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THE POINT-OUTERCOARSENESS OF COMPLETE n -PARTITE GRAPHS

by

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Introduction

A subdivision of a graph G is a graph G_1 obtained from G by replacing an edge $x = uv$ of G with a new vertex w together with edges uw and vw . Graph H is said to be homeomorphic from graph G if H can be obtained from G by a finite sequence of subdivisions. The *subgraph of G induced by a set W of vertices* has vertex set W and its edge set is the set of edges of G which are incident with two vertices of W . The *subgraph of G induced by an edge set Y* has Y as its edge set and contains all vertices incident with at least one edge of Y . For a real number r , $[r]$ denotes the greatest integer not exceeding r , and $\{r\}$ is the least integer not less than r .

Let $p_1 \leq p_2 \leq \dots \leq p_n$ be positive integers. Then the *complete n -partite graph $K(p_1, \dots, p_n)$* has $p = \sum_1^n p_i$ vertices, its vertex set can be partitioned into subsets V_i , $1 \leq i \leq n$, such that $|V_i| = p_i$, and two vertices are adjacent if and only if they are in different V_i . The sets V_1, \dots, V_n are called the partite sets of $K(p_1, \dots, p_n)$. If each $p_i = 1$, $K(p_1, \dots, p_n)$ is denoted by K_n and called the complete graph on n vertices.

An *outerplanar* graph is a graph which can be embedded in the plane so that every vertex of G lies on the exterior region. In [5] Chartrand and Harary have characterized outerplanar graphs as those graphs which contain no subgraph homeomorphic from K_4 or $K(2,3)$.

We define, for each positive integer n , the *vertex partition number* of a graph G , denoted by $\pi_n(G)$, as the maximum number of subsets into which the vertex set of G can be partitioned so that each set induces a graph which contains a subgraph homeomorphic from K_{n+1} or the complete 2-partite graph $K([\frac{n+2}{2}], \{\frac{(n+2)}{2}\})$. This general parameter was first introduced by Chartrand, Geller and Hedetniemi in [4].

For $i = 1, 2, 3, 4$, $\pi_i(g)$ is the maximum number of point induced disjoint subgraphs of G which are totally disconnected, acyclic, outerplanar, and planar, respectively.

The edge partition number $\pi'_n(G)$ is defined analogously to $\pi_n(G)$ with the word 'vertex' replaced by 'edge'. Then $\pi'_1(G)$ is simply the number

of lines of G . The only line partition number which has been given considerable study is $\pi'_4(G)$, which is called the coarseness of G . This has been investigated by Beineke [1], Beineke and Chartrand [2], Guy [9], and Beineke and Guy [3], with the last paper giving a partial formula for $\pi'_4(K(m, n))$.

The number $\pi_1(G)$ is the well-known line independence number, see Harary [10]. The number $\pi_2(G)$ has been studied by Corrádi and Hajnal [7], Dirac and Erdős [8], and Chartrand, Kronk, and Wall [6]. In this paper we investigate $\pi_3(G)$ which is called the point-outercoarseness of G and is now denoted simply $\pi(G)$.

Preliminary results

We make two easy observations and then commence the development of the formula for $\pi(K(p_1, \dots, p_n))$. Any non-outerplanar graph has at least 4 vertices and 6 edges. This implies

REMARK 1. If G is a graph with p points and q edges, then $\pi(G) \leq [p/4]$ and $\pi(G) \leq [q/6]$.

The maximum number of vertices in any complete subgraph of G is denoted $\omega(G)$ and is called the *clique number* of G .

REMARK 2. If G has p vertices and $\omega(G) \leq 3$, then $\pi(G) \leq [p/5]$.

THEOREM 1. Let $G = K(p_1, \dots, p_n)$ with $n \geq 2$. If $p_n \geq (\frac{3}{2})(p - p_n)$, then $\pi(G) = [(p - p_n)/2]$.

PROOF. In any decomposition of G into non-outerplanar subgraphs, each subgraph must include at least two vertices from $\bigcup_1^{n-1} V_i$. There are $p - p_n$ vertices in this set so that $\pi(G) \leq [(p - p_n)/2]$.

Any subgraph induced by a set consisting of three vertices from V_n and two vertices from $\bigcup_1^{n-1} V_i$ is not outerplanar. From the hypothesis that $p - p_n \leq (\frac{3}{2})p_n$ it follows that there are $[(p - p_n)/2]$ disjoint induced non-outerplanar subgraphs of G . Thus $\pi(G) = [(p - p_n)/2]$.

THEOREM 2. If $G = K(p_1, \dots, p_n)$ where $n = 2$ or 3 , and $p_n \leq (\frac{3}{2})(p - p_n)$, then $\pi(G) = [p/5]$.

PROOF. Since $\omega(G) \leq 3$, Remark 2 implies that $\pi(G) \leq [p/5]$. In order to show that $[p/5]$ non-outerplanar, mutually disjoint, induced subgraphs of G exist we consider two cases.

CASE (i). $n = 2$. Since $p_1 \leq p_2 \leq (\frac{3}{2})p_1$, there are $p_2 - p_1$ mutually disjoint sets of vertices such that each set contains three vertices from V_2 and two vertices from V_1 . Each of these sets induces a non-outerplanar

copy of $K(2, 3)$. There are $p_1 - 2(p_2 - p_1) = 3p_1 - 2p_2 \geq 0$ other vertices in V_1 and $p_2 - 3(p_2 - p_1) = 3p_1 - 2p_2 \geq 0$ other vertices in V_2 . Call these sets V'_1 and V'_2 respectively. If $3p_1 - 2p_2 = 0, 1,$ or $2,$ then we have partitioned G into $\lceil p/5 \rceil$ non-outerplanar subgraphs. If $3p_1 - 2p_2 \geq 3,$ then by alternating the use of three vertices from V'_2 and two vertices from V'_1 with two from V'_2 and three from $V'_1,$ we can complete the partition of $V(G)$ into $\lceil p/5 \rceil$ sets of cardinality five, each of which induces a non-outerplanar graph. Thus $\pi(G) = \lceil p/5 \rceil$ in this case.

CASE (ii). $n = 3.$ If $p_1 + p_2 \leq p_3$ we consider graph H which is G minus all edges joining V_1 to $V_2.$ From case (i). $\pi(G) \geq \pi(H) \geq \lceil p/5 \rceil.$ Thus we suppose $p_3 < p_1 + p_2.$ For $i = 1, 2, 3,$ let $V_i^0 = V_i.$ Form one copy of $K(2, 3)$ with three vertices $v_1, v_2, v_3,$ of V_3^0 and two vertices v_4, v_5 of $V_2^0.$ Let V_1^1, V_2^1 be an ordering of V_1 and $V_2 - \{v_4, v_5\}$ so that $|V_1^1| \leq |V_2^1|,$ and let $V_3^1 = V_3 - \{v_1, v_2, v_3\}.$ Then repeat this procedure with $V_1^1, V_2^1,$ and $V_3^1,$ and continue this procedure until reaching a non-negative integer j such that $V_3^j \leq V_2^j.$ (Note that j may be zero.) Let

$$V_1^{j+1}, V_2^{j+1}, V_3^{j+1}$$

be a reordering of V_1^j, V_2^j, V_3^j such that

$$|V_1^{j+1}| \leq |V_2^{j+1}| \leq |V_3^{j+1}|$$

and observe that

$$0 \leq |V_3^{j+1}| - |V_2^{j+1}| \leq 2.$$

Continue the partition of G into copies of $K(2, 3)$ by using three vertices $w_1, w_2, w_3,$ from V_3^{j+1} and two vertices w_4, w_5 from $V_2^{j+1}.$ Let

$$V_1^{j+1}, V_2^{j+1} - \{w_4, w_5\}, V_3^{j+1} - \{w_1, w_2, w_3\}$$

be reordered by

$$V_1^{j+2}, V_2^{j+2}, V_3^{j+2}$$

so that

$$|V_1^{j+2}| \leq |V_2^{j+2}| \leq |V_3^{j+2}|.$$

We stop this procedure when $|V_3^k| \leq 3$ for some $k \geq j + 1.$ If

$$|V_1^k| + |V_2^k| + |V_3^k| \leq 4,$$

then G has been partitioned into $\lceil p/5 \rceil$ non-outerplanar graphs. Otherwise induce one more non-outerplanar graph with the remaining vertices. Thus $\pi(G) \geq \lceil p/5 \rceil,$ which completes the proof of the theorem.

COROLLARY 3. *If $G = K(p_1, \dots, p_n)$ where $n = 2$ or 3 then $\pi(G) = \min \{ \lceil p/5 \rceil, \lfloor (p - p_n)/2 \rfloor \}.$*

THEOREM 4. *Let $G = K(p_1, \dots, p_n)$ where $p_n \leq (\frac{3}{2})(p - p_n).$ Then $\pi(G) \geq \lceil p/5 \rceil.$*

PROOF. We use induction and observe that Theorem 2 verifies the result for $n = 2$ or 3 . Assume Theorem 4 holds for $n \geq 3$ and let $G = K(p_1, \dots, p_{n+1})$ where $p_{n+1} \leq (\frac{3}{2})(p - p_{n+1})$. The subgraph of G formed by removing all edges joining V_1 with V_2 is a complete n -partite graph $H = K(p'_1, \dots, p'_n)$ where $p'_n = \max\{p_{n+1}, p_1 + p_2\}$. Since $p'_n \leq (3/2)(p'_1 + \dots + p'_{n-1})$, the inductive assumption applies and we have $\pi(G) \geq \pi(H) \geq \lfloor p/5 \rfloor$.

The following lemma will be helpful.

LEMMA 1. *Let c be an integer such that $1 < c \leq n$. If $p_n - p_{n-c+1} \leq 1$, then the complete n -partite graph $G = K(p_1, \dots, p_n)$ contains $\lfloor p/c \rfloor$ mutually disjoint copies of K_c .*

PROOF. We use induction on p . If the order of G is less than $n + c$, then $p_{n-c+1} = 1$. We form one copy of K_c by selecting one vertex from each V_i , $i = n - c + 1, \dots, n$. The remaining vertices of G induce a complete graph on $p - c$ vertices. Thus G contains $\lfloor p/c \rfloor$ mutually disjoint copies of K_c .

Let the order of G be $p \geq n + c$ and suppose the lemma is true for all complete n -partite graphs with less than p vertices. Form one copy of K_c by selecting one vertex from each of V_{n-c+1}, \dots, V_n . The graph H induced by the remaining vertices of G is a complete n -partite graph with $p'_n - p'_{n-c+1} \leq 1$ where p'_i is the order of the i th partite set of H . By the induction hypothesis H contains $\lfloor (p - c)/c \rfloor$ mutually disjoint copies of K_c and the lemma is proved.

THEOREM 5. *Let $G = K(p_1, \dots, p_n)$ where $n \geq 4$. If $p \geq 4p_n$, then $\pi(G) = \lfloor p/4 \rfloor$.*

PROOF. We use induction on p_n . If $p_n = 1$, G is the complete graph with $p = n$ vertices and $\pi(G) = \lfloor p/4 \rfloor$. Suppose the theorem holds if $p_n = k \geq 1$ and let $p_n = k + 1$. Remove one vertex from each V_n, V_{n-1}, V_{n-2} , and V_{n-3} . The resulting graph H is a complete m -partite graph with $n \geq m \geq 4$ and the largest partite set in H has $p_{n-1} = k$ or p_n vertices. The latter case implies that $p_n - p_{n-3} = 0$, and Lemma 1 proves the theorem. In the former case the inductive assumption implies $\pi(H) = \lfloor (p - 4)/4 \rfloor$ and thus $\pi(G) = \lfloor p/4 \rfloor$.

The principal result

Before stating the main theorem, we prove another lemma.

LEMMA 2. *Let $G = K(p_1, \dots, p_n)$ with $n \geq 3$. If r is a positive integer such that $p \geq 3r$, $p_1 + \dots + p_{n-1} \geq 2r$, and $p_1 + \dots + p_{n-2} \geq r$, then G contains at least r mutually disjoint triangles.*

PROOF. For $i = 1, \dots, n$, let $V_i^0 = V_i$. Form one triangle with vertices

$$v_{n-2}, v_{n-1}, v_n \text{ of } V_{n-2}^0, V_{n-1}^0, V_n^0$$

respectively. Let

$$V_n^1 = V_n^0 - \{v_n\} \text{ and } V_1^1, \dots, V_{n-1}^1$$

be a reordering of

$$V_1^0, \dots, V_{n-3}^0, V_{n-2}^0 - \{v_{n-2}\}, V_{n-1}^0 - \{v_{n-1}\}$$

such that

$$|V_i^1| \leq |V_{i+1}^1| \text{ for } i = 1, 2, \dots, n-2.$$

Repeat this procedure until either

$$|V_n^k| - |V_{n-2}^k| \leq 1 \text{ and } |V_{n-2}^k| \neq 0$$

for some k or $|V_{n-2}^k| = 0$ for some k . If the former occurs first, then from Lemma 1, it follows that G contains at least r mutually disjoint triangles. Thus suppose $|V_{n-2}^k| = 0$ for some j and consider two cases.

CASE (i) $|V_{n-1}^i| - |V_{n-2}^i| \leq 1$ for some $i < k$. Each of the k triangles which have been formed contain one vertex of V_n and two vertices from distinct $V_j, j = 1, \dots, n-1$. Since $|V_{n-1}^i| - |V_{n-2}^i| \leq 1$, Lemma 1 implies that at most one vertex of $\bigcup_1^{n-1} V_j$ is not included in one of the triangles. Thus $k = [(p_1 + \dots + p_{n-1})/2] \geq r$.

CASE (ii). $|V_{n-1}^i| - |V_{n-2}^i| > 1$ for all $i > k$. In this case

$$V_{n-1}^i \subset V_{n-1} \text{ for } i = 1, \dots, k-1.$$

Hence each of the k triangles contains exactly one vertex from $\bigcup_1^{n-2} V_j$. This implies

$$k = \left| \bigcup_1^{n-2} V_j \right| \geq r$$

and completes the proof.

THEOREM 6. Let $G = K(p_1, \dots, p_n)$ with $n \geq 2$, then

$$\pi(G) = \begin{cases} [(p-p_n)/2] & \text{if } p \leq (\frac{5}{3})p_n \\ [p/4] & \text{if } p \geq 4p_n \\ [(p+r)/5] & \text{if } (\frac{5}{3})p_n < p < 4p_n \end{cases}$$

where

$$r = \min \{ (p-p_n-p_{n-1}-p_{n-2}), [(p-p_n-p_{n-1})/2], [(3p-5p_n)/7] \}.$$

PROOF. If $p \leq (\frac{5}{3})p_n$ or $p \geq 4p_n$, the result follows from Theorems 1 and 5. Thus we consider only $(\frac{5}{3})p_n < p < 4p_n$ and distinguish three cases depending on r .

CASE (i). $r = p - p_n - p_{n-1} - p_{n-2}$. Since

$$p - p_n - p_{n-1} - p_{n-2} \leq (p - p_n - p_{n-1})/2$$

we have

$$p - p_n - p_{n-1} - p_{n-2} \leq p_{n-2} \leq p_{n-1} \leq p_n.$$

That is the cardinality of $\bigcup_1^{n-3} V_i$ does not exceed the cardinality of V_{n-2} . Thus there are r mutually disjoint copies of K_4 with one vertex in each of the sets

$$V_n, V_{n-1}, V_{n-2}, \bigcup_1^{n-3} V_i.$$

Let G minus these r copies of K_4 be denoted by H . Graph H has $p - 4r$ vertices, and we let $V_i^1 = V_i \cap V(H)$ for $i = n-2, n-1, n$. Since $r \leq (3p - 5p_n)/7$ we have $\frac{2}{3}(p_n - r) \leq p - p_n - 3r$, where $p_n - r = |V_n'|$ and

$$p - p_n - 3r = |V_{n-1}' \cup V_{n-2}'|.$$

Theorem 2 implies that $\pi(H) = [(p - 4r)/5]$. Hence

$$\pi(G) \geq r + [(p - 4r)/5] = [(p + r)/5].$$

Since G does not contain more than r copies of K_4 , it is clear that $\pi(G) = [(p + r)/5]$.

CASE (ii). $r = [(p - p_n - p_{n-1})/2] < [(3p - 5p_n)/7]$. In this case we consider the complete $(n-1)$ -partite graph $H = G - V_n$. By hypothesis

$$(1) \quad r \leq p - p_n - p_{n-1} - p_{n-2}$$

and

$$(2) \quad 2r \leq p - p_n - p_{n-1}$$

Inequality (2) together with $[(p - p_n - p_{n-1})/2] < [(3p - 5p_n)/7]$ imply

$$(3) \quad r \leq p_{n-1}.$$

Adding (2) and (3) we obtain

$$(4) \quad 3r \leq \sum_1^{n-1} p_i.$$

Since (1), (2), and (4) hold, Lemma 2 implies that H contains at least r mutually disjoint triangles. The set V_n contains $p_n \geq p_{n-1} \geq r$ vertices. Thus, G contains r mutually disjoint copies of K_4 , each of which has one vertex from V_n and three vertices from $\bigcup_1^{n-1} V_i$. There are $p - p_n - 3r$ other vertices in $\bigcup_1^{n-1} V_i$ and $p_n - r$ other vertices in V_n .

The graph G' induced by the remaining vertices of G is a complete m -partite graph, $m \leq n$. Since $r < (3p - 5p_n)/7$, we have

$$(5) \quad p - p_n - 3r > \frac{2}{3}(p_n - r).$$

That is the number of vertices in $V(G') - V_n$ is more than two-thirds the number of vertices in $V(G') \cap V_n$. From $r = [(p - p_n - p_{n-1})/2]$ it follows that

$$(6) \quad p_n - r + 1 \geq p - p_n - 3r.$$

If a maximum partite set of G' is $V(G') \cap V_n$, then (5) together with Theorem 4 imply that $\pi(G') \geq [(p - 4r)/5]$, and thus $\pi(G) \geq r + \pi(G') \geq [(p + r)/5]$. From (6) and the fact that $|V(G') - V_n| = p - p_n - 3r$ it follows that if $V(G') \cap V_n$ is not a largest partite set of G' , then a largest partite set contains exactly $p_n - r + 1 = p - p_n - 3r$. Thus G' is a bipartite graph with partite sets V'_1 and V'_2 where $|V'_2| = p - p_n - 3r$ and $|V'_1| = p_n - r$. According to Theorem 4, $\pi(G') \geq [(p - 4r)/5]$ and $\pi(G) \geq [(p + r)/5]$.

In order to show that equality holds suppose $\pi(G) > [(p + r)/5]$. Then there are more than r mutually disjoint copies of K_4 in G . Each copy of K_4 must contain two vertices from $\bigcup_1^{n-2} V_i$, so that $p - p_n - p_{n-1} \geq 2(r + 1)$. This implies that $[(p - p_n - p_{n-1})/2] > r$ which contradicts the hypothesis for this case. Hence $\pi(G) = [(p + r)/5]$.

CASE (iii). $r = [(3p - 5p_n)/7]$. In this case we let $H = G - V_n$. From the hypothesis for this case we have

$$(7) \quad r \leq p - p_n - p_{n-1} - p_{n-2} \quad \text{and}$$

$$(8) \quad 2r \leq p - p_n - p_{n-1}$$

Furthermore, $p - p_n - 3r \geq p - p_n - 3((3p - 5p_n)/7) = (\frac{8}{7})p_n - (\frac{2}{7})p > 0$. Thus $p - p_n > 3r$ which together with (7), (8) and Lemma 2 imply that H contains r mutually disjoint triangles.

Since $4p_n > p$, we have that $3p_n > p - p_n > 3r$. Thus V_n contains more than r vertices. Graph G has at least r mutually disjoint copies of K_4 each consisting of one vertex of V_n and three vertices of $\bigcup_1^{n-1} V_i$. There are $p - 4r$ other vertices in G . These vertices induce a complete m -partite subgraph G' of G with precisely $p_n - r$ vertices of V_n and $p - p_n - 3r$ vertices of $\bigcup_1^{n-1} V_i$. Since $r \leq (3p - 5p_n)/7$ we have

$$(9) \quad (\frac{3}{2})(p - p_n - 3r) \geq p_n - r > 0.$$

Let W be a maximum partite set of G' . If $W = V_n \cap V(G')$, then (9) together with Theorem 4 imply that $\pi(G') \geq [(p - 4r)/5]$, and $\pi(G) \geq [(p + r)/5]$.

Suppose $W \neq V_n \cap V(G')$; then let $k = 4p_n - p$. Thus,

$$r = [(3p - 5p_n)/7] = [p_n - 3k/7] = p_n - \{3k/7\}$$

where $\{x\}$ is the least integer not less than x . We have

$$(10) \quad p_n - r = \{3k/7\} \text{ and}$$

$$(11) \quad p - p_n - 3r = 3p_n - k - 3(p_n - \{3k/7\}) = -k + 3\{3k/7\}.$$

If $k = 1$, then the number of vertices in G' is $p - 4r = p - 4p_n + 4\{3k/7\} = -1 + 4 = 3$, and $\pi(G) \geq r + [(p - 4r)/5] = [(p + r)/5]$. If $k \geq 2$, then from (10) and (11) we have $p_n - r \geq (\frac{2}{3})(p - p_n - 3r)$. This implies that

$$|V(G') - W| \geq |V_n| - r = p_n - r \geq (\frac{2}{3})(p - p_n - 3r) \geq (\frac{2}{3})|W|.$$

Hence, according to Theorem 4,

$$\pi(G') \geq [(p - 4r)/5] \text{ and } \pi(G) \geq r + [(p - 4r)/5] = [(p + r)/5].$$

Suppose $\pi(G) > [(p + r)/5]$. Any decomposition of G into more than $[(p + r)/5]$ non-outerplanar graphs will necessarily contain $r + t$ mutually disjoint copies of K_4 , $t > 0$. Let $V_1^1, V_2^1, \dots, V_m^1$ be the partite sets of the complete m -partite graph H^1 which remains after deleting these $r + t$ copies of K_4 from G . The order of H^1 is $p - 4r - 4t$ and $|V_m^1| \geq |V_c^1|$ where $V_c^1 = V_n \cap V(H^1)$. We have $r + t \leq p/4 < p_n$, and thus

$$(12) \quad |V_m^1| \geq |V_c^1| \geq p_n - r - t > 0.$$

Then

$$(13) \quad |\bigcup_1^{m-1} V_i^1| \leq |V(H^1)| - (p_n - r - t) = p - p_n - 3r - 3t.$$

From the fact that $r > (3p - 5p_n)/7 - t$ we obtain

$$(14) \quad p - p_n - 3r - 3t < (\frac{2}{3})(p_n - r - t).$$

Using (12), (13), and (14) we have

$$|\bigcup_1^{m-1} V_i^1| \leq p - p_n - 3r - 3t < (\frac{2}{3})(p_n - r - t) \leq (\frac{2}{3})|V_m^1|.$$

According to Theorem 1,

$$\pi(H^1) = [|\bigcup_1^{m-1} V_i^1|/2] = s.$$

Suppose $t \geq 2$. Since $s \leq [(p - p_n - 3r - 3t)/2]$, the number of mutually disjoint non-outerplanar subgraphs in this decomposition does not exceed $r + t + [(p - p_n - 3r - 3t)/2] \leq [p - p_n - r - t]/2$.

However, $r + 2 > (3p - 5p_n)/7 + 1$, which implies

$$(15) \quad \left[\frac{p - p_n - (r + 2)}{2} \right] \leq \left[\frac{p - p_n - (3p - 5p_n + 7)/7}{2} \right] = \left[\frac{2p - p_n}{7} - \frac{1}{2} \right].$$

Also

$$(16) \quad \left\lceil \frac{p+r}{5} \right\rceil \geq \left\lceil \frac{(10p-5p_n-6)/7}{5} \right\rceil = \left\lceil \frac{2p-p_n}{7} - \frac{6}{35} \right\rceil.$$

Since the right side of (15) is not more than the right side of (16), we have $r+t+s \leq [(p-p_n-r-2)/2] \leq [(p+r)/5]$. That is this decomposition yields at most $[(p+r)/5]$ mutually disjoint non-outerplanar subgraphs of G .

If $t = 1$ and $s = 0$, then this decomposition yields $r+1$ mutually disjoint non-outerplanar graphs and since $|V_m^1| > 0$ there is at least one vertex which is not included in any of the $r+1$ copies of K_4 . Thus $r+1 \leq r + [(p-r)/5] = [(p+r)/5]$.

Finally, we suppose $t = 1$ and $s > 0$. Each of these s graphs has at least five vertices with two vertices in $\bigcup_1^{m-1} V_i^1$. Since

$$\left| \bigcup_1^{m-1} V_i^1 \right| < \frac{2}{3} |V_m^1|,$$

one of these s graphs has six or more vertices. That is in the decomposition of G into $r+t+s$ non-outerplanar mutually disjoint graphs one graph has more than 5 points. Thus there are $r+t$ copies of K_4 , one non-outerplanar graph with at least six vertices and at most $[(p-4r-4t-6)/5]$ other non-outerplanar graphs. Since $t = 1$, this decomposition has at most $r+2 + [(p-4r-10)/5] = [(p+r)/5]$ non-outerplanar graphs.

Thus, in this case, $\pi(G) = [(p+r)/5]$ and the theorem is proved.

REFERENCES

L. W. BEINEKE

- [1] Genus, thickness, coarseness, and a crossing number, Proc. 1966 Symp. on Graph Theory, Tihany, Acad. Sci. Hung., 1967.

L. W. BEINEKE and G. CHARTRAND

- [2] The coarseness of a graph, Comp. Math. 19 (1969), 290–298.

L. W. BEINEKE and R. K. GUY

- [3] The coarseness of the complete bigraph, Canad. J. Math. 21 (1969), 1086–1096.

G. CHARTRAND, D. GELLER and S. HEDETNIEMI

- [4] Graphs with forbidden subgraphs, J. Combinatorial Theory, 10 (1971), 12–41.

G. CHARTRAND and F. HARARY

- [5] Planar permutation graphs, Ann. Inst. H. Poincaré (Sect. B), 3 (1967), 433–438.

G. CHARTRAND, H. V. KRONK and C. E. WALL

- [6] The point-arboricity of a graph, Israel J. Math., 6 (1968) 168–175.

K. CORRÁDI and A. HAJNAL

- [7] On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hungar. 14 (1963), 423–439.

G. DIRAC and P. ERDÖS

- [8] On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hungar. 14 (1963), 79–93.

R. K. GUY

[9] A coarseness conjecture of Erdős, *J. Comb. Theory*, 3 (1967), 38–42.

F. HARARY

[10] *Graph Theory*, Addison-Wesley, Reading, Mass., 1969, 94–97.

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