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**ON A SERIES OF COSECANTS RELATED TO
 A PROBLEM IN ERGODIC THEORY**

by

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In investigating the spectrum of the transformation induced on the space consisting of $[0, 1)$ with a "second floor" above $[0, \beta)$ by translation mod 1 by an irrational α , one is led to consider [9] convergence of the series

$$(1) \quad \sum_{k \neq 0} \frac{1}{k^2} \frac{\|k\beta\|^2}{\|k\alpha\|^2},$$

where $\| \cdot \|$ denotes distance to the nearest integer. Since there are constants c, c', d, d' for which $c|\sin \pi x| \leq \|x\| \leq c'|\sin \pi x|$ and $d|1 - e^{2\pi i x}| \leq \|x\| \leq d'|1 - e^{2\pi i x}|$ for all $x \in \mathbf{R}$, convergence of this series is equivalent to convergence of

$$(2) \quad \sum_{k \neq 0} \frac{1}{k^2} \frac{\sin^2 \pi k \beta}{\sin^2 \pi k \alpha}$$

and to convergence of

$$(3) \quad \sum_{k \neq 0} \frac{1}{k^2} \frac{|1 - e^{2\pi i k \beta}|^2}{|1 - e^{2\pi i k \alpha}|^2}.$$

Series similar to (2) have been mentioned in an earlier paper of Kac and Salem [6], and problems of convergence of series with small denominators are well known in celestial mechanics.

Let $f(x) = \chi_{[0, \beta)}(x) - \beta$ for $x \in [0, 1)$, and let

$$f_n(x) = \sum_{k=0}^{n-1} f\langle x + k\alpha \rangle$$

for $n = 1, 2, \dots$, where $\langle y \rangle$ denotes the fractional part of $y \in \mathbf{R}$. From the equidistribution mod 1 of $\{\langle x + k\alpha \rangle : k \in \mathbf{Z}\}$ it follows that $|f_n(x)| = o(n)$ for each $x \in [0, 1)$. Kesten [7] has proved that $\{|f_n(0)| : n = 1, 2, \dots\}$ is bounded if and only if $\beta \in \mathbf{Z}\alpha \pmod{1}$, and recently a simple proof of this and related theorems along with an application to topological

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dynamics have been given by Shapiro [10] and Furstenberg, Keynes, and Shapiro [3]. It will develop that convergence of (1) is equivalent to boundedness of the sequence of L_2 norms $\{\|f_n\|_2 : n = 1, 2, \dots\}$, and that (1) converges if and only if $\beta \in \mathbf{Z}\alpha \pmod{1}$. For earlier literature concerning boundedness of $\{\|f_n\| : n = 1, 2, \dots\}$, see [1, p. 226 ff.], [5], and [8].

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THEOREM. *Let $\alpha, \beta \in [0, 1)$ with α irrational, let $f(x) = \chi_{[0, \beta)}(x) - \beta$ for $x \in [0, 1)$, and let $f_n(x) = \sum_{k=0}^{n-1} f(x+k\alpha)$ for $n = 1, 2, \dots$. Then the following statements are equivalent.*

- (i) $\sum_{k \neq 0} \frac{1}{k^2} \frac{\|k\beta\|^2}{\|k\alpha\|^2} < \infty$.
- (ii) $\sup_n \sum_{k \neq 0} \frac{1}{k^2} \frac{\|k\beta\|^2}{\|k\alpha\|^2} \|kn\alpha\|^2 < \infty$.
- (iii) $\sup_n \|f_n\|_2 < \infty$.
- (iv) *There is $g \in L^2[0, 1)$ such that $f(x) = g(x) - g(x+\alpha)$ a.e.*
- (v) $\beta \in \mathbf{Z}\alpha \pmod{1}$.
- (vi) *There is an $x \in [0, 1)$ for which $\sup_n |f_n(x)| < \infty$.*
- (vii) $\sup_{x, n} |f_n(x)| < \infty$.

PROOF. The series in (ii) converges for each n because always $\|kn\alpha\| \leq n\|k\alpha\|$. Since $\|kn\alpha\|^2 \leq 1$ for all n and k , the implication from (i) to (ii) is clear. In order to see that (ii) implies (iii), note that f has the Fourier expansion

$$f(x) = \sum_{k \neq 0} \frac{1}{2\pi ik} (1 - e^{-2\pi ik\beta}) e^{2\pi ikx},$$

so that

$$\begin{aligned} f_n(x) &= \sum_{j=0}^{n-1} f(x+j\alpha) = \sum_{k \neq 0} \frac{1}{2\pi ik} (1 - e^{-2\pi ik\beta}) e^{2\pi ikx} \sum_{j=0}^{n-1} e^{2\pi ijk\alpha} \\ &= \sum_{k \neq 0} \frac{1}{2\pi ik} (1 - e^{-2\pi ik\beta}) \frac{1 - e^{2\pi ink\alpha}}{1 - e^{2\pi ijk\alpha}} e^{2\pi ikx}, \end{aligned}$$

and thus

$$\|f_n\|_2^2 = \sum_{k \neq 0} \frac{1}{4\pi^2 k^2} \frac{|1 - e^{-2\pi ik\beta}|^2}{|1 - e^{2\pi ik\alpha}|^2} |1 - e^{2\pi ink\alpha}|^2,$$

which lies between two constant multiples of

$$\sum_{k \neq 0} \frac{1}{k^2} \frac{\|k\beta\|^2}{\|k\alpha\|^2} \|kn\alpha\|^2.$$

We prove now that (iii) implies (iv). For $h \in L^2[0, 1]$, let $Uh(x) = h\langle x + \alpha \rangle$, and define $V : L^2[0, 1] \rightarrow L^2[0, 1]$ by $Vh = f + Uh$. Let K denote the norm-closed convex cover of $\{f_1, f_2, \dots\}$, so K is weakly compact. We claim that $VK \subset K$. For if $h \in K$, then there is a sequence of finite convex combinations $\Sigma a_\nu f_{n_\nu}$ ($a_\nu \geq 0, \Sigma a_\nu = 1$) converging to h . Since V is continuous, $V\Sigma a_\nu f_{n_\nu}$ converges to Vh . But, using linearity of U , we see that

$$\begin{aligned} V\Sigma a_\nu f_{n_\nu} &= f + U\Sigma a_\nu f_{n_\nu} = \Sigma a_\nu f + \Sigma a_\nu Uf_{n_\nu} \\ &= \Sigma a_\nu (f + Uf_{n_\nu}) = \Sigma a_\nu f_{n_\nu + 1} \in K, \end{aligned}$$

so $Vh \in K$. Therefore, by the Schauder-Tychonoff Theorem, there is $g \in K$ with $Vg = g$.

Suppose now that (iv) holds and let $\tau(x) = e^{2\pi i g(x)}$ for $x \in [0, 1]$. Then $\tau\langle x + \alpha \rangle = e^{2\pi i [g(x) - f(x)]} = \tau(x)e^{2\pi i [\beta - \chi_{[0, \beta)}(x)]} = e^{2\pi i \beta} \tau(x)$, so τ is an eigenfunction with eigenvalue $e^{2\pi i \beta}$ of the transformation $x \rightarrow x + \alpha \pmod{1}$. All eigenvalues of this transformation are known to be of the form $e^{2\pi i n \alpha}$, $n \in \mathbf{Z}$; therefore $\beta \in \mathbf{Z}\alpha \pmod{1}$.

Since $\|nx\| \leq n\|x\|$ for all $x \in \mathbf{R}$, that (v) implies (i) is immediate. For the sake of completeness, we include Hecke's proof [5, p. 70] that (v) implies (vi). Suppose $\beta = \langle j\alpha \rangle$ with $j > 0$; we will show that $|f_n(0)| \leq j$ for all n (the proof in case $j \leq 0$ is similar). Note first that

$$\langle (k - j)\alpha \rangle = \langle k\alpha \rangle - \langle j\alpha \rangle + \chi_{[0, \beta)}\langle k\alpha \rangle$$

for $k = 0, 1, 2, \dots$. Then we have

$$\begin{aligned} f_n(0) &= \sum_{k=0}^{n-1} [\chi_{[0, \beta)}\langle k\alpha \rangle - \langle j\alpha \rangle] = \sum_{k=0}^{n-1} [\langle (k - j)\alpha \rangle - \langle k\alpha \rangle] \\ &= \sum_{k=-j}^{-1} \langle k\alpha \rangle - \sum_{k=n-j}^{n-1} \langle k\alpha \rangle, \end{aligned}$$

so $|f_n(0)| \leq j$ for all n .

Suppose now that (vi) holds, so that there are x and M with $|f_n(x)| \leq M$ for all $n = 1, 2, \dots$. Then

$$|f_n\langle x + j\alpha \rangle| = |f_{n+j}(x) - f_j(x)| \leq 2M,$$

and $|f_n|$ is bounded by $2M$ on $\{\langle x + j\alpha \rangle : j = 0, 1, 2, \dots\}$, a dense subset of $[0, 1)$. But each f_n is a step function with only finitely many jumps; therefore we must have $|f_n(y)| \leq 2M$ for all $y \in [0, 1)$, and we have proved (vii).

Since the implication from (vii) to (iii) is obvious, the proof is complete.

The convergence of (1) only in case $\beta \in \mathbf{Z}\alpha \pmod{1}$ has some obvious applications to the theory of Diophantine approximation; for example, if α is irrational and $\beta \notin \mathbf{Z}\alpha \pmod{1}$, then for each $\varepsilon > 0$ and $c > 0$ there are infinitely many $k \in \mathbf{Z}$ for which $\|k\beta\| > ck^{\frac{1}{2}-\varepsilon}\|k\alpha\|$. Also, it is apparent that the foregoing theorem contains another easy proof of the theorem of Kesten mentioned above.

The equivalence of conditions (iii) and (iv), which is analogous to a theorem of Gottschalk and Hedlund [4, Theorem 14.11], is easily generalized to the case of any linear operator U acting continuously on a reflexive Banach space B : an element $f \in B$ has $\|f + Uf + \cdots + U^{n-1}f\|$ bounded in n only if there is $g \in \overline{co}\{f + Uf + \cdots + U^{n-1}f : n = 1, 2, \cdots\}$ such that $f = g - Ug$. Essentially the same result has been obtained earlier by Butzer and Westphal [2]. Further generalizations to cases such as B locally convex and $\{f + Uf + \cdots + U^{n-1}f : n = 1, 2, \cdots\}^-$ compact are also possible.

The proof that (iii) \Rightarrow (iv) \Rightarrow (v) applies also to the case of a general measure-preserving transformation $T : X \rightarrow X$ of a probability space (X, \mathcal{B}, μ) . Let $A \subset X$ be a measurable set with $\mu(A) = \beta$, and let

$$f_n(x) = \sum_{k=0}^{n-1} [\chi_A(T^k x) - \beta]$$

for $n = 1, 2, \cdots$. If $\{\|f_n\|_2 : n = 1, 2, \cdots\}$ is bounded, then $e^{2\pi i\beta}$ must be in the spectrum of T . The same observation has been made independently by Furstenberg, Keynes, and Shapiro [3, Theorem 2.4].

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