COMPOSITIO MATHEMATICA

CHARLES E. CLEAVER

A characterization of spectral operators of finite type

Compositio Mathematica, tome 26, nº 2 (1973), p. 95-99

<http://www.numdam.org/item?id=CM_1973__26_2_95_0>

COMPOSITIO MATHEMATICA, Vol. 26, Fasc. 2, 1973, pag. 95-99
Noordhoff International Publishing
Printed in the Netherlands

A CHARACTERIZATION OF SPECTRAL OPERATORS OF FINITE TYPE

by

Charles E. Cleaver [1]

A spectral operator is said to be of finite type if it is the sum of a scalar and a nilpotent operator. In this paper, these operators are characterized as operators which can be expressed as the product of a generalized scalar $\mathscr{A}$-unitary operator with a regular generalized scalar whose spectrum is contained in the nonnegative real axis.

Using results of S. R. Foguel [3] and some properties of semi-inner products, T. V. Pachapagesan [6] obtained a decomposition of scalar operators into the product of a unitary operator (under some equivalent norm) and a scalar operator with nonnegative spectrum. In the present paper, it is shown that spectral operators of finite type can be expressed in the same way with scalar replace by regular generalized scalar.

Most of the theorems and definitions used in this paper can be found in the book, 'Theory of Generalized Spectral Operators', by I. Colojoara and C. Foias [1]. Some of the definitions are stated below for easy reference.

Throughout the rest of the paper, X will denote a fixed Banach space and $B(X)$ the space of bounded linear operators on X. The algebra of all infinitely differentiable functions on $R^2(=C)$ with the topology of uniform convergence of the functions and all their derivates, will be denoted by C^∞. The functions $F(\lambda) = \lambda$, $(\lambda = s+it)$, and $f(\lambda) \equiv 1$ will be shortened to λ and 1 respectively.

Definition (1). A continuous algebraic homomorphism U from C^∞ into $B(X)$ such that $U_1 = I$ (identity operator) is called a *spectral distribution.*

Definition (2). Let $\mathscr{A}$ be an algebra of functions defined on a set D in the complex plane. If U is an algebraic homomorphism from $\mathscr{A}$ into $B(X)$ such that $U_1 = I$ and the $B(X)$ valued function $\xi \to U_{f_\xi}$ is analytic

[1] This paper was written at the U. S. A. F. Aerospace Research Laboratories while in the capacity of an Ohio State University Research Foundation Visiting Research Associate under contract F 33615 67 C 1758. AMS Subject Classification: Primary 47 B 40, Secondary 46 F 99 (1970).

on $C\setminus\mathrm{supp}\,(f)$, where

$$f(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi-\lambda} & \text{if } \lambda \in D\setminus\{\xi\} \\ 0 & \lambda \in \Omega \cap \{\xi\} \end{cases}$$

then the operator $T = U_\lambda$ is called an *$\mathscr{A}$-scalar* operator. If $\mathscr{A} = C^\infty$ and U is continuous, then T is a *generalized scalar operator*. The operator T is *$\mathscr{A}$-unitary* if D is the unit circle.

For an operator T, $C(T, T)A$ will denote the operator $TA-AT$.

DEFINITION (3). A spectral distribution U is called regular if $C(U_\lambda, U_\lambda)A = 0$ implies $C(U_f, U_f)A = 0$ for all $f \in C^\infty$. A generalized scalar operator is regular if it has a regular spectral distribution.

To see that every spectral operator $T = P+Q$, with P scalar and Q nilpotent of order n, is a regular generalized scalar operator, define $U: C^\infty \to B(X)$ by,

$$U_f = \sum_{k=0}^{n} \frac{Q^k}{k!} \int D^k f(\lambda)\,dE(\lambda), \qquad f \in C^\infty$$

where the operator $D = \frac{1}{2}(\partial/\partial s - i\partial/\partial t)$, $(\lambda = s+it)$, and E is the resolution of the identity corresponding to T. It follows that T is regular since any operator which commutes with T also commutes with $E(\sigma)$ for any Borel set σ (cf: [2], theorem 5), and hence commutes with $\int f(\lambda)\,dE(\lambda)$ for all $f \in C^\infty$.

THEOREM (1). *Let T be a spectral operator of finite type. Then there exists operators R and S satisfying,*

i) *S is a generalized scalar $\mathscr{A}$-unitary operator*

ii) *R is a regular generalized scalar with its spectrum contained in the nonnegative real axis*

iii) $T = RS = SR$.

PROOF. There exists operators P and Q such that $T = P+Q, PQ = QP$, Q is nilpotent and P is scalar. According to lemma 2 ([3], page 60) there exist two commuting scalar operators T_1, T_2 such that $P = T_1T_2$ and $\sigma(T_1)$ is a subset of positive real axis, $\sigma(T_2)$ is a subset of the unit circle.

It follows from the construction of T_2 that $QT_2 = T_2Q$ and hence $QT_2^{-1} = T_2^{-1}Q$. Since Q is a nilpotent operator, so is $T_2^{-1}Q$. Thus the operator $R = T_1 + T_2^{-1}Q$ is a spectral operator of finite type and it follows that T is a regular generalized scalar (cf: [1], theorem 3.6, page 107) and $\sigma(R) = \sigma(T_1)$ is a subset of positive real axis.

Finally note that $T = T_2R = RT_2$ and T_2 is a generalized scalar $\mathscr{A}$-unitary operator. Thus letting $S = T_2$, properties i), ii) and iii) are satisfied.

COROLLARY (1). *If T is a spectral operator, then there exists operators R and S such that S is a generalized scalar $\mathscr{A}$-unitary operator, R is a spectral operator with spectrum contained in the nonnegative real axis, and $T = RS = SR$.*

PROOF. If $T = P+Q$, P scalar and Q quasi-nilpotent, then $T = SR_1+Q$ where SR_1 is the decomposition of P from theorem 1. Thus $T = S(R_1+S^{-1}Q)$ and since S commutes with Q, $S^{-1}Q$ is quasi-nilpotent which implies that $\sigma(R) = \sigma(R_1+S^{-1}Q) = \sigma(R_1)$ is contained in the nonnegative real axis.

Using the fact that a scalar operator is unitary under some equivalent norm iff its spectrum is contained in the unit circle, we obtain the following decomposition.

COROLLARY (2). *If T is a spectral operator of finite type, then $T = RS = SR$ where S is unitary under some equivalent norm and R is a regular generalized scalar operator with spectrum contained in the nonnegative real axis. Further, if T is scalar, then R is also scalar.*

Before obtaining the converse to theorem 1, we first establish the following lemma.

LEMMA (1). *Let T be an operator in $B(X)$ such that $T = P+Q$ where P is a regular generalized scalar which commutes with every operator A that commutes with T and Q is quasi-nilpotent. If $T = RS = SR$ where R and S are generalized scalar operators with commuting spectral distributions, then Q is nilpotent.*

PROOF. If U, V are commuting spectral distribution for R, S respectively, then there exists a sepctral distribution W such that $W_\lambda = U_\lambda V_\lambda = T$. Letting Y be a regular spectral distribution for P, it follows that W_f commutes with $Y_\lambda = P$ for every $f \in C^\infty$ and hence Y_g for every $g \in C^\infty$. Therefore, W, Y are commuting spectral distributions and thus $Q = W_\lambda - Y_\lambda$ is a generalized scalar which implies it is nilpotent (cf: [1], page 106).

THEOREM (2). *Let T be a spectral operator such that $T = RS = SR$ with S a generalized scalar $\mathscr{A}$-unitary and R a regular generalized scalar. Then T is of finite type.*

PROOF. Let U and V be spectral distributions of R and S respectively with U regular. Since the spectrum of S is contained in the unit circle and hence thin, S is regular (cf: [1], theorem 1.11, page 100). We may assume that V is a regular spectral distribution for S. Since $U_\lambda V_\lambda = V_\lambda U_\lambda$, $C(U_\lambda, U_\lambda)V_\lambda = 0$ and thus $C(U_f, U_f)V_\lambda = 0$ for all $f \in C^\infty$ because U is regular. Similarly $C(V_f, V_f)U_\lambda = 0$, $f \in C^\infty$. Let $g \in C^\infty$, then

$C(V_g, V_g)U_\lambda = 0$ implies $C(U_\lambda, U_\lambda)V_g = 0$ and hence $C(U_f, U_f)V_g = 0$, $f \in C^\infty$. Therefore U and V are commuting spectral distributions and since T is spectral, it follows from the lemma that T is of finite type.

An operator having the above decomposition does not have to be spectral. For example, consider the operator T defined on $L_p(0, 1)$ by

$$Tf(x) = xf(x) + \int_0^x f(t)dt.$$

Kantorovitz [5] has shown that T is not spectral. However, letting U be the spectral distribution defined by

$$U_\varphi f(x) = \varphi(x)f(x) + \varphi'(x)\int_0^x f(t)dt.$$

We see that T is a generalized scalar and since its spectrum is contained in [0, 1], the operators T and I satisfy the conditions of theorem 1. Further, since the operator $Qf(x) = \int_0^x f(t)dt$ for $f \in L_p(0, 1)$ is quasi-nilpotent (cf: [4], page 668), it follows that the condition that P commute with every operator which commutes with T is necessary in lemma 1.

In some cases, we can find a decomposition without using the fact that they are spectral.

THEOREM (3). *Let E be a Banach space of complex-valued functions contained in $C^\infty(\Omega)$ with the property that $fg \in E$ for each bounded function g on Ω and each $f \in E$. If T is a bounded linear operator on E defined by $Tf(x) = f(x)h(x)$ for some bounded function h, then there exists operators R and S satisfying properties i), ii), iii) of theorem* 1.

PROOF. For each $\varphi \in C^\infty$, let U_φ be the operator defined by $U_\varphi f(x) = \varphi(h(x))f(x)$, then U is a spectral distribution for T. Let C_1 be the unit circle in complex plane, then define $V : C^\infty(C_1) \to B(X)$ by

$$V_\varepsilon f(x) = \begin{cases} \varepsilon(\operatorname{sgn} h(x))f(x) & h(x) \neq 0 \\ f(x) & h(x) = 0 \end{cases}$$

for each $\varepsilon \in C^\infty(C_1)$.

Then V_λ is a generalized scalar $\mathscr{A}$-unitary operator and $V_{\bar\lambda} U_\lambda f(x) = |h(x)|f(x)$. Letting $W_\lambda = V_{\bar\lambda} U_\lambda$, it follows that $T = V_\lambda W_\lambda$ and V_λ, W_λ satisfy properties i), ii), iii) of theorem 1.

The author wishes to give credit to the referee for suggesting a shorter proof of theorem 1.

REFERENCES

I. COLOJOARA and C. FOIAS
[1] Theory of Generalized Spectral Operators. New York: Gordon and Breach, 1968

N. Dunford
[2] Spectral operators, Pacific J. Math. 4 (1954), 321–354.

S. R. Foguel
[3] The relation between a spectral operator and its scalar part, Pacific J. Math. 8 (1958) 51–65.

E. Hille and R. S. Phillips
[4] Functional Analysis and Semi-Groups, Amer. Math. Soc. Coll. Publ., Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

S. Kantorovitz
[5] The C^k-classification of certain operators in L_p ( ), Trans. Amer. Math. Soc. 132 (1968), 323–333.

T. V. Pachapagesan
[6] Unitary operators in Banach spaces, Pacific J. Math. 22 (1967). 465–475.

(Oblatum 19–IV–71 & 29–I–72)

Department of Mathematics
Kent State University
Kent, Ohio 44242
USA