a space and $V \subseteq X$, then $B(V)$ denotes the boundary of $V$. If $X$ is a space, $M \subseteq X$, and $H \subseteq M$, then $B(M, H)$ denotes the boundary in the subspace $M$ of $H$.

**Definitions:** The collection $G$ of subsets of the space $X$ is discrete means every point of $X$ is contained in an open set that intersects at most one element of $G$.

*Covering dimension* is denoted by $\dim$. $\dim X \leq n$ means if $G$ is a finite open cover of $X$, then there exists an open cover $R$ of $X$ such that $R$ refines $G$ and $\text{ord } R \leq n + 1$.

*Boundary covering dimension* is denoted by $\text{bcd}$. For $n \geq 1$, $\text{bcd } X \leq n$ means if $H$ is a closed set, $W$ is an open set, $H \subseteq W$, and $G$ is a finite open cover of $X$, then there are an open set $V$ and discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H \subseteq V \subseteq W$, $\bigcup_{j=1}^n G_j$ refines $G$, and $B(V) = \bigcup (\bigcup_{j=1}^n G_j)$. Now $\text{bcd } X = n$ means $\text{bcd } X \leq n$ and $\text{bcd } X \leq n - 1$.

*Complete covering dimension* is denoted by $\text{complete dim}$. Complete $\dim X \leq n$ means if $G$ is an open cover of $X$, then there exists an open cover $R$ of $X$ such that $R$ refines $G$ and $\text{ord } R \leq n + 1$.

*Complete boundary covering dimension* is denoted by $\text{complete bcd}$. For $n \geq 1$, complete $\text{bcd } X \leq n$ means if $H$ is a closed set, $W$ is an open set, $H \subseteq W$, and $G$ is an open cover of $X$, then there exist an open set $V$ and discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H \subseteq V \subseteq W$, $\bigcup_{j=1}^n G_j$ refines $G$, and $B(V) = \bigcup (\bigcup_{j=1}^n G_j)$.

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1 The work for that paper was done while the author was on a Cottrell College Science Grant for Research Corporation.
Remark: What is meant by $\text{bcd } X \leq 0$? Let us note that the definition for $\text{bcd}$ can be written another way. $\text{bcd } X \leq n$ means if $H$ is a closed set, $W$ is an open set, $H \subset W$, and $G$ is a finite open cover of $X$, then there are an open set $V$ and a collection $T$ of at most $n$ elements such that $H \subset V \subset W$, each element of $T$ is a discrete collection of closed sets, $\bigcup T$ refines $G$, and $B(V) = \bigcup (\bigcup T)$. So when $n = 0$, $T = \emptyset$ and $B(V) = \emptyset$.

Thus, $\text{bcd } X \leq 0$ means $\text{Ind } X \leq 0$. In our proofs we will not be considering the case where $n = 0$ since it will be evident what the proof would be for $n = 0$.

Lemma 1: If $X$ is a topological space, $\text{bcd } X \leq n$, and $M$ is a closed subset of $X$, then $\text{bcd } M \leq n$. (The proof is straight-forward and will not be given.)

Lemma 2: If $X$ is a topological space and $\text{bcd } X \leq n$, then if $H$ is a closed set, $W$ is an open set, $H \subset W$, and $G$ is a finite open cover of $X$, then there are an open set $V$ and finite discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H \subset V \subset W$, $\bigcup_{i=1}^{n} G_i$ refines $G$, and $B(V) = \bigcup (\bigcup_{i=1}^{n} G_i)$.

(The proof is straight-forward and will not be given.)

Lemma 3: If each of $G_1, G_2, \cdots, G_n$ is a finite open cover of the topological space $X$, then there is a finite open cover $G$ of $X$ such that for every $i \in \{1, \cdots, n\}$, $G$ refines $G_i$.

Proof: For every $p \in X$, let $T(p) = \{g | \exists i \in \{1, \cdots, n\} \text{ such that } g \in G_i \text{ and } p \in g\}$. Let $G = \{\bigcap T(p) | p \in X\}$. $G$ is a finite open cover of $X$ such that for every $i \in \{1, \cdots, n\}$. $G$ refines $G_i$.

Lemma 4: If $X$ is a paracompact $T_2$-space, $M \subset X$, $M$ is closed, $n$ is a positive integer, $G$ is a collection of open sets of $X$ covering $M$, and no point of $M$ belongs to $n+1$ elements of $G$, then there exist discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $\bigcup_{i=1}^{n} G_i$ refines $G$ and $\bigcup (\bigcup_{i=1}^{n} G_i) = M$.

Proof: Since every paracompact $T_2$-space is collectionwise normal, Theorem 2 of [1] can be applied to prove the Lemma.

Lemma 5: If $X$ is a normal topological space, $M \subset X$, $M$ is closed, $n$ is a positive integer, $G$ is a finite collection of open sets of $X$ covering $M$, and no point of $M$ belongs to $n+1$ elements of $G$, then there exist discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $\bigcup_{j=1}^{n} G_j$ refines $G$ and $\bigcup (\bigcup_{j=1}^{n} G_j) = M$.

Proof: The proof is similar to the proofs of Theorem 1 and Theorem 2 of [1]. Only normality is needed instead of collectionwise normality since the open cover $G$ is finite.
\textbf{THEOREM 1:} If $X$ is a normal topological space, then $bcd X = \dim X$.  

\textbf{PROOF:}

\textit{Part 1:} Show $\dim X \leq bcd X$. Assume $n$ is a positive integer and $bcd X \leq n$. Assume $G$ is a finite open cover of $X$. Let $G = \{g_1, \cdots, g_m\}$. Let $H_1 = g_1 - (\bigcup_{j=2}^{m} g_j) = X - \bigcup_{j=2}^{m} g_j$. Now $g_1$ is an open set containing the closed set $H_1$. Since $bcd X \leq n$, by Lemma 2, there exist an open set $V_1$, and finite discrete collections $L_1, L_2, \cdots, L_n$ of closed sets such that $H_1 \subset V_1 \subset g_1$, $\bigcup_{j=1}^{n} L_j$ refines $G$, and $B(V_1) = \bigcup (\bigcup_{j=1}^{n} L_j)$. For every $j \in \{1, \cdots, n\}$, let $S(1, j) = L_j$. Let $X_1 = X$.

Assume $k$ is a positive integer such that $1 \leq k \leq m$ and for every $i \in \{1, \cdots, k\}$,

(a) $X_i = X - \bigcup_{j=1}^{i-1} V_j = X_{i-1} - V_{i-1}$

(b) $H_i = X_i - \bigcup_{j=i+1}^{m} g_j$

(c) $H_i \subset V_i \subset g_i$, $V_i \subset X_i$, $V_i$ open in $X_i$ (Hence $X_i$ and $H_i$ are closed in $X$)

(d) $\forall j \in \{1, \cdots, n\}$, $S(i, j)$ is a finite discrete collection of closed sets and $S(i, j)$ refines $G$, and

\[(e) \quad \bigcup \left( \bigcup_{j=1}^{n} S(i, j) \right) = \bigcup_{j=1}^{i} B(X_j, V_j).\]

Now let $X_{k+1} = X - \bigcup_{j=1}^{k} V_j = X_{k} - V_k$ and let $H_{k+1} = X_{k+1} - \bigcup_{j=k+2}^{m} g_j$

Now $H_{k+1} \subset g_{k+1}$. For every $j \in \{1, \cdots, n\}$, let $E_j = \{f(j, w) | w \in S(k, j)\}$ and $F_j = \{f(j, w) | w \in S(k, j)\}$ be finite discrete collections of open sets such that $F_j$ refines $G$, $\forall w \in S(k, j)$ $w \subset e(j, w) \subset e(j, j) \subset f(j, w)$ and $f(j, w)$ intersects only one element of $S(k, j)$, and let $T_j = \{f(j, w) \cap X_{k+1} | w \in S(k, j)\} \cup \{g - (\bigcup E_j) \cap X_{k+1} | g \in G\}$. By Lemma 3, there is a finite cover $T$ of $X_{k+1}$ such that each element of $T$ is open in $X_{k+1}$, and for every $j \in \{1, \cdots, n\}$, $T$ refines $T_j$. By Lemma 1, $bcd X_{k+1} \leq n$ so by Lemma 2 there exist a set $V_{k+1}$, open in $X_{k+1}$, and finite discrete collections $G_1, G_2, \cdots, G_n$ of closed sets such that $H_{k+1} \subset V_{k+1} \subset g_{k+1}$, $\bigcup_{j=1}^{n} G_j$ refines $T$, and $B(X_{k+1}, V_{k+1}) = \bigcup (\bigcup_{j=1}^{n} G_j)$. For every $j \in \{1, \cdots, n\}$, $\forall w \in S(k, j)$, let $b(j, w) = \{w\} \cup \{h | h \in G_j \text{ and } h \subset f(j, w)\}$. For every $j \in \{1, \cdots, n\}$, let $M_j = \{h | h \in G_j \text{ and } \forall w \in S(k, j) \text{ and } h \subset f(j, w)\}$ and let $S(k+1, j) = \{b(j, w) \cup M_j \cup S(k+1, j) \}$. Assume $j \in \{1, \cdots, n\}$. It will now be shown that $S(k+1, j)$ is discrete. Since $S(k+1, j)$ is finite, we need only to show that no two elements of $S(k+1, j)$ intersect. It should be clear that no two elements of $M_j$ intersect and no two elements of $\{\bigcup b(j, w) | w \in S(k, j)\}$ intersect. Assume $\exists w_0 \in S(k, j)$ and $h_0 \in M_j$ such that $\bigcup b(j, w_0)$ intersects $h_0$.

\textit{Case 1:} $\exists h_1 \in G_j$ such that $h_1 \subset f(j, w_0)$ and $h_1$ intersects $h_0$. Since no two elements of $G_j$ intersect, $h_0 = h_1$. $\forall w \in S(k, j)$, $h_0 \notin b(j, w)$ since
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h_0 \in M_j. But h_0, which is h_1, is an element of b(j, w_0). Contradiction.

Case 2: h_0 intersects w_0. Since G_j refines T which refines T_j, there is an

element g_0 of T_j such that h_0 \subset g_0. Thus g_0 intersects w_0, and w_0 \subset \bigcup E_j.

No element of \{[g - (\bigcup E_j)] \cap X_{k+1} \mid g \in G\} intersects \bigcup E_j so g_0 \in
\{f(j, w) \cap X_{k+1} \mid w \in S(k, j)\}. Thus \exists w_1 \in S(k, j) such that g_0 = f(j, w_1) \cap
X_{k+1}. This means h_0 \subset f(j, w_1). So h_0 \in b(j, w_1). Since h_0 \in M_j, we know
that \forall w \in S(k, j), h_0 \notin b(j, w). This means h_0 \notin b(j, w_1), but h_0 \in b(j, w_1).

Contradiction. Therefore, no two elements of S(k+1, j) intersect.

It follows that \bigcup (\bigcup_{j=1}^{k} S(k+1, j)) = \bigcup_{j=1}^{k+1} B(X_j, V_j). We have now
completed our inductive definition. Thus each of S(m, 1), S(m, 2), \ldots,
S(m, n) is a finite discrete collection of closed sets that refines G. \forall j \in
\{1, \ldots, n\}, let Z_j be a finite discrete collection of open sets such that
S(m, j) refines Z_j and Z_j refines G. \forall i \in \{1, \ldots, m\}, let V'_i = V_i - [\bigcup (\bigcup_{j=1}^{i} Z_j)].

Now \{V'_1, V'_2, \ldots, V'_m\} is a finite collection of mutually exclusive closed
sets such that \forall i \in \{1, \ldots, m\}, V'_i \subset g_i. Let Z_{n+1} = \{a_1, \ldots, a_m\} be a finite
discrete collection of open sets such that \forall i \in \{1, \ldots, m\}, V'_i \subset a_i \subset g_i.
Let Z = \bigcup_{j=1}^{n+1} Z_j. Z is an open cover of X such that Z refines G and
ord Z = n + 1. Thus dim X \leq n.

Part II: Show bcd X \leq dim X. Assume n is a positive integer and
dim X \leq n. Assume H is a closed set, W is an open set H \subset W, and G is a
finite open cover of X. Let F be a finite open cover of X such that F
refines G and every element of F that intersects H is a subset of W. Let
T = \{t_i \mid i = 1, \ldots, k\} be a finite open cover of X such that T refines F,
ord T \leq n + 1, and if i \neq j, then t_i \neq t_j. Let R = \{r_i \mid i = 1, \ldots, k\} be an
open cover of X such that \forall i \in \{1, \ldots, k\}, r_i \subset t_i. Let V = \bigcup \{r_i \mid i \in \{1, \ldots, k\}\}
and r_i intersects H. Assume p \in B(V) and n + 1 elements of R contain p.

There exist positive integers j_1, j_2 < \cdots < j_{n+1} \leq k such that \forall i \in
\{1, \ldots, n+1\}, p \in r_{j_i}. Since R is finite, \exists j_{n+2} \in \{1, \ldots, k\} such that
p \in B(r_{j_{n+2}}). \forall i \in \{1, \ldots, n+2\}, p \in t_{j_i} since r_{j_i} \subset t_{j_i}. Thus, n + 2 elements
of T contain p, which is a contradiction. Therefore no point of B(V) is
contained by n + 1 elements of R. By Lemma 5, there exist discrete collections
G_1, G_2, \ldots, G_n of closed sets such that \bigcup_{j=1}^{n} G_j refines G and
B(V) = \bigcup (\bigcup_{j=1}^{n} G_j). So bcd X \leq n.

THEOREM 2: If X is a paracompact T_2-space, then bcd X = dim X =
complete bcd X = complete dim X.

PROOF: Assume X is a paracompact T_2-space. Theorem II.6 page 22
of [2] makes it clear dim X = complete dim X, Theorem 1 gives us
bcd X = dim X. It is trivial that bcd X \leq complete bcd X. It will now
be shown that complete bcd X \leq bcd X. Assume n is positive integer
and bcd X \leq n. Thus dim X \leq n, and hence complete dim X \leq n.
Assume $H$ is a closed set, $W$ is an open set, $H \subset W$, and $G$ is an open cover of $X$. Let $F$ be an open cover of $X$ such that $F$ refines $G$ and every element of $F$ that intersects $H$ is a subset of $W$. Let $T = \{t_b | b \in B\}$ be a locally finite open cover of $X$ such that $T$ refines $F$, ord $T \leq n + 1$, and if $b_1, b_2 \in B$ and $b_1 \neq b_2$ then $t_{b_1} \neq t_{b_2}$ (Theorem 3 of [1] assures the existence of such a $T$). Let $R = \{r_b | b \in B\}$ be an open cover of $X$ such that $\forall b \in B, r_b \subset t_b$. Let $V = \bigcup \{r_b | b \in B\}$ and $r_b$ intersects $H$. Assume $p \in B(V)$ and $n+1$ elements of $R$ contain $p$. There exist $n+1$ elements $b_1, b_2, \ldots, b_{n+1}$ of $B$ such that $\forall i \in \{1, \ldots, n+1\}, p \in r_{b_i}$. Since $R$ is locally finite, there exists $b_{n+2} \in B$ such that $p \in B(r_{b_{n+2}})$. $\forall i \in \{1, \ldots, n+2\}, p \in t_{b_i}$, since $r_{b_i} \subset t_{b_i}$. Thus $n+2$ elements of $R$ contain $p$, which is a contradiction. Therefore, no point of $B(V)$ is contained by $n+1$ elements of $R$. By Lemma 4, there exist discrete collections $G_1, G_2, \ldots, G_n$ of closed sets such that $H \subset G_1$, $K \subset G_n$, every element of $T$ is a discrete collection of closed sets, $\bigcup T$ refines $G$, and $X - (G_1 \cup G_n) = \bigcup (\bigcup T)$.

**Proof:** The proof follows from Theorem 1 (If $X$ is $T_2$-paracompact and the open cover $G$ is not necessarily finite, then the proof follows from Theorem 2).

**Remark:** Note the similarity between the above Corollary and the following familiar theorem on large inductive dimension (denoted Ind): For $X$ normal, $\text{Ind} X \leq n$ if and only if for all mutually exclusive closed sets $H$ and $K$, for every finite (the word 'finite' can be deleted for $X$ a paracompact $T_2$-space) open cover $G$ of $X$, there exist mutually exclusive open sets $D_H$ and $D_K$ and a collection $T$ of at most $n$ elements such that $H \subset D_H$, $K \subset D_K$, every element of $T$ is a discrete collection of closed sets, $\bigcup T$ refines $G$, and $X - (D_H \cup D_K) = \bigcup (\bigcup T)$.

**Corollary:** Assume $X$ is a normal topological space. Then $\text{dim} X \leq n$ if and only if for all mutually exclusive closed sets $H$ and $K$, for every finite (the word 'finite' can be deleted for $X$ a paracompact $T_2$-space) open cover $G$ of $X$, there exist mutually exclusive open sets $D_H$ and $D_K$ and a collection $T$ of at most $n$ elements such that $H \subset D_H$, $K \subset D_K$, every element of $T$ is a discrete collection of closed sets, $\bigcup T$ refines $G$, and $X - (D_H \cup D_K) = \bigcup (\bigcup T)$.

**Proof:** The proof follows from Theorem 1 (If $X$ is $T_2$-paracompact and the open cover $G$ is not necessarily finite, then the proof follows from Theorem 2).

**References**


(Oblatum 1–X–1973)