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ON EXISTENCE IN THE CAUCHY PROBLEM FOR THE BIHARMONIC EQUATION

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Abstract

A necessary condition and sufficient conditions for the existence of a solution to the improperly posed Cauchy problem for the biharmonic equation are determined.

1. Introduction

A well-posed problem in partial differential equations is a boundary value and/or initial value problem for which a solution exists, is unique, and depends continuously on the data. It is well known [1] that the Cauchy problem for the Laplace equation is not well-posed in the sense that a slight change in the data may result in a large variation in the solution. Because of their appearance in applications [2], much attention has been given recently to improperly posed problems (see the bibliography to Payne's paper in [3]); little of this attention has been devoted to the question of existence, however.

In [6], Pucci cites a necessary and sufficient condition for the existence of a solution to the Cauchy problem for the Laplace equation for a rectangle in a neighborhood of the initial axis. Payne and Sather [5] also obtain a necessary and sufficient condition for the existence of a solution to the Cauchy problem for the elliptic equation

$$U_{yy}+y^{\alpha}U_{xx} = 0, \ \alpha \geqq 0, \ y > 0,$$

with initial data given on the x-axis. These results for specific equations, data, and geometry are partially because of the extreme difficulty of the existence question (see [4], p. 240, and [3], p. 7).

In [7, 8] we deduced uniqueness, continuous dependence on the data, and pointwise estimates on the solution to the Cauchy problem for a class of fourth order elliptic equations under the assumption that the solution

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is uniformly bounded. Here we present a necessary condition and sufficient conditions for existence of a solution to the Cauchy problem when that fourth order equation is the biharmonic equation.

2. Results

Let $[a, b]$ be an arbitrary but fixed closed interval of the x-axis in the plane and let Σ denote a sufficiently smooth curve in the upper half plane which intersects the x-axis at $x = a$ and $x = b$. Let R_Σ denote the domain in the upper half plane bounded by Σ and $[a, b]$.

Consider the problem of determining a function u which is $C^4(R_\Sigma)$ and $C^3(R_\Sigma \cup \Sigma \cup [a, b])$ such that u satisfies

$$\Delta^2 u = 0, \text{ in } R_\Sigma,$$

$$\left.\begin{array}{ll} u(x,0) = f(x), & u_y(x,0) = g(x) \\ \Delta u(x,0) = h(x), & (\Delta u)_y(x,0) = k(x) \end{array}\right\} \text{ on } a < x < b, \tag{2.1}$$

where Δ is the two dimensional Laplace operator and f''', g'', h', k are continuous on $[a, b]$.

Let

$$v \equiv v(\xi, \eta; x, 0) = r^2 \log r \tag{2.2}$$

denote the singular part of the fundamental solution of the biharmonic operator Δ^2, where $r^2 = (\xi - x)^2 + \eta^2$ is the square of the distance from the variable point (ξ, η) in R_Σ to the fixed point $(x, 0)$ for $a < x < b$. Further, let C_ε, for $\varepsilon > 0$ sufficiently small, be the semi-circular arc $r = \varepsilon$ in the upper half plane and denote by D_ε the domain bounded by Σ, $[a, x-\varepsilon]$, C_ε, and $[x+\varepsilon, b]$.

We now utilize Green's second identity for the biharmonic operator, namely,

$$\iint [U\Delta^2 V - V\Delta^2 U]\, d\xi\, d\eta = \oint \left[U \frac{\partial \Delta V}{\partial n} - \Delta V \frac{\partial U}{\partial n} + \Delta U \frac{\partial V}{\partial n} - V \frac{\partial \Delta U}{\partial n}\right] ds, \tag{2.3}$$

where $\partial/\partial n$ denotes the outward normal derivative operator, to deduce

$$0 = \left(\int_\Sigma + \int_a^{x-\varepsilon} + \int_{C\varepsilon} + \int_{x+\varepsilon}^b\right) \{u, v\}\, ds, \tag{2.4}$$

for the solution u of (2.1) and v given by (2.2), where

$$\{u, v\} = u \frac{\partial \Delta v}{\partial n} - \Delta v \frac{\partial u}{\partial n} + \Delta u \frac{\partial v}{\partial n} - v \frac{\partial \Delta u}{\partial n}.$$

Since $\partial/\partial n = -\partial/\partial r$ on C_ε and

$$\begin{aligned} v_r &= r(1+2\log r), \\ \Delta v &= 4(1+\log r), \\ (\Delta v)_r &= \frac{4}{r}, \end{aligned}$$

it follows by the boundedness properties of u and its derivatives and an integral mean value theorem that

$$\int_{C\varepsilon} \{u, v\}\, \mathrm{d}s \to -4\pi u(x, 0) = -4\pi f(x), \text{ as } \varepsilon \to 0. \tag{2.5}$$

Moreover, on $[a, x-\varepsilon]$ and $[x+\varepsilon, b]$, where $\partial/\partial n = -\partial/\partial\eta$, we have

$$\begin{aligned} v(\xi, 0; x, 0) &= (\xi-x)^2 \log|\xi-x|, \\ -v_\eta(\xi, 0; x, 0) &= 0, \\ \Delta v(\xi, 0; x, 0) &= 4(1+\log|\xi-x|), \\ -(\Delta v)_\eta(\xi, 0; x, 0) &= 0, \end{aligned}$$

so that

$$\lim_{\varepsilon\to 0} \left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right) \{u, v\}\, \mathrm{d}s = \int_a^b [4(1+\log|\xi-x|)g(\xi) + (\xi-x)^2(\log|\xi-x|)k(\xi)]\, \mathrm{d}\xi. \tag{2.6}$$

Consequently, letting $\varepsilon \to 0$ in (2.4), we have

$$\int_\Sigma \{u, v\}\, \mathrm{d}s = 4\pi f(x) - \int_a^b [4(1+\log|\xi-x|)g(\xi)+(\xi-x)^2(\log|\xi-x|)k(\xi)]\, \mathrm{d}\xi. \tag{2.7}$$

Since the left side of (2.7) is analytic in x for $a < x < b$, we conclude the right side must be also. Thus we obtain the following result which imposes a certain compatibility condition on the Cauchy data.

THEOREM 1: *If u is a solution of* (2.1), *then*

$$N(x) = f(x) - \frac{1}{4\pi}\int_a^b [4(1+\log|\xi-x|)g(\xi)+(\xi-x)^2(\log|\xi-x|)k(\xi)]\, \mathrm{d}\xi \tag{2.8}$$

is analytic in x for $a < x < b$.

The omission of the data term $h(x)$ in (2.8) is a consequence of the geometry. For a more general domain with sufficiently smooth boundary, one can easily obtain an analogous necessary condition. In this case, each of the initial data functions would usually appear in the condition as v and its appropriate derivatives would not normally vanish identically along the initial curve.

To establish sufficient conditions for the existence of a solution u of (2.1), we consider the function

$$w(x,y) = \frac{1}{4\pi}\int_a^b [4(1+\log\sqrt{(x-\xi)^2+y^2})\, g(\xi) + [(x-\xi)^2+y^2](\log\sqrt{(x-\xi)^2+y^2})k(\xi)]\,\mathrm{d}\xi, \tag{2.9}$$

which satisfies the biharmonic equation in $y > 0$. Moreover, proceeding formally,

$$w(x,0) = \lim_{y\to 0} w(x,y) = -N(x)+f(x),$$

$$w_y(x,0) = \lim_{y\to 0} w_y(x,y) = \lim_{y\to 0}\frac{1}{4\pi}\int_a^b \left[\frac{4g(\xi)y}{(x-\xi)^2+y^2} + yk(\xi)(1+2\log\sqrt{(x-\xi)^2+y^2})\right]\mathrm{d}\xi$$

$$= \lim_{y\to 0}\frac{y}{\pi}\int_{x-\delta}^{x+\delta}\frac{g(\xi)}{(x-\xi)^2+y^2}\,\mathrm{d}\xi = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{t^2+1}\,\mathrm{d}t = g(x),$$

for $\delta < \min\{|x-a|, |x-b|\}$,

$$\Delta w(x,0) = \lim_{y\to 0}\Delta w(x,y) = \lim_{y\to 0}\frac{1}{\pi}\int_a^b (1+\log\sqrt{(x-\xi)^2+y^2})k(\xi)\mathrm{d}\xi$$
$$= -S(x)+h(x),$$

where

$$S(x) = h(x) - \frac{1}{\pi}\int_a^b (1+\log|x-\xi|)k(\xi)\,\mathrm{d}\xi,$$

and

$$(\Delta w)_y(x,0) = \lim_{y\to 0}(\Delta w)_y(x,y) = \lim_{y\to 0}\frac{1}{\pi}\int_a^b \frac{k(\xi)y}{(x-\xi)^2+y^2}\,\mathrm{d}\xi$$
$$= \lim_{y\to 0}\frac{y}{\pi}\int_{x-\delta}^{x+\delta}\frac{k(\xi)}{(x-\xi)^2+y^2}\,\mathrm{d}\xi = k(x).$$

Now, if we assume that $N(x)$ and $S(x)$ are analytic functions of x, then by the Cauchy-Kowaleski theorem we have that there exists a solution $W(x,y)$ of

$$\Delta^2 W = 0, \text{ in } y > 0$$

$$\left.\begin{aligned} W(x,0) &= N(x), \quad W_y(x,0) = 0 \\ \Delta W(x,0) &= S(x), (\Delta W)_y(x,0) = 0 \end{aligned}\right\}\ a < x < b \tag{2.10}$$

in a neighborhood of the initial axis. Thus we arrive at the following result.

THEOREM 2: *If*

$$N(x) = f(x) - \frac{1}{4\pi}\int_a^b [4(1+\log|\xi-x|)g(\xi)+(\xi-x)^2(\log|\xi-x|)k(\xi)]\,d\xi$$

and

$$S(x) = h(x) - \frac{1}{\pi}\int_a^b (1+\log|\xi-x|)k(\xi)\,d\xi$$

are both analytic in x *on* $a < x < b$ *for* f, g, h, k *the Cauchy data prescribed in* (2.1), *then* $u(x, y) = w(x, y)+W(x, y)$ *solves* (2.1) *in a neighborhood of the initial axis* (a, b), *where* $w(x, y)$ *is given by* (2.9) *and* $W(x, y)$ *is the solution of* (2.10).

Clearly, the determination of sufficient conditions for a more general geometry would not be as easy to obtain as analogous necessary conditions would be.

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