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AN IMPROVEMENT OF A RESULT OF I. N. STEWART

by

Jan de Ruiter

1. Introduction

In this paper we shall present a sharpening of a theorem of I. N. Stewart [1] which states that if $L$ is a Lie algebra over a field of arbitrary characteristic such that every subalgebra of $L$ is an $n$-step subideal of $L$, then $L$ is nilpotent of order $\leq \mu(n)$ for some function $\mu : \mathbb{N} \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. The dimension of $L$ may be finite or infinite.

It turns out to be possible to shorten Stewart's argument considerably by replacing lemma 3.3.14 [1] by a stronger one. Then some definitions and theorems can be omitted and we obtain a better bound for the order of nilpotency of $L$. Nevertheless our bound also takes astronomical values for $n = 3, 4, \cdots$ but in the case of $n = 2$ the value of the bound is 7, an improvement of which will be given in the last section.

In order to prove our result we shall first give an exposition of the requisite basic concepts in a preceding chapter. We remark that notation and terminology in the domain of infinite-dimensional Lie algebras are non-standard in part and sometimes we shall use other symbols than Stewart does.

2. Preliminary definitions

Let $L$ be a Lie algebra (possibly of infinite dimension) over a field $k$ of arbitrary characteristic. If $x, y \in L$ we write $[x, y]$ for the Lie product of $x$ and $y$.

If $H, K$ are subspaces of $L$ we write $H + K$ for the subspace of $L$ consisting of all sums $h + k$, $h \in H$, $k \in K$ and $[H, K]$ for the subspace of $L$ consisting of all finite sums $\sum [h_i, k_i]$, $h_i \in H$, $k_i \in K$. A useful notation is $[H, \underbrace{K, \cdots, K}]$ instead of $[\cdots[[H, K], K], \cdots, K]$.

A subspace $S$ of $L$ is a subalgebra of $L$ if $[S, S] \subseteq S$ and a subspace $I$ of $L$ is an ideal of $L$ if $[I, L] \subseteq I$. We write respectively $S \leq L$ and $I \lhd L$. 

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$H$ is called an $n$-step subideal of $L$ if we have

$$H = H_n \Leftarrow H_{n-1} \Leftarrow \cdots \Leftarrow H_0 = L.$$ 

In this case we write $H \Leftarrow nL$.

Suppose $A, B \subseteq L$. $\langle A \rangle$ is the smallest subalgebra of $L$ containing $A$ and we say $A$ generates $\langle A \rangle \cdot \langle A^B \rangle$ is the smallest subalgebra of $L$ which contains $A$ and which is invariant under multiplication by elements of $B$.

$L$ is finitely generated if $L = \langle X \rangle$ for some finite subset $X$ of $L$.

The lower central series $L_1, L_2, \cdots$ of $L$ is inductively defined by

$L_1 = L$, $L_1^{n+1} = [L^n, L]$. We say $L$ is nilpotent if $L^{n+1} = 0$ for some $n$. The least such $n$ is the order of nilpotency of $L$.

A concept which is weaker than the preceding one now follows. $L$ is locally nilpotent if every finitely generated subalgebra of $L$ is nilpotent.

The derived series $L^{(0)}, L^{(1)}, \cdots$ of $L$ is inductively defined in a similar way by $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$ and $L$ is solvable if $L^{(n)} = 0$ for some $n$. In this case the least such $n$ is called the derived length of $L$. It is a well-known fact (see [2] theorem 0.1) that $L^{(n)} \Leftarrow L^{2n}, n = 0, 1, \cdots$ and consequently

$L^{(n)} \Leftarrow L^n, n = 1, 2, \cdots$

where $n$ is the least integer $\geq \log_2 n$.

The upper central series $C(L) = C_1(L), C_2(L), \cdots$ of $L$ is defined by

$C_n(L) = \{x \in L : [x, L] = 0\}$.

All $C_n(L)$ are characteristic ideals of $L$.

Finally we introduce a number of classes of Lie algebras which we shall need in the following.

The classes of Lie algebras we consider are

$FD_r = \text{the class of Lie algebras of dimension } \leq r$

$FG_s = \text{the class of Lie algebras generated by } \leq s \text{ elements}$

$NIL_c = \text{the class of nilpotent Lie algebras of order } \leq c$

$SOL_d = \text{the class of solvable Lie algebras of derived length } \leq d$

$SI_d : L \in SI_d \text{ iff } H \leq L \Rightarrow H \Leftarrow nL$

$NC_m : L \in NC_m \text{ iff } H \leq L \Rightarrow \langle H^L \rangle^m \leq H.$

### 3. A sharpening of Stewart's theorem

In this chapter we shall derive a result which yields a better bound for the order of nilpotency of a Lie algebra, all of whose subalgebras are $n$-step subideals. The fundamental step in our argument is theorem 1. All results of Stewart used by us will be called lemmas and proofs are to be found in [1].
LEMMA 1. If $H \triangleleft L$, $H \in FD_n$ and $L$ is locally nilpotent, then $H \triangleleft C_n(L)$.

LEMMA 2. If $H \triangleleft L$, $H \in NIL_c$ and $L/H^2 \in NIL_d$, then $L \in NIL_{M_1(c,d)}$ where $M_1(c,d) = cd + (c-1)(d-1)$.

We note that this bound is best possible. See [1] p. 318.

LEMMA 3. If $L \subseteq FG_r \cap NIL_u$ then $L \subseteq FD_{M_2(r,s)}$ where $M_2(r,s) = r + r^2 + \cdots + r^s$.

We now state and prove an important theorem.

THEOREM 1. $NC_n \subseteq NIL_{M_3(n)}$ where $M_3(1) = 1$ and $M_3(n) = n - 1 + M_2(n, n^2 - n)$ for $n = 2, 3, \cdots$.

PROOF. Let $L \in NC_1$. If $H \triangleleft L$ then $H = L$ and therefore $L \in SI_1$. The converse is also true and consequently $NC_1 = SI_1$. We show that $L$ is Abelian if $L \in SI_1$. Suppose $x, y \in L$ then $kx, ky \in L$ since $L \in SI_1$. If $x$ and $y$ are linearly independent then $[x, y] \subseteq kx \cap ky = 0$. If $x$ and $y$ are linearly dependent then $[x, y] = 0$ by the definition of the Lie product.

Let now $L \in NC_n$ where $n > 1$.

We assert $x \in L \Rightarrow \langle x^L \rangle \in NIL_{n-1}$.

The proof is as follows. If $x \in L$ then $kx \leq L$ and consequently $\langle x^L \rangle \subseteq L$. Now suppose $\langle x^L \rangle \neq 0$, then we have $kx = \langle x^L \rangle^r \subseteq L$ and therefore $kx = \langle x^L \rangle = \langle x^L \rangle^r$, but this is impossible. Hence we conclude $\langle x^L \rangle = 0$.

Let $x_1, \cdots, x_n \in L$, then $X = \langle x_1, \cdots, x_n \rangle \subseteq \langle x_1^L \rangle + \cdots + \langle x_n^L \rangle \in NIL_{n^2 - n}$ since $\langle x_i^L \rangle \in NIL_{n-1}$. By applying lemma 3 we obtain $X \in FD_r$ where $r = M_2(n, n^2 - n)$.

Now we have $\langle [x_1, \cdots, x_n]^L \rangle \subseteq \langle X^L \rangle \subseteq X$ since $L \in NC_n$ and therefore $\langle [x_1, \cdots, x_n]^L \rangle \subseteq FD_r$.

Since $X$ is an arbitrary finitely generated subalgebra of $L$ we have also proved $L$ is locally nilpotent. Hence by lemma 1 $\langle [x_1, \cdots, x_n]^L \rangle \triangleleft C_r(L)$ and consequently $L^n = \Sigma_{x_i \in L} \langle [x_1, \cdots, x_n]^L \rangle \triangleleft C_r(L)$; thus $L^{n+r} = [L^n, L] = 0$ and this concludes the proof of the theorem.

LEMMA 4. $SOL_2 \cap SI_n \subseteq NC_n$.

THEOREM 2. $SOL_k \cap SI_n \subseteq NIL_{M_4(k,n)}$ where $M_4(1, n) = 1$ and $M_4(k+1, n) = M_1(M_4(k, n), M_3(n))$.

PROOF. This theorem is the same as lemma 3.3.10 of Stewart [1] p. 320, but our bound is something better because in our proof we can refer to theorem 1. For the sake of completeness the proof now follows.

We use induction on $k$.

$k = 1$: $SOL_1 \cap SI_n \subseteq NIL_1$

$k = 2$: $SOL_2 \cap SI_n \subseteq NC_n \subseteq NIL_{M_3(n)}$ by lemma 4 and theorem 1
2 \leq k \Rightarrow k + 1: \text{ If } L \in SOL_{k+1} \cap SI_n \text{ then } H = L^{(k-1)} \in SOL_2 \cap SI_n \text{ and therefore } H \in NIL_{M_4(n)}^1. \text{ But } L_{/H^2} \in SOL_k \cap SI_n \text{ and consequently by induction } L_{/H^2} \in NIL_{M_4(k,n)}. \text{ Finally we apply lemma 2.}

Suppose \( H \leq L \).

The series \( L = H_0 \triangleright H_1 \triangleright \cdots \), inductively defined by \( H_0 = L \), \( H_{i+1} = \langle H^{H_i} \rangle \), is called the \textit{ideal closure series} of \( H \).

\textbf{Lemma 5.} \( H \leq ^*L \) \iff \( H = H_n \).

\textbf{Lemma 6.} \( H \leq L \in SI_n \Rightarrow H_i \triangleright H_{i+1} \in SI_{n-1} \) for \( i = 0, \cdots, n-1 \).

This lemma is of the first importance for the proof of the main theorem.

\textbf{Theorem 3.} \( SI_n \subset NIL_{M_4(n)} \) where \( M(1) = 1 \) and \( M(n+1) = M_3(M_4(nM(n)+1, n+1)+1) \).

\textbf{Proof.} By induction on \( n \).

\( n = 1: SI_1 = NC_1 \subset NIL_1 \)

\( n \Rightarrow n+1: \text{ Let } L \in SI_{n+1}. \text{ If } H \leq L \text{ then because of lemma 6 and by induction } H_{i+1} \in SI_{n+1-i} \subset SI_n \subset NIL_{M_4(n)} \text{ for } i = 1, \cdots, n. \text{ Therefore } H_{i+1} \in SOL_{M_4(n+1)}. \text{ By lemma 5 } H = H_n \leq H_{n-1} \leq \cdots \leq H_0 = L \text{ where } (H_i) \text{ is the ideal closure series of } H \text{ and it follows now easily that } H_1^{(r)} \leq H \text{ where } r = nM(n)+1. \text{ Moreover we have } H_1^{(r)} \in SOL_r \cap SI_{n+1} \text{ since } SI_{n+1} \text{ is closed under taking subalgebras and quotient algebras. By applying theorem 2 we now obtain } H_1^{(r)} \in NIL_s \text{ where } s = M_4(r, n+1) \text{ and therefore } H_1^{(r+1)} \leq H_1^{(r)} \leq H. \text{ Thus } \langle H^L \rangle^{s+1} \leq H. \text{ Hence } L \in NC_{s+1}. \text{ We finish the proof by applying theorem 1.}

4. The class \( SI_2 \)

\( M(2) = M_3(M_4(1, 2)+1) = M_3(2) = 1+M_2(2, 2) = 1+2+2^2 = 7, \) but this bound can be improved still further.

For the following result I am indebted to my referee.

\textbf{Proposition.} If the characteristic of the field \( k \) is not 3, then \( SI_2 = NIL_2 \) and if the characteristic is 3, then \( NIL_2 \subset SI_2 \subset NIL_3 \).

\textbf{Proof.} Let \( x \in L \in SI_2, \) then \( kx \ll K \ll L \) for some \( K. \) Therefore \( kx \) is a minimal ideal of \( H = \langle x^L \rangle, \) which we know already is nilpotent (theorem 3); so by Lemma 1 \( x \in C_1(H) \) which is a characteristic ideal in \( H \) and hence \( \ll L. \) Therefore \( H = C_1(H) \) is Abelian.
If now $y \in L$ it follows that $[[x, y], x] = 0$, so $L$ has Engel 2-condition. By a result of Higgins [3] we now conclude that $L \in NIL_2$, if char. $\neq 3$ and $L \in NIL_3$ if char. $= 3$.

REFERENCES

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