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CENTRAL SIMPLICITY AND CHEVALLEY ALGEBRAS

by

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1. Introduction

Our object is to present simplified and more conceptual proofs of some theorems of Hurley [3, 4] on 'Chevalley algebras' over commutative rings, and simultaneously to generalise his results. Our methods are derived from Jacobson [5] p. 109, Kaplansky [6] p. 150 and were used by us in [9]. It turns out that [9] can be modified to cover the present situation; and for reasons of compatibility we shall follow the notation of that paper.

Hurley starts from the theorem of Chevalley [2] that every finite-dimensional split semisimple Lie algebra L over a field of characteristic zero has a basis with respect to which the structure constants are integers – the so-called *Chevalley basis*. Thus the $\boldsymbol{Z}$-module generated by this basis has a natural ring structure; we denote the ring by $L_{\boldsymbol{Z}}$. Interpreting the structure constants as elements of a commutative ring-with-1 R we obtain the *Chevalley algebra* L_R which is a Lie algebra over R. Clearly $L_R \cong R \otimes L_{\boldsymbol{Z}}$, where the tensor product is over $\boldsymbol{Z}$.

Suppose now that L is simple. What conditions on R will ensure that every Lie ideal of L_R arises as $J \otimes L_{\boldsymbol{Z}}$ for an ideal J of R? The main theorem of Hurley [3] states that if 2 and 3 are not zero-divisors in R then a necessary and sufficient condition is that two integers should be invertible in R, namely: the determinant of the Cartan matrix of L, and the square of the ratio of the lengths of long and short roots in L. The proof given in [3] involves detailed calculations using a Chevalley basis.

In another paper [4] Hurley shows that a necessary and sufficient condition for L_R to have a composition series of ideals is that R should have a composition series of ideals. In these bare terms the result is rather easy to prove, as we shall show. Of course, [4] provides much detailed information about Chevalley algebras which is not obtainable from the more general viewpoint of the present paper.

2. Notation and terminology

We shall show that Hurley's results are typical of the general situation of a tensor product $R \otimes B$ where B is a certain special kind of (non-associative) ring, having properties closely allied to central simplicity. In order to state our theorems concisely we must introduce some terminology.

From now on the terms 'ring', 'algebra', and 'ideal' will be interpreted in the non-associative sense unless suitably qualified.

Let B be a ring, with additive group B^+. Let $\mathfrak{E}(B)$ denote the (associative) ring of endomorphisms of B^+. The *multiplication ring* $\mathfrak{M}(B)$ is the associative subring of $\mathfrak{E}(B)$ generated by the right and left multiplications

$$b_R : x \mapsto xb \quad (x \in B)$$
$$b_L : x \mapsto bx \quad (x \in B)$$

for all $b \in B$.

We shall usually assume that B satisfies the following hypotheses:

(H1) $\qquad$ *B^+ is a finitely generated free* $\mathbf{Z}$*-module.*

(H2) $\qquad$ *$Q \otimes B$ is a central simple Q-algebra.*

In H2 (and elsewhere) the tensor product is over $\mathbf{Z}$. The Chevalley algebra $L_{\mathbf{Z}}$ certainly satisfies H1; and it is well-known that it also satisfies H2. (By direct computation (exactly as in section 4) it follows that the multiplication algebra of $Q \otimes L_{\mathbf{Z}}$ is the full algebra of endomorphisms, which obviously implies central simplicity of $Q \otimes L_{\mathbf{Z}}$.)

Notice that if H1 holds, then $Q \otimes B$ is finite-dimensional.

We shall also work with a second ring R, which will satisfy

(H3) *R is a commutative associative ring-with-*1.

Given a set π of primes we shall say that R is *π-regular* if every $p \in \pi$ is invertible considered as an element of R; and R is *π-singular* otherwise. If $\pi = \{p\}$ we use the terms *p-singular*, *p-regular*.

For any ring T we let $I(T)$ denote the set of ideals of T. There is a natural map

$$\tau : I(R) \to I(R \otimes B)$$

defined by $\tau(J) = J \otimes B$ for $J \in I(R)$. We seek conditions under which τ is a bijection.

3. Relevant Primes

In one direction, our results will follow from:

THEOREM 3.1. *Let R and B satisfy hypotheses* H1, H2, H3. *Then there exists an integer $\lambda > 0$ such that*

$$\lambda \mathfrak{E}(B) \subseteq \mathfrak{M}(B).$$

For such an integer λ, every ideal I of $R\otimes B$ satisfies

$$\lambda J\otimes B \subseteq I \subseteq J\otimes B$$

for a suitable ideal J of R.

PROOF: We modify the proof of the 'Sandwich Lemma' of [9]. Let $\mathfrak{M} = \mathfrak{M}(B)$, $\mathfrak{E} = \mathfrak{E}(B)$, $\mathfrak{M}_Q = \mathfrak{M}(Q\otimes B)$. Let $\{p_\alpha\}$ be a set of free generators for B^+, where α runs through a finite index set A. Such generators exist by H1. By H2, $Q\otimes B$ is central simple; so by Jacobson's Density Theorem we can find elements $t_{\alpha\beta} \in \mathfrak{M}_Q$ ($\alpha, \beta \in A$) such that

$$(1) \qquad \begin{aligned} p_\alpha t_{\alpha\beta} &= p_\beta \\ p_\gamma t_{\alpha\beta} &= 0 \quad (\alpha \neq \gamma \in A). \end{aligned}$$

By multiplying together the denominators of the coefficients involved in the expression of $t_{\alpha\beta}$ in terms of $(p_\delta)_L$ and $(p_\delta)_R$ ($\delta \in A$) we can find integers $\lambda_{\alpha\beta} > 0$ such that

$$\lambda_{\alpha\beta} t_{\alpha\beta} = u_{\alpha\beta} \in \mathfrak{M}.$$

If we set

$$\lambda = \operatorname{lcm}_{\alpha,\beta} \lambda_{\alpha\beta}$$

then the elements

$$v_{\alpha\beta} = \lambda t_{\alpha\beta}$$

lie in $\mathfrak{M}$, and

$$\begin{aligned} p_\alpha v_{\alpha\beta} &= \lambda p_\beta \\ p_\gamma v_{\alpha\beta} &= 0 \qquad (\alpha \neq \gamma \in A). \end{aligned}$$

Now let I be any ideal of $R\otimes B$, and let J be the ideal of R generated by the coefficients m of elements

$$m = \sum m_\alpha \otimes p_\alpha$$

of I. The m_α are uniquely determined since $R\otimes B$ is a free R-module (by H1). The elements of J take the form $\sum m_\alpha r$ for all possible $\alpha \in A$, $m \in I$, $r \in R$.

As in the first part of the proof of lemma 2.1 of [9] we can develop from the $v_{\alpha\beta}$ elements $w_{\alpha\beta} \in \mathfrak{M}(R\otimes B)$ such that if $a, b \in R$ then

$$\begin{aligned} \sum a \otimes p_\alpha \cdot w_{\alpha\beta} &= \sum \lambda ar \otimes p_\beta \\ \sum b \otimes p_\gamma \cdot w_{\alpha\beta} &= 0 \qquad\qquad (\alpha \neq \gamma \in A) \end{aligned}$$

for all $\alpha, \beta \in A$. It follows that I contains $\lambda J\otimes B$. Clearly I is contained in $J\otimes B$.

We now define the set π of *relevant primes* for B to be the set of prime divisors of the smallest integer $\lambda > 0$ for which $\mathfrak{M}(B)$ contains $\lambda\mathfrak{E}(B)$.

We may then draw the following:

COROLLARY 3.2. *Let R, B satisfy hypotheses* H1, H2, H3; *and let π be the set of relevant primes for B. Then π is a finite set. If R is π-regular, then the natural map*

$$\tau : I(R) \to I(R \otimes B)$$

is a bijection.

PROOF: Since R is π-regular, λ is invertible. But then $\lambda J = J$.

This corollary may be seen as a qualitative version of half of theorem 3.3 of Hurley [3]. In the next section we discuss how it may be made quantitative; and in section 5 we consider the other half of his theorem.

4. Some computations

To find the relevant primes for Chevalley algebras we must compute a suitable integer λ, but in the most economical fashion possible. Hurley's calculations in [3], can be interpreted as choices of the elements $t_{\alpha\beta}$ required for theorem 3.1, and yield:

PROPOSITION 4.1 *If $B = L_Z$ is a Chevalley algebra, then the relevant primes are*

(a) *The prime divisors of* $\det(C)$, *where C is the Cartan matrix,*

(b) *The ratio of the squares of the lengths of long and short roots, if this is not* 1.

It is easy to see that the set of relevant primes consists *at most* of 2, 3, and the divisors of det (C), and we sketch the proof.

We take $\{p_\alpha\}$ to be a Chevalley basis for the simple Lie algebra L. In the notation of Hurley [3] it consists of $\{h_i\} \bigcup \{e_r\}$ where i runs through a system of fundamental roots, r runs through a system of roots, and the multiplication is given by

$$\begin{aligned} e_r e_{-r} &= h_r \\ h_i h_j &= 0 \\ e_r e_s &= \pm N_{rs} e_{r+s} \quad (r \neq -s) \\ h_r e_s &= c(s, r) e_s . \end{aligned}$$

The structure constants N_{rs} and $c(r, s)$ are integers of absolute value $\leqq 3$, and $N_{rs} = 0$ if $r+s$ is not a root. The Cartan matrix of L is the matrix $C = (c(i,j))$ where i, j run through a system of fundamental roots.

If r and s are roots there is a sequence of roots $r_1, \cdots, r_k$ such that

$$s = r + r_1 + \cdots + r_k$$

and such that each

$$r + r_1 + \cdots + r_i$$

$(i \leqq k)$ is a root, by [3] 4.1. So

$$e_s = Ke_r e_{r_1} \cdots e_{r_k}$$

for a non-zero integer K, which is a product of certain N_{tu}.

Let ρ be the highest root, and let r be any root. As above we can find $r_1, \cdots, r_k$ such that

$$e_\rho = Ke_r e_{r_1} \cdots e_{r_k}.$$

Since adding a root r_i preserves the ordering of the roots, we know that for all roots $s > r$ we have

$$e_s e_{r_1} \cdots e_{r_k} = 0.$$

Therefore the element

$$e_{r_1} \cdots e_{r_k} e_{-\rho} e_{-\rho} e_\rho$$

sends e_r to $K'h_\rho$, where K' is a product of structure constants, and kills all the other basis vectors.

Now the elements $e_s e_{-s}$ map each e_t to a scalar multiple, and map h_t to $c(t, s)h_s$. Hence some integer linear combination of these maps h_ρ to $\det(C)h_i$ for any chosen i. Finally, h_i may be mapped by e_s to $c(s, i)e_s$. By composing the above maps we can find an element of $\mathfrak{M}(L_{\mathbf{Z}})$ mapping e_r to an integer multiple of any given basis vector and annihilating all other basis vectors. The integer concerned will be a product of $\det(C)$ and certain structure constants.

In a similar manner we can find an element of $\mathfrak{M}(L_{\mathbf{Z}})$ with the analogous effect on h_i. So we can find an integer λ of the form $2^a 3^b \det(C)$, and so may take π to be a subset of $\{2, 3\} \cup \{\text{divisors of } \det(C)\}$.

To get exactly Hurley's results involves traversing the root system in a more careful manner; and this is essentially what is done in [3]: for example see section 4.2 of that paper.

5. Bad Primes

Assume that B satisfies hypotheses H1, H2. A prime p is a *bad prime* for B if $\mathbf{Z}_p \otimes B$ is not a simple $\mathbf{Z}_p$-algebra. It follows immediately that if p is a bad prime and $\mathfrak{k}$ is any field of characteristic p, then

$$\mathfrak{k} \otimes B \cong \mathfrak{k} \otimes \mathbf{Z}_p \otimes B$$

is not simple; so the natural map $\tau : I(\mathfrak{k}) \to I(\mathfrak{k} \otimes B)$ cannot be a surjection.

The next result, though quite simple, is decisive for our purposes.

THEOREM 5.1. *Let B,R satisfy hypotheses* H1, H2, H3. *Suppose that p is a bad prime for B, and R is p-singular. Then the natural map*

$$\tau : I(R) \to I(R \otimes B)$$

is not surjective.

PROOF: Since R is p-singular, pR is a proper ideal of R. Now R has an identity, so we may find a maximal ideal K of R such that K contains pR. Then R/K is a field $\mathfrak{k}$ of characteristic p. The quotient

$$(R\otimes B)/(K\otimes B)$$

is isomorphic to $\mathfrak{k}\otimes B$. Since p is bad, this quotient ring is not simple; therefore there is an ideal I of $R\otimes B$ such that $K\otimes B \subsetneqq I \subsetneqq R\otimes B$. Since K is maximal, τ cannot be surjective.

This result shows that every bad prime is relevant. If the converse also holds, we have a characterisation theorem:

THEOREM 5.2. *Let B,R satisfy hypotheses* H1, H2, H3. *Suppose that every relevant prime for B is bad. Then the natural map $\tau : I(R) \to I(R\otimes B)$ is a bijection if and only if R is π-regular, where π is the set of relevant primes.*

Steinberg [8] (p.1120) discusses the simplicity, or otherwise, of the algebras $\mathfrak{k}\otimes L_{\mathbf{Z}}$ where $\mathfrak{k}$ is a field of characteristic $p > 0$. From his remarks on page 1121, together with the known non-simplicity of the Lie algebra of type A_r over a field of characteristic p dividing $l+1$, it follows that when $B = L_{\mathbf{Z}}$ the bad primes are precisely the relevant primes described in proposition 4.1. Thus 5.2 applies and we recover theorem 3.3 of Hurley [3] in its entirety.

If B is an $n \times n$ matrix ring over $\mathbf{Z}$ then there are no relevant or bad primes, and we recover the well-known result that under hypothesis H3 the ideals of the matrix ring over R are in bijective correspondence with the ideals of R under the natural map. (See e.g. McCoy [7], p. 37).

The question of whether in general the relevant primes are bad is still open.

6. Composition Series

In [4] Hurley considers composition series of ideals in Chevalley algebras, and proves that L_R has such a composition series if and only if R has a composition series of ideals. It should be remarked that 'ideal' here must be interpreted as 'R-algebra ideal', that is, 'ring ideal and R-submodule'. For if we take L to be of type A_n and R a field of characteristic $p|n+1$, then the Chevalley algebra $R\otimes L_{\mathbf{Z}}$ is the algebra of $(n+1)(n+1)$ matrices of trace 0 over R; and any additive subgroup of the centre, which consists of the scalar matrices, is a ring ideal. Now R, being a field, has a composition series of ideals; but we can easily choose R so that R^+ has neither maximal nor minimal condition on subgroups, and hence $R\otimes L_{\mathbf{Z}}$ has no composition series of ring ideals.

It is therefore necessary to consider only R-algebra ideals. Now if R does not have a composition series then $R \otimes B$ certainly does not have a composition series (see Hurley [4] p. 430). But it is well-known that if R is a commutative associative Artinian (Noetherian) ring then any finitely-generated R-module is Artinian (Noetherian); see for example Atiyah and MacDonald [1], or Zariski and Samuel [10] p. 158. From this we immediately have:

PROPOSITION 6.1 *Let B, R satisfy hypotheses* H1, H2, H3. *Then $R \otimes B$ has a composition series of R-algebra ideals if and only if R has a composition series of ideals.*

Note that theorems 3.1 and 5.2 refer to *ring* ideals.

7. Extending the Results

The bad primes for a simple Lie algebra of type A_n are the prime divisors of $n+1$. If B is the corresponding Chevalley algebra over $\mathbf{Z}$, and $\mathfrak{k}$ is a field of characteristic p dividing $n+1$, then $\mathfrak{k} \otimes B$ has non-trivial centre, so is not simple. However, the central quotient, say K, is simple, indeed central simple. In contrast to this, the bad primes for the other classical Lie algebras have a much more drastic effect on the structure of the tensor product.

For any commutative associative ring-with-1 R, we have $pR \otimes K = 0$. It is therefore reasonable to confine our attention to rings R such that $pR = 0$, which are therefore algebras over $\mathbf{Z}_p$. By Kaplansky [6] p. 150 it follows that the natural map $\tau : I(R) \to I(R \otimes K)$ is a bijection, although p is certainly not invertible in R! Thus by passing to the central quotient we may recover good behaviour of the ideals in the case of A_n.

We may also consider what happens if R does not have an identity. We can still define $R \otimes B$; and by the methods of [9] it follows that with λ as before every ideal of $R \otimes B$ lies between $J \otimes B$ and $\lambda J^n \otimes B$ for an ideal J of R and an integer $\lambda > 0$. Using this and the induction argument of [9] theorem 4.1 we may show that every *subideal* (with the obvious definition) of $R \otimes B$ lies between $J \otimes B$ and $\lambda^r J^n \otimes B$ for suitable $r, n > 0$ and an ideal J of R. Returning to the case where R has an identity, and assuming λ invertible in R, then every subideal of $R \otimes B$ lies between $J \otimes B$ and $J^n \otimes B$. It is now possible to carry over most of the results of [9] sections 4 and 5; in particular to give examples of Lie rings in which the join of any pair of subideals is a subideal, or which satisfy the maximal condition on subideals.

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