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M. H. EGGAR

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EX-HOMOTOPY THEORY

M. H. Eggar

The ex-homotopy category has been investigated in recent years by I. M. James, J. Becker, L. Smith, J. F. McClendon, C. A. Robinson and others. In [5] I. M. James develops homotopy theory for ex-spaces as far as Puppe sequences, and in [6] he examines the Puppe sequence in a special case in order to calculate ex-homotopy groups.

Further developments are hampered by the fact that no sufficiently helpful homology theory for ex-spaces is known. In this paper we obviate the need for one and use results from [2] and [3] to derive an EHP-sequence. The technique is to mimic globally the *constructions* of ordinary homotopy theory and then to apply comparison theorems to deduce *theorems* in exhomotopy theory from the corresponding theorems in ordinary homotopy theory. As an example of our main result we calculate in § 6 some ex-homotopy groups involving the Hopf bundle which, to my knowledge, were not previously obtainable.

I am most grateful to Professor James for his help and encouragement during the preparation of this work.

Throughout the paper we consider only Hausdorff spaces and adopt the terminology of [2] and [3]. Let B be a connected, locally finite CW-complex (although our arguments in Sections 1–3 pertain more generally for B any locally contractible, para-compact, locally compact and path connected space). Recall that an ex-space (E, ρ, σ) (over B) consists of a space E and maps $\rho : E \rightarrow B$, $\sigma : B \rightarrow E$ such that $\rho \cdot \sigma = 1_B$ and $\sigma(B)$ is closed in E .

For ex-spaces E, X the ex-spaces $E \vee X$ (wedge sum), $E \times X$ (direct product), ΣE (reduced suspension), and $E \# X$ (smash product) are defined in [5]. The loop ex-space $(\Omega E, \rho', \sigma')$ of the ex-space (E, ρ, σ) has total space the subspace

$$\{\omega \in E^I : \omega(0) = \omega(1) = \sigma\rho\omega(0), \rho(\omega(t)) = \rho(\omega(0)) \text{ for all } t \in I\}$$

of E^I where E^I has the compact open topology. The ex-structure is defined by

$$\rho'(\omega) = \rho(\omega(0)), (\sigma'(b))(t) = \sigma(b) \quad (t \in I, b \in B).$$

Recall from [2] that an ex-space (E, ρ, σ) is said to be *docile* if for each point $b \in B$ there is a closed neighbourhood $W(b)$ such that the restriction ex-space $(\rho^{-1}W(b), \rho|_{\rho^{-1}W(b)}, \sigma|_{W(b)})$ over $W(b)$ is ex-homotopically equivalent to the product ex-space $W(b) \times F$, where F is a well-pointed space.

DEFINITION: An ex-space (E, ρ, σ) over B is a *placid ex-space* if σ is a cofibration, ρ is a Hurewicz fibration and the fibre of ρ has the pointed homotopy type of a locally finite CW-complex.

By [2] Theorem 3.6 a placid ex-space is docile. The class of placid ex-spaces over B is closed under product and smash product ([2] Corollary 3.4 and [4] Lemma 8.1).

1. Reduced product ex-spaces

DEFINITION (1.1): An ex-space (E, ρ, σ) is *distance-based* if there exists a map $\psi: E \rightarrow [0, 1]$ such that $\psi^{-1}(0) = \sigma(B)$. An ex-space (E, ρ, σ) where the total space E is normal and $\sigma(B)$ is a closed (G, δ) -set of E is distance-based ([10] p. 134). Thus any ex-space with a metrizable total space is distance-based. If E is a distance-based ex-space then so are the ex-spaces ΣE and ΩE .

Let E_n ($n \geq 2$) denote the ex-space obtained from the direct product of n copies of the ex-space (E, ρ, σ) by making the identifications, for each $1 \leq i \leq n-1$,

$$(e_1, e_2, \dots, e_{n-1}, \sigma\rho(e_1)) \sim (e_1, \dots, e_{i-1}, \sigma\rho(e_1), e_i, \dots, e_{n-1}) (e_j \in E).$$

Since $E \setminus \sigma(B)$ is open in E the inclusion ex-map $i_n: E_n \rightarrow E_{n+1}$, $i_n(e_1, \dots, e_n) = (e_1, \dots, e_n, \sigma\rho(e_1))$, is open and embeds E_n naturally in E_{n+1} . Define the reduced product ex-space to be $E_\infty = \varinjlim E_n$.

If E is distance-based by the function ψ an ex-map $f: E_\infty \rightarrow \Omega\Sigma E$ may be defined as follows. For $(e_1, \dots, e_n) \in E_n \setminus E_{n-1}$ ($n \geq 1$) set $a_i = \psi(e_i) / \sum_{j=1}^n \psi(e_j)$ and define

$$(1.2) \quad (f(e_1, \dots, e_n))(t) = \begin{cases} [e_1, t/a_1] & 0 \leq t \leq a_1 \\ [e_2, (t-a_1)/a_2] & a_1 \leq t \leq a_1 + a_2 \\ \vdots & \\ [e_n, (t - \sum_{i=1}^{n-1} a_i)/a_n] & \sum_{i=1}^{n-1} a_i \leq t \leq \sum_{i=1}^n a_i. \end{cases}$$

Take $E_0 = \sigma(B)$, $E_1 = E$. Then (1.2) defines f on $E_\infty \setminus E_0$. The map f is extended to an ex-map $f: E_\infty \rightarrow \Omega\Sigma E$ by defining $(f(\sigma(b)))(t) = \sigma_{\Sigma E}(b)$, $0 \leq t \leq 1$.

Let F be the fibre $\rho^{-1}(b)$ of E at the point $b \in B$. We then have $E_n \cap \rho_n^{-1}(b) = F_n$, $E_\infty \cap \rho_\infty^{-1}(b) = F_\infty$ where the right-hand sides are obtained by the reduced product construction for the pointed space $(F, \sigma(b))$. By D. Puppe's refinement [11] (p. 234 Theorem 17.3) of the theorem of I. M. James [8] we know that $f|_b$ is a homotopy equivalence if $(F, \sigma(b))$ is an h -well-pointed, path-connected space which admits a numerable null-homotopic covering. The same method of proof as in [2] Proposition 3.8 establishes that E_n , E_∞ and ΩE are docile ex-spaces if E is a docile ex-space. By [2] Theorem 3.9 we then obtain

PROPOSITION (1.3): *Let (E, ρ, σ) be a docile distance-based ex-space with fibre having the pointed homotopy type of a connected locally finite CW-complex. Then the ex-map $f: E_\infty \rightarrow \Omega \Sigma E$ in (1.2) is an ex-homotopy equivalence.*

2. The reduced join of ex-spaces

Let (E, ρ, σ) and (X, ρ', σ') be ex-spaces. The total space of the reduced join $E * X$ is obtained from $E \times X \times I$ by making the identifications

$$\begin{aligned} (e, x, 0) &\sim (\sigma\rho(e), x, 0) & e \in E, x \in X \\ (e, x, 1) &\sim (e, \sigma'\rho'(x), 1) & e \in E, x \in X \\ (\sigma(b), \sigma'(b), t) &\sim (\sigma(b), \sigma'(b), 0) & b \in B, t \in I. \end{aligned}$$

The projection $E * X \rightarrow B$ takes $(e, x, t)/\sim$ to $\rho(e)$, and the section $B \rightarrow E * X$ takes b to $(\sigma(b), \sigma'(b), 0)/\sim$. There is a natural collapsing ex-map $E * X \rightarrow \Sigma(E \# X)$, which, by [15] p. 239, induces a homotopy equivalence between the fibres of $E * X$ and $\Sigma(E \# X)$ over any point $b \in B$ if the fibres of E and X over b are polyhedra.

By [2] Proposition 3.8 and Theorem 3.9 one has

PROPOSITION (2.1): *Let E, X be docile ex-spaces with fibres having the pointed homotopy type of locally finite CW-complexes. Then the collapsing ex-map $E * X \rightarrow \Sigma(E \# X)$ is an ex-homotopy equivalence.*

I remark in passing that by an application of [2] Theorem 3.9 similar to that in (1.3) or (2.1) a Hilton-Milnor theorem for ex-spaces may be obtained (see [4]).

3. Ex-homotopy exact sequences

The material in this section is a straightforward generalization of the corresponding results in homotopy theory. The reader is referred to [4] for more detailed proofs.

Let $(Z, Z', (X, X'), (W, W'))$ be ex-space pairs ([3] Part 1 Section 4).

Composition on the left by an ex-map $f : (Z, Z') \rightarrow (X, X')$ induces a pointed function $f_{\#} : \pi(W, W'; Z, Z') \rightarrow \pi(W, W'; X, X')$, and composition on the right induces a pointed function $f^{\#} : \pi(X, X'; W, W') \rightarrow \pi(Z, Z'; W, W')$. By restricting the domain and codomain of an ex-map $g : (W, W') \rightarrow (Z, Z')$ to W' and Z' respectively one obtains an ex-map $\delta(g) : W' \rightarrow Z'$. This boundary operation respects ex-homotopy and defines a pointed function $\delta : \pi(W, W'; Z, Z') \rightarrow \pi(W', Z')$.

By a Puppe sequence argument one deduces

PROPOSITION (3.1): (*Exact ex-homotopy sequence of a pair*).

Let W be an ex-space and (X, X') be an ex-space pair. Then the sequence $\cdots \rightarrow \pi(\Sigma W, X') \xrightarrow{i^{\#}} \pi(\Sigma W, X) \xrightarrow{j^{\#}} \pi(CW, W; X, X') \xrightarrow{\delta} \pi(W, X') \xrightarrow{i^{\#}} \pi(W, X)$, where $i : X' \rightarrow X$ and $j : (X, \sigma_X(b)) \rightarrow (X, X')$ are inclusions, is exact.

The proof of [11] p. 378 Theorem 15 generalizes to yield

PROPOSITION (3.2): (*Exact ex-homotopy sequence of a triple*).

Let W, X, X' and X'' be ex-spaces such that (X, X') and (X', X'') are ex-space pairs. Then the sequence

$\cdots \rightarrow \pi(C\Sigma W, \Sigma W; X, X') \xrightarrow{\delta} \pi(CW, W; X', X'') \xrightarrow{i^{\#}} \pi(CW, W; X, X'') \xrightarrow{j^{\#}} \pi(CW, W; X, X')$ is exact, where $i : (X', X'') \rightarrow (X, X'')$ and $j : (X, X'') \rightarrow (X, X')$ are the inclusions.

DEFINITION (3.3): Let E, X and K be ex-spaces. An ex-map $q : E \rightarrow X$ has the *ex-homotopy lifting property for K* if, given an ex-map $g : K \rightarrow E$ and an ex-homotopy $F : K \times I \rightarrow X$ such that $F_0 = q \cdot g$, there exists an ex-homotopy $G : K \times I \rightarrow E$ such that $G_0 = g$ and $q \cdot G = F$.

DEFINITION (3.4): The ex-map $q : E \rightarrow X$ is an *ex-fibration* if it has the ex-homotopy lifting property for all ex-spaces K . If B is a CW-complex the ex-map $q : E \rightarrow X$ is a *Serre ex-fibration* if it has the ex-homotopy lifting property for all ex-complexes K . (Recall from [7] Section 5 that the ex-space (K, ρ, σ) over B is an *ex-complex* if K is a CW-complex with sub-complex $\sigma(B)$. An ex-complex is *proper* if the projection ρ is a cellular map. If K is a proper ex-complex then CK and ΣK are proper ex-complexes.)

Example of an ex-fibration: Let (X, ρ, σ) be an ex-space. Set $P(X) = \{w \in X^I : \rho(w(t)) = \rho(w(0)) \text{ for all } t \in I\}$, and assign to $P(X)$ the subspace topology from X^I , where X^I has the compact open topology. The space $P(X)$ possesses a natural projection onto $B(w \mapsto \rho(w(0)))$ and also a section $(b \mapsto \text{constant path at } \sigma(b), (b \in B))$. Hence $P(X)$ is an ex-space. The map $q : P(X) \rightarrow X, q(w) = w(1)$, is an ex-fibration.

The proof ([4] Proposition 5.4) of the next proposition is lengthy but not difficult.

PROPOSITION (3.5): *Let E be an ex-space and (X, X') an ex-space pair over the [CW-complex] space B . Let $q : E \rightarrow X$ be a [Serre] ex-fibration, and write E' for the subex-space $q^{-1}(X')$ of E . Then for any [proper ex-complex] ex-space K the pointed function $q_{\#} : \pi(CK, K; E, E') \rightarrow \pi(CK, K; X, X')$ is bijective.*

PROPOSITION (3.6): *(Exact ex-homotopy sequence of an ex-fibration).*

Let E and X be ex-spaces over the [CW-complex] space B , and let $q : E \rightarrow X$ be a [Serre] ex-fibration. For any [proper ex-complex] ex-space K the sequence

$$\cdots \xrightarrow{q_{\#}} \pi(\Sigma^2 K, X) \xrightarrow{\delta'} \pi(\Sigma K, D) \xrightarrow{i_{\#}} \pi(\Sigma K, E) \xrightarrow{q_{\#}} \pi(\Sigma K, X) \xrightarrow{\delta'} \pi(K, D) \xrightarrow{i_{\#}} \pi(K, E)$$

is exact, where D is the subex-space $q^{-1}(\sigma_X(B))$ of E , i is the inclusion ex-map: $D \subset E$, and $\delta' = \delta \cdot \tilde{q}_{\#}^{-1}$

$$(\tilde{q}_{\#} : \pi(CK, K; E, D) \rightarrow \pi(CK, K; X, \sigma_X(b)), \delta : \pi(CK, K; E, D) \rightarrow \pi(K, D)).$$

Proposition 3.6 may be proved by applying Proposition 3.5 to the exact sequence of the ex-space pair (E, D) . As with the analogous exact sequences in homotopy theory except near their tails the exact sequences of (3.1), (3.2) and (3.6) are exact sequences of abelian groups.

4. A relative comparison theorem

Let K be a proper ex-complex over B .

THEOREM (4.1): *(Relative Comparison Theorem)*

Let (E_1, E_2) , (X_1, X_2) be ex-space pairs over B where the projections $\rho_{E_1}, \rho_{E_2}, \rho_{X_1}, \rho_{X_2}$ are Serre fibrations.

Suppose that, for some $n \geq 1$, $f : (E_1, E_2) \rightarrow (X_1, X_2)$ is an ex-map whose restriction to a fibre is n -connected.¹ Then the function $f_{\#} : \pi(CK, K; E_1, E_2) \rightarrow \pi(CK, K; X_1, X_2)$ is bijective for $\dim K < n-1$, surjective for $\dim K \leq n-1$.

PROOF: We construct the ex-space $(P, \bar{\rho}, \bar{\sigma})$ where $P = \{w \in E_1^I | w(1) \in E_2, \rho(w(t)) = \rho(w(0)) \text{ for all } t \in [0, 1]\}$, $\bar{\rho}(w) = \rho_{E_1}(w(0))$, and $(\bar{\sigma}(b))(t) = \sigma_{E_1}(b)$ for all $t \in I$. The ex-map $p : P \rightarrow E_1$, $p(w) = w(0)$, is an ex-fibration. Set $D = p^{-1}(\sigma_{E_1}(B))$ and regard D as a subex-space of P over B . Since ρ_{E_2} and ρ_{E_1} are Serre fibrations the projection, ρ_D say, of D is a Serre fibration.

¹ i.e. if F_1, F_2, Y_1, Y_2 are the fibres of E_1, E_2, X_1, X_2 respectively over some point of B , then $f|_{p^{-1}\#} : \pi_i(F_1, F_2) \rightarrow \pi_i(Y_1, Y_2)$ is bijective for $i < n$, surjective for $i \leq n$.

Define the subex-space $P'E_1$ of $P(E_1)$ over B to be the subex-space with total space $P'E_1 = \{w \in E_1^I \mid w(0) \in \sigma_{E_1}(B), \rho(w(t)) = \rho(w(0)) \text{ for all } t \in I\}$. The map $p' : P'E_1 \rightarrow E_1$, $p'(w) = w(1)$ is an ex-fibration, and $D = p'^{-1}(E_2)$. Since $P'E_1$ is ex-contractible, by Proposition 3.1 $\pi(CK, K; P'E_1, D) \xrightarrow{\delta} \pi(K, D)$ is bijective regardless of $\dim K$. Also, by Proposition 3.5, $\pi(CK, K; P'E_1, D) \xrightarrow{p'^\#} \pi(CK, K; E_1, E_2)$ is bijective (regardless of $\dim K$).

The ex-map f induces a commutative diagram

$$\begin{array}{ccccc} \pi(CK, K; E_1, E_2) & \xleftarrow[p' \#]{\approx} & \pi(CK, K; P'E_1, D) & \xrightarrow[\approx]{\delta} & \pi(K, D) \\ \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# \\ \pi(CK, K; X_1, X_2) & \xleftarrow[p' \#]{\approx} & \pi(CK, K; P'X_1, W) & \xrightarrow[\approx]{\delta} & \pi(K, W) \end{array}$$

where W bears the same relation to X_1, X_2 as D does to E_1, E_2 . By [7] Theorem 6.3 the right-hand $f_\#$ is bijective if $\dim K < n-1$, surjective if $\dim K \leq n-1$, and Theorem 4.1 follows.

The following collapsing theorem is immediate from Theorem 4.1 and [12] p. 487 Corollary 6.

COROLLARY (4.2): *Let (E_1, E_2) be an ex-space pair over B , where ρ_{E_1}, ρ_{E_2} and ρ_{E_1/E_2} are Serre fibrations. Let F_1, F_2 be the fibres of E_1, E_2 over some point of B . Suppose that F_2 is m -connected, $m \geq 1$, and (F_1, F_2) has the homotopy type of an n -connected relative CW-complex, $n \geq 2$. Then the function*

$$k_\# : \pi(CK, K; E_1, E_2) \rightarrow \pi(\Sigma K, E_1/E_2)$$

induced by the collapsing ex-map $k : (E_1, E_2) \rightarrow (E_1/E_2, \sigma_{E_1/E_2}(B))$ is bijective for $\dim K < m+n$, surjective for $\dim K \leq m+n$.

We remark in passing that Theorem 4.1 in conjunction with [12] p. 484 Theorem 5 yields an ex-homotopy excision theorem.

5. The EHP-sequence

Our objective is to investigate the suspension functor Σ in the metastable range.

PROPOSITION (5.1): *(Stability theorem) (c.f. [7] Theorem 6.4)*

Let (E, ρ, σ) be a Hurewicz ex-space over B , where σ is a cofibration, and let K be an ex-complex over B . If the fibre of E is m -connected, then the suspension function

$$\Sigma : \pi(K, E) \rightarrow \pi(\Sigma K, \Sigma E)$$

is injective if $\dim K \leq 2m$, surjective if $\dim K \leq 2m+1$.

PROOF: By [2] Corollary 3.4 the projection of the ex-space $\Omega\Sigma E$ is a fibration, and so [7] Theorem 6.3 may be applied in the same way as in the proof of [7] Theorem 6.4 to prove our proposition.

THEOREM (5.2) (EHP-sequence): *Let (E, ρ, σ) be a placid, distance-based ex-space whose fibre F is m -connected ($m \geq 1$) and let K be a proper ex-complex of dimension k . Then there are pointed functions H, P (all H, P except the final P are homomorphisms between ex-homotopy groups) such that the following sequence is exact:*

$$\begin{aligned} \pi(\Sigma^{3m-k+1}K, E) &\xrightarrow{\Sigma} \pi(\Sigma^{3m-k+2}K, \Sigma E) \xrightarrow{H} \pi(\Sigma^{3m-k+2}K, E * E) \xrightarrow{P} \cdots \\ \cdots \rightarrow \pi(\Sigma K, E) &\xrightarrow{\Sigma} \pi(\Sigma^2 K, \Sigma E) \xrightarrow{H} \pi(\Sigma^2 K, E * E) \xrightarrow{P} \pi(K, E) \\ &\xrightarrow{\Sigma} \pi(\Sigma K, \Sigma E). \end{aligned}$$

PROOF: The restriction to the fibres of the inclusion ex-map: $E_2 \subset E_\infty$ (see Section 1 for notation) is the inclusion map: $F_2 \subset F_\infty$. The function inclusion $_{\#} : \pi_r(F_2) \rightarrow \pi_r(F_\infty)$ is bijective for $r < 3m+2$ and surjective for $r \leq 3m+2$. By the comparison theorem ([7]Theorem 6.3) inclusion $_{\#} : \pi(K, E_2) \rightarrow \pi(K, E_\infty)$ is injective if $\dim K < 3m+2$, surjective if $\dim K \leq 3m+2$. (Observe that the projections of E_2 and E_∞ are Hurewicz fibrations by a proof similar to that of [2] Corollary 3.4. Namely, if λ is a special lifting function for ρ with respect to σ then the composite map

$$\text{quotient} \cdot \lambda^n : \tilde{\Omega}_\rho n \rightarrow (E^n)^I \rightarrow (E_n)^I$$

factors through $\tilde{\Omega}_\rho n$.)

From the exact ex-homotopy sequence of the pair (E_∞, E_2) (Proposition 3.1) we have $\pi(CK, K; E_\infty, E_2) = 0$ for $\dim K < 3m+2$. By Proposition 3.2

$$\begin{aligned} \pi(C\Sigma K, \Sigma K; E_\infty, E_2) &\xrightarrow{\delta} \pi(CK, K; E_2, E) \xrightarrow{\text{incln}_{\#}} \pi(CK, K; E_\infty E) \\ &\xrightarrow{\text{incln}_{\#}} \pi(CK, K; E_\infty, E_2) \end{aligned}$$

is exact, and so

$$(5.3) \quad \text{inclusion}_{\#} : \pi(CK, K; E_2, E) \rightarrow \pi(CK, K; E_\infty, E)$$

is injective for $\dim K < 3m+2$, and surjective for $\dim K \leq 3m+2$.

The collapsing ex-map: $(E_2, E) \rightarrow (E_2/E, \sigma_{E_2/E}(B))$ induces a pointed function

$$(5.4) \quad \text{collapse}_{\#} : \pi(CK, K; E_2, E) \rightarrow \pi(\Sigma K, E_2/E) = \pi(\Sigma K, E \# E)$$

which by Corollary 4.2 is injective for $\dim K < 3m+1$ and surjective for $\dim K \leq 3m+1$.

The pointed space $F \# F$ is $2m$ -connected, and so by Propositions 5.1 and 2.1

$$(5.5) \quad \Sigma : \pi(K, E \# E) \rightarrow \pi(\Sigma K, \Sigma(E \# E)) \cong \pi(\Sigma K, E * E)$$

is injective for $\dim K \leq 4m$, and surjective for $\dim K \leq 4m + 1$.

Consider the ex-homotopy exact sequence (3.1) of the pair (E_∞, E)

$$(5.6) \quad \cdots \xrightarrow{\delta} \pi(\Sigma K, E) \xrightarrow{i\#} \pi(\Sigma K, E_\infty) \xrightarrow{j\#} \pi(CK, K; E_\infty, E) \xrightarrow{\delta} \pi(K, E) \xrightarrow{i\#} \pi(K, E_\infty)$$

where i, j are inclusion ex-maps. By Proposition 1.3 there is a bijection: $\pi(K, E_\infty) \approx \pi(K, \Omega\Sigma E)$ for all $\dim K$, and (5.3), (5.4) and (5.5) provide a bijection: $\pi(CK, K; E_\infty, E) \rightarrow \pi(\Sigma^2 K, E * E)$ for $\dim K < 3m + 1$. Inserting these bijections in (5.6) and observing that the diagram

$$\begin{array}{ccc} \pi(K, E) & \xrightarrow{\text{inclusion}\#} & \pi(K, E_\infty) \\ \Sigma \downarrow & & \cong \\ \pi(\Sigma K, \Sigma E) & \xrightarrow[\approx]{\text{adjoint}} & \pi(K, \Omega\Sigma E) \end{array}$$

commutes, one deduces Theorem 5.2.

When the fibre F of ρ is a sphere one can obtain information about Σ irrespective of the dimension of K .

THEOREM (5.7.i): *Let (E, ρ, σ) be a placid distance-based ex-space over B with fibre S^m , m odd. Let K be a proper ex-complex over B . Then there are pointed functions H, P (homomorphisms except for the final P) such that the sequence*

$$\cdots \rightarrow \pi(\Sigma K, E) \xrightarrow{\Sigma} \pi(\Sigma^2 K, \Sigma E) \xrightarrow{H} \pi(\Sigma^2 K, E * E) \xrightarrow{P} \pi(K, E) \xrightarrow{\Sigma} \pi(\Sigma K, \Sigma E)$$

is exact.

PROOF: Write $c : (E_2, E) \rightarrow (E \# E, B)$ for the collapsing ex-map, and $C : (E_\infty, E) \rightarrow ((E \# E)_\#, B)$ for the combinatorial extension of c (see [8] p. 176 Lemma 2.5). By the theorem of I. M. James ([9] Theorem 1.2 or [14] Theorem 2.4), $(C|_{\text{fibre}})_\# : \pi_i((S^m)_\infty, S^m) \rightarrow \pi_i((S^{2m})_\infty)$ is isomorphism for all i . By Theorem 4.1 $C_\# : \pi(CK, K; E_\infty, E) \rightarrow \pi(\Sigma K, (E \# E)_\infty)$ is bijective. Theorem 5.7 (i) follows from the ex-homotopy exact sequence of the pair (E_∞, E) as in the last paragraph of the proof of Theorem 5.2.

Let $\mathcal{C}$ be the Serre class of torsion abelian groups of odd order.

THEOREM (5.7ii): *Let (E, ρ, σ) be a placid distance-based ex-space over a finite CW-complex B with fibre S^m , m even. Let K be a proper placid ex-complex over B with compact total space. Then there is a $\mathcal{C}$ -exact sequence*

$$\cdots \rightarrow \pi(\Sigma^3 K, E) \xrightarrow{\Sigma} \pi(\Sigma^4 K, \Sigma E) \xrightarrow{H} \pi(\Sigma^4 K, E * E) \xrightarrow{P} \pi(\Sigma^2 K, E) \xrightarrow{\Sigma} \pi(\Sigma^3 k, \Sigma E).$$

PROOF: We use the notation in Theorem 5.7(i). We know that $(C|\text{fibre})_{\#} : \pi_i((S^m)_{\infty}, S^m) \rightarrow \pi_i((S^m \# S^m)_{\infty}, *)$ is a $\mathcal{C}$ -isomorphism for all i (by [9] Theorem 1.3 or [14] Theorem 2.4). To deduce that

$$C_{\#} : \pi(C\Sigma^2 K, \Sigma^2 K; E_{\infty}, E) \rightarrow \pi(\Sigma^3 K, (E \# E)_{\infty})$$

is a $\mathcal{C}$ -isomorphism one uses [3] Theorem 3.1 made relative by the argument of Theorem 4.1.

6. Some calculations

Let E^3 denote the fibre suspension of the Hopf bundle over S^2 , and regard E^3 as an ex-space over S^2 by choosing a cross-section (in one of the obvious ways). For $r > 3$ inductively define the ex-space E^r over S^2 by $E^r = \Sigma E^{r-1}$. (By [7] Theorem 6.1 E^r is ex-homotopically equivalent to the sphere bundle (with section) associated to the Whitney sum of the canonical complex line bundle over $\mathbb{C}P^1$, regarded as a real vector bundle, and the product bundle over $\mathbb{C}P^1$ with fibre $\mathbb{R}^{r-2}$.)

$$(6.1) \quad \pi(E^6, E^5) \approx Z \oplus Z_{24}, \pi(E^7, E^6) \approx Z_2 \oplus Z_{24}.$$

By [6] Theorem (1.6), (1.8) there is an exact sequence

$$\cdots \rightarrow \pi_6 S^4 \xrightarrow{\Psi} \pi_7 S^4 \xrightarrow{\Theta} \pi(E^6, E^3) \xrightarrow{\varphi'} \pi_5 S^4 \xrightarrow{\Psi'} \pi_6 S^4$$

where

$$\begin{aligned} \Psi(\alpha) &= \alpha \cdot S^4 J(\beta) - S^2 J(\beta) \cdot S\alpha & \alpha \in \pi_6 S^4 \\ \Psi'(\alpha') &= \alpha' \cdot S^3 J(\beta) - S^2 J(\beta) \cdot S\alpha' & \alpha' \in \pi_5 S^4 \end{aligned}$$

where $\beta \in \pi_1(O_2)$ is the classifying element for the Hopf bundle. We adopt standard notation (Toda [14]); then $\pi_6 S^4 \approx Z_2$ is generated by $\langle \eta \cdot \eta \rangle$ and $\pi_5 S^4 \approx Z_2$ is generated by $\langle \eta \rangle$. Since $\Psi(\eta \cdot \eta) = 0$ and $\Psi'(\eta) = 0$ we deduce that $\pi(E^6, E^5)$ is an extension of $Z \oplus Z_{12}$ by Z_2 , but at this stage we do not know which. Similarly $\pi(E^7, E^6)$ is either $Z_2 \oplus Z_{24}$ or Z_{48} .

Apply Theorem 5.2 (the EHP-sequence) with “ K ” = E^5 , “ E ” = E^5 , “ F ” = S^4 , “ m ” = 3, “ B ” = S^2 , “ k ” = $4+2 = 6$. We have $3m-k+1 = 4$ and so the following sequence is exact

$$\begin{aligned} \pi(E^8, E^6) &\xrightarrow{H} \pi(E^8, \Sigma(E^5 \# E^5)) \xrightarrow{P} \pi(E^6, E^5) \rightarrow \pi(E^7, E^6) \\ &\xrightarrow{H} \pi(E^7, \Sigma(E^5 \# E^5)) \text{ i.e. } Z_2 \xrightarrow{H} Z \xrightarrow{P} \pi(E^6, E^5) \xrightarrow{\Sigma} \pi(E^7, E^6) \xrightarrow{H} 0 \end{aligned}$$

Comparing this sequence with the previous results we obtain (6.1).

$$(6.2) \quad \pi(E^5, E^4) \approx Z_{24}.$$

From the exact sequence of [6]

$$\cdots \rightarrow \pi_5 S^3 \xrightarrow{0} \pi_6 S^3 \longrightarrow \pi(E^5, E^4) \longrightarrow \pi_4 S^3 \xrightarrow{0} \pi_5 S^3$$

we deduce $\pi(E^5, E^4) \approx Z_{12} \oplus Z_2$ or Z_{24} . By the EHP-sequence with “ K ” = E^4 , “ E ” = E^4 , “ m ” = 2, “ k ” = 5, $3m - k + 1 = 2$ the sequence

$$\cdots \xrightarrow{H} \pi(E^7, \Sigma(E^4 \# E^4)) \xrightarrow{P} \pi(E^5, E^4) \xrightarrow{\Sigma} \pi(E^6, \pi(E^4 \# E^4)) \xrightarrow{P} \pi(E^4, E^4) \rightarrow \pi(E^5, E^5)$$

is exact. This sequence is:

$$\cdots \xrightarrow{H} 0 \xrightarrow{P} \pi(E^5, E^4) \xrightarrow{\Sigma} \pi(E^6, E^5) \xrightarrow{H} Z \xrightarrow{P} Z_2 \oplus Z \xrightarrow{\Sigma} Z_2 \oplus Z.$$

The final Σ is surjective since it is the last ex-suspension before the stable range, and by the specific nature of the groups it is an isomorphism. From (6.1) and the exact sequence we deduce that $\pi(E^5, E^4) \approx Z_{24}$.

REMARK: All auxiliary ex-homotopy groups used in this calculation can be computed using [6] and standard results on the homotopy groups of spheres.

Further remarks:

Let D, E, X be ex-spaces, and let $u : \Sigma D \rightarrow X, v : \Sigma E \rightarrow X$ be ex-maps. We define an ex-map $[u, v] : D * E \rightarrow X$ as follows. Regard u, v as ex-maps of pairs

$$u : (CD, D) \rightarrow (X, \sigma_*(B)) \quad v : (CE, E) \rightarrow (X, \sigma_* B).$$

The map $u \times v$ determines an ex-map $CD \times CE \rightarrow X \times X$, and the restriction to the subex-space $CD \times E \rightarrow D \times CE = D * E$ maps into $X \vee X$ and determines an ex-map $D * E \rightarrow X \vee X$. The composite of this ex-map with the folding ex-map $X \vee X \rightarrow X$ is denoted $[u, v]$. If the ex-maps $u, u' : \Sigma D \rightarrow X$ are ex-homotopic then so are $[u, v]$ and $[u', v]$, and the analogous statement holds for v . Thus a pairing,

$$\pi(\Sigma D, X) \times \pi(\Sigma E, X) \xrightarrow{[\cdot, \cdot]} \pi(D * E, X)$$

called the *Whitehead product*, is induced at the ex-homotopy level.

As one would expect, the function P in Theorem 5.2 is related to the Whitehead product. Suppose in Theorem 5.2 that $E = \Sigma E'$ where E' is a placid ex-space. Then one can show by a naturality argument (see [4] Theorem 6.5) that the diagram

$$\begin{array}{ccc} \pi(\Sigma^2 K, \Sigma^3(E' \# E')) & \xrightarrow{P} & \pi(K, \Sigma E') \\ \uparrow \approx & \nearrow [i, i] \circ (\cdot) & \\ \pi(K, \Sigma(E' \# E')) & & \end{array}$$

commutes, where $i : \Sigma E' \rightarrow \Sigma E'$ is the identity ex-map. One corollary of this result is that, with appropriate conditions on the ex-spaces E_1 and E_2 , if $\alpha \in \pi(\Sigma E_1, X)$ and $\beta \in \pi(\Sigma E_2, X)$ then $P(\cdot[\alpha, \beta])$ is precisely the join $\alpha * \beta \in \pi(\Sigma E_1 * \Sigma E_2, X * X)$.

One can easily derive many properties of the Whitehead product along the lines of [1] by introducing an ex-homotopy theory analogue of the Samelson product (see [4]). However at later stages difficulties arise, some of which may be discussed in a future paper.

REFERENCES

- [1] M. ARKOWITZ: The Generalized Whitehead Product, *Pacific J. of Math.*, vol. 12 (1962) 7–22.
- [2] M. H. EGGAR: The Piecing Comparison Theorem, *Nederl. Akad. Wetensch. Proc. Ser. A.* (to appear).
- [3] M. H. EGGAR: On structure preserving maps between fibre spaces with cross-sections, *London, Journal of Math.* (to appear).
- [4] M. H. EGGAR: D. Phil. thesis (Oxford 1971).
- [5] I. M. JAMES: Ex-homotopy theory I, *Illinois Journal of Math.*, vol. 15 (1971) 324–337.
- [6] I. M. JAMES: On the maps of one fibre space into another, *Compos. Math.*, vol. 23 (1971) 317–328.
- [7] I. M. JAMES: Bundles with Special Structure I, *Ann. of Math.*, vol. 89 (1969) 359–390.
- [8] I. M. JAMES: Reduced product spaces, *Ann. of Math.*, vol. 62 (1955) 170–197.
- [9] I. M. JAMES: The Suspension Traid of a Sphere, *Ann. of Math.*, vol. 63 (1956) 407–429.
- [10] J. KELLEY: General Topology, van Nostrand (1955).
- [11] D. PUPPE, K. KAMPS, T. TOM DIECK, Homotopietheorie, Springer Lecture Notes 157 (1970).
- [12] E. SPANIER: Algebraic Topology, McGraw Hill (1966).
- [13] A. STRØM: Note on Cofibrations II, *Math. Scand.*, vol. 19 (1966) 11–14.
- [14] H. TODA: Composition methods in homotopy groups of spheres, Annals Studies 49, Princeton Univ. Press (1962).
- [15] J. H. C. WHITEHEAD: Combinatorial Homotopy I, *Bull. Amer. Math. Soc.*, vol. 55 (1949) 213–245

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Dept. Pure Math.
University of Hull
Hull, Yorkshire, England