COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 26, nº 2 (1973), p. 101-110
<http://www.numdam.org/item?id=CM_1973__26_2_101_0>

COMPOSITIO MATHEMATICA, Vol. 26, Fasc. 2, 1973, pag. 101-110
Noordhoff International Publishing
Printed in the Netherlands

THE POINT-OUTERCOARSENESS OF COMPLETE n-PARTITE GRAPHS

by

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Introduction

A subdivision of a graph G is a graph G_1 obtained from G by replacing an edge $x = uv$ of G with a new vertex w together with edges uw and vw. Graph H is said to be homeomorphic from graph G if H can be obtained from G by a finite sequence of subdivisions. The *subgraph* of G *induced by a set* W *of vertices* has vertex set W and its edge set is the set of edges of G which are incident with two vertices of W. The *subgraph* of G *induced by an edge set* Y has Y as its edge set and contains all vertices incident with at least one edge of Y. For a real number r, $[r]$ denotes the greatest integer not exceeding r, and $\{r\}$ is the least integer not less than r.

Let $p_1 \leqq p_2 \leqq \cdots \leqq p_n$ be positive integers. Then the *complete n-partite graph* $K(p_1, \cdots, p_n)$ has $p = \sum_1^n p_i$ vertices, its vertex set can be partitioned into subsets V_i, $1 \leqq i \leqq n$, such that $|V_i| = p_i$, and two vertices are adjacent if and only if they are in different V_i. The sets $V_1, \cdots, V_n$ are called the partite sets of $K(p_1, \cdots p_n)$. If each $p_1 = 1$, $K(p_1, \cdots, p_n)$ is denoted by K_n and called the complete graph on n vertices.

An *outerplanar* graph is a graph which can be embedded in the plane so that every vertex of G lies on the exterior region. In [5] Chartrand and Harary have characterized outerplanar graphs as those graphs which contain no subgraph homeomorphic from K_4 or $K(2,3)$.

We define, for each positive integer n, the *vertex partition number* of a graph G, denoted by $\pi_n(G)$, as the maximum number of subsets into which the vertex set of G can be partitioned so that each set induces a graph which contains a subgraph homeomorphic from K_{n+1} or the complete 2-partite graph $K([n+2)/2], \{(n+2)/2\})$. This general parameter was first introduced by Chartrand, Geller and Hedetniemi in [4].

For $i = 1, 2, 3, 4$, $\pi_i(g)$ is the maximum number of point induced disjoint subgraphs of G which are totally disconnected, acyclic, outerplanar, and planar, respectively.

The edge partition number $\pi'_n(G)$ is defined analogously to $\pi_n(G)$ with the word 'vertex' replaced by 'edge'. Then $\pi'_1(G)$ is simply the number of lines of G. The only line partition number which has been given considerable study is $\pi_4'(G)$, which is called the coarseness of G. This has been investigated by Beineke [1], Beineke and Chartrand [2], Guy [9], and Beineke and Guy [3], with the last paper giving a partial formula for $\pi_4'(K(m, n))$.

The number $\pi_1(G)$ is the well-known line independence number, see Harary [10]. The number $\pi_2(G)$ has been studied by Corrádi and Hajnal [7], Dirac and Erdös [8], and Chartrand, Kronk, and Wall [6]. In this paper we investigate $\pi_3(G)$ which is called the point-outercoarseness of G and is now denoted simply $\pi(G)$.

Preliminary results

We make two easy observations and then commence the development of the formula for $\pi(K(p_1, \cdots, p_n))$. Any non-outerplanar graph has at least 4 vertices and 6 edges. This implies

REMARK 1. If G is a graph with p points and q edges, then $\pi(G) \leqq [p/4]$ and $\pi(G) \leqq [q/6]$.

The maximum number of vertices in any complete subgraph of G is denoted $\omega(G)$ and is called the *clique number* of G.

REMARK 2. If G has p vertices and $\omega(G) \leqq 3$, then $\pi(G) \leqq [p/5]$.

THEOREM 1. *Let* $G = K(p_1, \cdots, p_n)$ *with* $n \geqq 2$. If $p_n \geqq (\frac{3}{2})(p-p_n)$, *then* $\pi(G) = [(p-p_n)/2]$.

PROOF. In any decomposition of G into non-outerplanar subgraphs, each subgraph must include at least two vertices from $\bigcup_1^{n-1} V_i$. There are $p-p_n$ vertices in this set so that $\pi(G) \leqq [(p-p_n)/2]$.

Any subgraph induced by a set consisting of three vertices from V_n and two vertices from $\bigcup_1^{n-1} V_i$ is not outerplanar. From the hypothesis that $p-p_n \leqq (\frac{2}{3})p_n$ it follows that there are $[(p-p_n)/2]$ disjoint induced non-outerplanar subgraphs of G. Thus $\pi(G) = [(p-p_n)/2]$.

THEOREM 2. *If* $G = K(p_1, \cdots, p_n)$ *where* $n = 2$ *or* 3, *and* $p_n \leqq (\frac{3}{2})(p-p_n)$, *then* $\pi(G) = [p/5]$.

PROOF. Since $\omega(G) \leqq 3$, Remark 2 implies that $\pi(G) \leqq [p/5]$. In order to show that $[p/5]$ non-outerplanar, mutually disjoint, induced subgraphs of G exist we consider two cases.

CASE (i). $n = 2$. Since $p_1 \leqq p_2 \leqq (\frac{3}{2})p_1$, there are p_2-p_1 mutually disjoint sets of vertices such that each set contains three vertices from V_2 and two vertices from V_1. Each of these sets induces a non-outerplanar copy of $K(2, 3)$. There are $p_1 - 2(p_2 - p_1) = 3p_1 - 2p_2 \geqq 0$ other vertices in V_1 and $p_2 - 3(p_2 - p_1) = 3p_1 - 2p_2 \geqq 0$ other vertices in V_2. Call these sets V_1' and V_2' respectively. If $3p_1 - 2p_2 = 0$, 1, or 2, then we have partitioned G into $[p/5]$ non-outerplanar subgraphs. If $3p_1 - 2p_2 \geqq 3$, then by alternating the use of three vertices from V_2' and two vertices from V_1' with two from V_2' and three from V_1', we can complete the partition of $V(G)$ into $[p/5]$ sets of cardinality five, each of which induces a non-outerplanar graph. Thus $\pi(G) = [p/5]$ in this case.

CASE (ii). $n = 3$. If $p_1 + p_2 \leqq p_3$ we consider graph H which is G minus all edges joining V_1 to V_2. From case (i). $\pi(G) \geqq \pi(H) \geqq [p/5]$. Thus we suppose $p_3 < p_1 + p_2$. For $i = 1, 2, 3$, let $V_i^0 = V_i$. Form one copy of $K(2, 3)$ with three vertices v_1, v_2, v_3, of V_3^0 and two vertices v_4, v_5 of V_2^0. Let V_1^1, V_2^1 be an ordering of V_1 and $V_2 - \{v_4, v_5\}$ so that $|V_1^1| \leqq |V_2^1|$, and let $V_3^1 = V_3 - \{v_1, v_2, v_3\}$. Then repeat this procedure with V_1^1, V_2^1, and V_3^1, and continue this procedure until reaching a non-negative integer j such that $V_3^j \leqq V_2^j$. (Note that j may be zero.) Let

$$V_1^{j+1},\ V_2^{j+1},\ V_3^{j+1}$$

be a reordering of V_1^j, V_2^j, V_3^j such that

$$|V_1^{j+1}| \leqq |V_2^{j+1}| \leqq |V_3^{j+1}|$$

and observe that

$$0 \leqq |V_3^{j+1}| - |V_2^{j+1}| \leqq 2.$$

Continue the partition of G into copies of $K(2, 3)$ by using three vertices w_1, w_2, w_3, from V_3^{j+1} and two vertices w_4, w_5 from V_2^{j+1}. Let

$$V_1^{j+1},\ V_2^{j+1} - \{w_4, w_5\},\ V_3^{j+1} - \{w_1, w_2, w_3\}$$

be reordered by

$$V_1^{j+2},\ V_2^{j+2},\ V_3^{j+2}$$

so that

$$|V_1^{j+2}| \leqq |V_2^{j+2}| \leqq |V_3^{j+2}|.$$

We stop this procedure when $|V_3^k| \leqq 3$ for some $k \geqq j+1$. If

$$|V_1^k| + |V_2^k| + |V_3^k| \leqq 4,$$

then G has been partitioned into $[p/5]$ non-outerplanar graphs. Otherwise induce one more non-outerplanar graph with the remaining vertices. Thus $\pi(G) \geqq [p/5]$, which completes the proof of the theorem.

COROLLARY 3. *If* $G = K(p_1, \cdots, p_n)$ *where* $n = 2$ *or* 3 *then* $\pi(G) = \min\{[p/5], [(p - p_n)/2]\}$.

THEOREM 4. *Let* $G = K(p_1, \cdots, p_n)$ *where* $p_n \leqq (\frac{3}{2})(p - p_n)$. *Then* $\pi(G) \geqq [p/5]$.

PROOF. We use induction and observe that Theorem 2 verifies the result for $n = 2$ or 3. Assume Theorem 4 holds for $n \geqq 3$ and let $G = K(p_1, \cdots, p_{n+1})$ where $p_{n+1} \leqq (\frac{3}{2})(p-p_{n+1})$. The subgraph of G formed by removing all edges joining V_1 with V_2 is a complete n-partite graph $H = K(p'_1, \cdots, p'_n)$ where $p'_n = \max\{p_{n+1}, p_1+p_2\}$. Since $p'_n \leqq (3/2)(p'_1+\cdots+p'_{n-1})$, the inductive assumption applies and we have $\pi(G) \geqq \pi(H) \geqq [p/5]$.

The following lemma will be helpful.

LEMMA 1. *Let c be an integer such that $1 < c \leqq n$. If $p_n-p_{n-c+1} \leqq 1$, then the complete n-partite graph $G = K(p_1, \cdots, p_n)$ contains $[p/c]$ mutually disjoint copies of K_c.*

PROOF. We use induction on p. If the order of G is less than $n+c$, then $p_{n-c+1} = 1$. We form one copy of K_c by selecting one vertex from each V_i, $i = n-c+1, \cdots, n$. The remaining vertices of G induce a complete graph on $p-c$ vertices. Thus G contains $[p/c]$ mutually disjoint copies of K_c.

Let the order of G be $p \geqq n+c$ and suppose the lemma is true for all complete n-partite graphs with less than p vertices. Form one copy of K_c by selecting one vertex from each of $V_{n-c+1,\ldots,}V_n$. The graph H induced by the remaining vertices of G is a complete n-partite graph with $p'_n-p'_{n-c+1} \leqq 1$ where p'_i is the order of the ith partite set of H. By the induction hypothesis H contains $[(p-c)/c]$ mutually disjoint copies of K_c and the lemma is proved.

THEOREM 5. *Let $G = K(p_1, \cdots, p_n)$ where $n \geqq 4$. If $p \geqq 4p_n$, then $\pi(G) = [p/4]$.*

PROOF. We use induction on p_n. If $p_n = 1$, G is the complete graph with $p = n$ vertices and $\pi(G) = [p/4]$. Suppose the theorem holds if $p_n = k \geqq 1$ and let $p_n = k+1$. Remove one vertex from each V_n, V_{n-1}, V_{n-2}, and V_{n-3}. The resulting graph H is a complete m-partite graph with $n \geqq m \geqq 4$ and the largest partite set in H has $p_{n-1} = k$ or p_n vertices. The latter case implies that $p_n-p_{n-3} = 0$, and Lemma 1 proves the theorem. In the former case the inductive assumption implies $\pi(H) = [(p-4)/4]$ and thus $\pi(G) = [p/4]$.

The principal result

Before stating the main theorem, we prove another lemma.

LEMMA 2. *Let $G = K(p_1, \cdots, p_n)$ with $n \geqq 3$. If r is a positive integer such that $p \geqq 3r$, $p_1+\cdots+p_{n-1} \geqq 2r$, and $p_1+\cdots+p_{n-2} \geqq r$, then G contains at least r mutually disjoint triangles.*

PROOF. For $i = 1, \cdots n$, let $V_i^0 = V_i$. Form one triangle with vertices

$$v_{n-2}, v_{n-1}, v_n \text{ of } V_{n-2}^0, V_{n-1}^0, V_n^0$$

respectively. Let

$$V_n^1 = V_n^0 - \{v_n\} \text{ and } V_1^1, \cdots, V_{n-1}^1$$

be a recordering of

$$V_1^0, \cdots V_{n-3}^0, V_{n-2}^0 - \{v_{n-2}\}, V_{n-1}^0 - \{v_{n-1}\}$$

such that

$$|V_i^1| \leqq |V_{i+1}^1| \text{ for } i = 1, 2, \cdots, n-2.$$

Repeat this procedure until either

$$|V_n^k| - |V_{n-2}^k| \leqq 1 \text{ and } |V_{n-2}^k| \neq 0$$

for some k or $|V_{n-2}^k| = 0$ for some k. If the former occurs first, then from Lemma 1, it follows that G contains at least r mutually disjoint triangles. Thus suppose $|V_{n-2}^k| = 0$ for some j and consider two cases.

CASE (i) $|V_{n-1}^i| - |V_{n-2}^i| \leqq 1$ for some $i < k$. Each of the k triangles which have been formed contain one vertex of V_n and two vertices from distinct $V_j, j = 1, \cdots, n-1$. Since $|V_{n-1}^i| - |V_{n-2}^i| \leqq |$, Lemma 1 implies that at most one vertex of $\bigcup_1^{n-1} V_j$ is not included in one of the triangles. Thus $k = [(p_1 + \cdots + p_{n-1})/2] \geqq r$.

CASE (ii). $|V_{n-1}^i| - |V_{n-2}^i| > |$ for all $i > k$. In this case

$$V_{n-1}^i \subset V_{n-1} \text{ for } i = 1, \cdots, k-1.$$

Hence each of the k triangles contains exactly one vertex from $\bigcup_1^{n-2} V_j$. This implies

$$k = |\bigcup_{1}^{n-2} V_j| \geqq r$$

and completes the proof.

THEOREM 6. *Let $G = K(p_1, \cdots, p_n)$ with $n \geqq 2$, then*

$$\pi(G) = \begin{cases} [(p-p_n)/2] & \text{if } p \leqq (\frac{5}{3})p_n \\ [p/4] & \text{if } p \geqq 4p_n \\ [(p+r)/5] & \text{if } (\frac{5}{3})p_n < p < 4p_n \end{cases}$$

where

$$r = \min\{(p-p_n-p_{n-1}-p_{n-2}), [(p-p_n-p_{n-1})/2], [(3p-5p_n)/7]\}.$$

PROOF. If $p \leqq (\frac{5}{3})p_n$ or $p \geqq 4p_n$, the result follows from Theorems 1 and 5. Thus we consider only $(\frac{5}{3})p_n < p < 4\,p_n$ and distinguish three cases depending on r.

CASE (i). $r = p-p_n-p_{n-1}-p_{n-2}$. Since

$$p-p_n-p_{n-1}-p_{n-2} \leqq (p-p_n-p_{n-1})/2$$

we have

$$p-p_n-p_{n-1}-p_{n-2} \leqq p_{n-2} \leqq p_{n-1} \leqq p_n.$$

That is the cardinality of $\bigcup_1^{n-3} V_i$ does not exceed the cardinality of V_{n-2}. Thus there are r mutually disjoint copies of K_4 with one vertex in each of the sets

$$V_n,\ V_{n-1},\ V_{n-2},\ \bigcup_1^{n-3} V_i.$$

Let G minus these r copies of K_4 be denoted by H. Graph H has $p-4r$ vertices, and we let $V_i^1 = V_i \cap V(H)$ for $i = n-2, n-1, n$. Since $r \leqq (3p-5p_n)/7$ we have $\frac{2}{3}(p_n-r) \leqq p-p_n-3r$, where $p_n-r = |V_n'|$ and

$$p-p_n-3r = |V_{n-1}' \cup V_{n-2}'|.$$

Theorem 2 implies that $\pi(H) = [(p-4r)/5]$. Hence

$$\pi(G) \geqq r+[(p-4r)/5] = [(p+r)/5].$$

Since G does not contain more than r copies of K_4, it is clear that $\pi(G) = [(p+r)/5]$.

CASE (ii). $r = [(p-p_n-p_{n-1})/2] < [(3p-5p_n)/7]$. In this case we consider the complete $(n-1)$-partite graph $H = G-V_n$. By hypothesis

$$r \leqq p-p_n-p_{n-1}-p_{n-2} \tag{1}$$

and

$$2r \leqq p-p_n-p_{n-1} \tag{2}$$

Inequality (2) together with $[(p-p_n-p_{n-1})/2] < [(3p-5p_n)/7]$ imply

$$r \leqq p_{n-1}. \tag{3}$$

Adding (2) and (3) we obtain

$$3r \leqq \sum_1^{n-1} p_i. \tag{4}$$

Since (1), (2), and (4) hold, Lemma 2 implies that H contains at least r mutually disjoint triangles. The set V_n contains $p_n \geqq p_{n-1} \geqq r$ vertices. Thus, G contains r mutually disjoint copies of K_4, each of which has one vertex from V_n and three vertices from $\bigcup_1^{n-1} V_i$. There are $p-p_n-3r$ other vertices in $\bigcup_1^{n-1} V_i$ and p_n-r other vertices in V_n.

The graph G' induced by the remaining vertices of G is a complete m-partite graph, $m \leqq n$. Since $r < (3p-5p_n)/7$, we have

$$p-p_n-3r > \tfrac{2}{3}(p_n-r). \tag{5}$$

That is the number of vertices in $V(G')-V_n$ is more than two-thirds the number of vertices in $V(G') \cap V_n$. From $r = [(p-p_n-p_{n-1})/2]$ it follows that

$$(6) \qquad p_n - r + 1 \geqq p - p_n - 3r.$$

If a maximum partite set of G' is $V(G') \cap V_n$, then (5) together with Theorem 4 imply that $\pi(G') \geqq [(p-4r)/5]$, and thus $\pi(G) \geqq r + \pi(G') \geqq [(p+r)/5]$. From (6) and the fact that $|V(G')-V_n| = p-p_n-3r$ it follows that if $V(G') \cap V_n$ is not a largest partite set of G', then a largest partite set contains exactly $p_n - r + 1 = p - p_n - 3r$. Thus G' is a bipartite graph with partite sets V_1' and V_2' where $|V_2'| = p - p_n - 3r$ and $|V_1'| = p_n - r$. According to Theorem 4, $\pi(G') \geqq [(p-4r)/5]$ and $\pi(G) \geqq [(p+r)/5]$.

In order to show that equality holds suppose $\pi(G) > [(p+r)/5]$. Then there are more than r mutually disjoint copies of K_4 in G. Each copy of K_4 must contain two vertices from $\bigcup_1^{n-2} V_i$, so that $p-p_n-p_{n-1} \geqq 2(r+1)$. This implies that $[(p-p_n-p_{n-1})/2] > r$ which contradicts the hypothesis for this case. Hence $\pi(G) = [(p+r)/5]$.

Case (iii). $r = [(3p-5p_n)/7]$. In this case we let $H = G - V_n$. From the hypothesis for this case we have

$$(7) \qquad r \leqq p - p_n - p_{n-1} - p_{n-2} \quad \text{and}$$

$$(8) \qquad 2r \leqq p - p_n - p_{n-1}$$

Furthermore, $p-p_n-3r \geqq p-p_n-3((3p-5p_n)/7) = (\frac{8}{7})p_n - (\frac{2}{7})p > 0$. Thus $p - p_n > 3r$ which together with (7), (8) and Lemma 2 imply that H contains r mutually disjoint triangles.

Since $4p_n > p$, we have that $3p_n > p - p_n > 3r$. Thus V_n contains more than r vertices. Graph G has at least r mutually disjoint copies of K_4 each consisting of one vertex of V_n and three vertices of $\bigcup_1^{n-1} V_i$. There are $p-4r$ other vertices in G. These vertices induce a complete m-partite subgraph G' of G with precisely $p_n - r$ vertices of V_n and $p-p_n-3r$ vertices of $\bigcup_1^{n-1} V_i$. Since $r \leqq (3p-5p_n)/7$ we have

$$(9) \qquad (\tfrac{3}{2})(p-p_n-3r) \geqq p_n - r > 0.$$

Let W be a maximum partite set of G'. If $W = V_n \cap V(G')$, then (9) together with Theorem 4 imply that $\pi(G') \geqq [(p-4r)/5]$, and $\pi(G) \geqq [(p+r)/5]$.

Suppose $W \neq V_n \cap V(G')$; then let $k = 4p_n - p$. Thus,

$$r = [(3p-5p_n)/7] = [p_n - 3k/7] = p_n - \{3k/7\}$$

where $\{x\}$ is the least integer not less than x. We have

$$p_n - r = \{3k/7\} \text{ and} \tag{10}$$

$$p - p_n - 3r = 3p_n - k - 3(p_n - \{3k/7\}) = -k + 3\{3k/7\}. \tag{11}$$

If $k = 1$, then the number of vertices in G' is $p-4r = p-4p_n+4\{3k/7\} = -1+4 = 3$, and $\pi(G) \geqq r+[(p-4r)/5] = [(p+r)/5]$. If $k \geqq 2$, then from (10) and (11) we have $p_n - r \geqq (\frac{2}{3})(p-p_n-3r)$. This implies that

$$|V(G')-W| \geqq |V_n|-r = p_n - r \geqq (\tfrac{2}{3})(p-p_n-3r) \geqq (\tfrac{2}{3})|W|.$$

Hence, according to Theorem 4,

$$\pi(G') \geqq [(p-4r)/5] \text{ and } \pi(G) \geqq r+[(p-4r)/5] = [(p+r)/5].$$

Suppose $\pi(G) > [(p+r)/5]$. Any decomposition of G into more than $[(p+r)/5]$ non-outerplanar graphs will necessarily contain $r+t$ mutually disjoint copies of K_4, $t > 0$. Let $V_1^1, V_2^1, \cdots, V_m^1$ be the partite sets of the complete m-partite graph H^1 which remains after deleting these $r+t$ copies of K_4 from G. The order of H^1 is $p-4r-4t$ and $|V_m^1| \geqq |V_c^1|$ where $V_c^1 = V_n \cap V(H^1)$. We have $r+t \leqq p/4 < p_n$, and thus

$$|V_m^1| \geqq |V_c^1| \geqq p_n - r - t > 0. \tag{12}$$

Then

$$\Big|\bigcup_1^{m-1} V_i^1\Big| \leqq |V(H^1)| - (p_n - r - t) = p - p_n - 3r - 3t. \tag{13}$$

From the fact that $r > (3p-5p_n)/7 - t$ we obtain

$$p - p_n - 3r - 3t < (\tfrac{2}{3})(p_n - r - t). \tag{14}$$

Using (12), (13), and (14) we have

$$\Big|\bigcup_1^{m-1} V_i^1\Big| \leqq p - p_n - 3r - 3t < (\tfrac{2}{3})(p_n - r - t) \leqq (\tfrac{2}{3})|V_m^1|.$$

According to Theorem 1,

$$\pi(H^1) = \Big[\Big|\bigcup_1^{m-1} V_i^1\Big|/2\Big] = s.$$

Suppose $t \geqq 2$. Since $s \leqq [(p-p_n-3r-3t)/2]$, the number of mutually disjoint non-outerplanar subgraphs in this decomposition does not exceed $r+t+[(p-p_n-3r-3t)/2] \leqq [p-p_n-r-t)/2]$.
However, $r+2 > (3p-5p_n)/7+1$, which implies

$$\left[\frac{p-p_n-(r+2)}{2}\right] \leqq \left[\frac{p-p_n-(3p-5p_n+7)/7}{2}\right] = \left[\frac{2p-p_n}{7} - \frac{1}{2}\right]. \tag{15}$$

Also

$$(16)\quad \left[\frac{p+r}{5}\right] \geqq \left[\frac{(10p-5p_n-6)/7}{5}\right] = \left[\frac{2p-p_n}{7} - \frac{6}{35}\right].$$

Since the right side of (15) is not more than the right side of (16), we have $r+t+s \leqq [(p-p_n-r-2)/2] \leqq [(p+r)/5]$. That is this decomposition yields at most $[(p+r)/5]$ mutually disjoint non-outerplanar subgraphs of G.

If $t = 1$ and $s = 0$, othen this decomposition yields $r+1$ mutually disjoint non-outerplanar graphs and since $|V_m^1| > 0$ there is at least one vertex which is not included in any of the $r+1$ copies of K_4. Thus $r+1 \leqq r+[(p-r)/5] = [(p+r)/5]$.

Finally, we suppose $t = 1$ and $s > 0$. Each of these s graphs has at least five vertices with two vertices in $\bigcup_1^{m-1} V_i^1$. Since

$$|\bigcup_1^{m-1} V_i^1| < \tfrac{2}{3}|V_m^1|,$$

one of these s graphs has six or more vertices. That is in the decomposition of G into $r+t+s$ non-outerplanar mutually disjoint graphs one graph has more than 5 points. Thus there are $r+t$ copies of K_4, one non-outerplanar graph with at least six vertices and at most $[(p-4r-4t-6)/5]$ other non-outerplanar graphs. Since $t = 1$, this decomposition has at most $r+2+[(p-4r-10)/5] = [(p+r)/5]$ non-outerplanar graphs.

Thus, in this case, $\pi(G) = [(p+r)/5]$ and the theorem is proved.

REFERENCES

L. W. Beineke

[1] Genus, thickness, coarseness, and a crossing number, Proc. 1966 Symp. on Graph Theory, Tihany, Acad. Sci. Hung., 1967.

L. W. Beineke and G. Chartrand

[2] The coarseness of a graph, Comp. Math. 19 (1969), 290–298.

L. W. Beineke and R. K. Guy

[3] The coarseness of the complete bigraph, Canad. J. Math. 21 (1969), 1086–1096.

G. Chartrand, D. Geller and S. Hedetniemi

[4] Graphs with forbidden subgraphs, J. Combinatorial Theory, 10 (1971), 12–41.

G. Chartrand and F. Harary

[5] Planar permutation graphs, Ann. Inst. H. Poincaré (Sect. B), 3 (1967), 433–438.

G. Chartrand, H. V. Kronk and C. E. Wall

[6] The point-arboricity of a graph. Israel J. Math., 6 (1968) 168–175.

K. Corrádi and A. Hajnal

[7] On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hungar. 14 (1963), 423–439.

G. Dirac and P. Erdös

[8] On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hungar. 14 (1963), 79–93.

R. K. Guy
[9] A coarseness conjecture of Erdös, J. Comb. Theory, 3 (1967), 38–42.

F. Harary
[10] Graph Theory, Addison-Wesley, Reading, Mass., 1969, 94–97.

(Oblatum 3-I-1972)

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