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AN IMPROVEMENT OF A RESULT OF I. N. STEWART

by

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1. Introduction

In this paper we shall present a sharpening of a theorem of I. N. Stewart [1] which states that if L is a Lie algebra over a field of arbitrary characteristic such that every subalgebra of L is an n-step subideal of L, then L is nilpotent of order $\leqq \mu(n)$ for some function $\mu : N \to N$, where N is the set of all positive integers. The dimension of L may be finite or infinite.

It turnes out to be possible to shorten Stewart's argument considerably by replacing lemma 3.3.14 [1] by a stronger one. Then some definitions and theorems can be omitted and we obtain a better bound for the order of nilpotency of L. Nevertheless our bound also takes astronomical values for $n = 3, 4, \cdots$ but in the case of $n = 2$ the value of the bound is 7, an improvement of which will be given in the last section.

In order to prove our result we shall first give an exposition of the requisite basic concepts in a preceding chapter. We remark that notation and terminology in the domain of infinite-dimensional Lie algebras are non-standard in part and sometimes we shall use other symbols than Stewart does.

2. Preliminary definitions

Let L be a *Lie algebra* (possibly of infinite dimension) over a *field* $\mathbf{k}$ of arbitrary characteristic. If $x, y \in L$ we write $[x, y]$ for the *Lie product* of x and y.

If H, K are subspaces of L we write $H+K$ for the subspace of L consisting of all sums $h+k$, $h \in H$, $k \in K$ and $[H, K]$ for the subspace of L consisting of all finite sums $\Sigma[h_i, k_i]$, $h_i \in H$, $k_i \in K$. A useful notation is $[H, {}_iK]$ instead of

$$[\cdots[[H, \underbrace{K], K], \cdots, K}_{i \text{ times } K}].$$

A subspace S of L is a *subalgebra* of L if $[S, S] \subset S$ and a subspace I of L is an *ideal* of L if $[I, L] \subset I$. We write respectively $S \leqq L$ and $I \lhd L$.

H is called an *n-step subideal* of L if we have

$$H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_0 = L.$$

In this case we write $H \lhd {}^n L$.

Suppose $A, B \subset L$. $\langle A \rangle$ is the smallest subalgebra of L containing A and we say A *generates* $\langle A \rangle \cdot \langle A^B \rangle$ is the smallest subalgebra of L which contains A and which is invariant under multiplication by elements of B. L is *finitely generated* if $L = \langle X \rangle$ for some finite subset X of L.

The *lower central series* $L^1, L^2, \cdots$ of L is inductively defined by $L^1 = L$, $L^{n+1} = [L^n, L]$. We say L is *nilpotent* if $L^{n+1} = 0$ for some n. The least such n is the *order of nilpotency* of L.

A concept which is weaker than the preceding one now follows. L is *locally nilpotent* if every finitely generated subalgebra of L is nilpotent.

The *derived series* $L^{(0)}, L^{(1)}, \cdots$ of L is inductively defined in a similar way by $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$ and L is *solvable* if $L^{(n)} = 0$ for some n. In this case the least such n is called the *derived length* of L. It is a well-known fact (see [2] theorem 0.1) that $L^{(n)} \lhd L^{2^n}$, $n = 0, 1, \cdots$ and consequently

$$L^{(\bar{n})} \lhd L^n, \quad n = 1, 2, \cdots$$

where $\bar{n}$ is the least integer $\geqq \log_2 n$.

The *upper central series* $C(L) = C_1(L), C_2(L), \cdots$ of L is defined by

$$C_n(L) = \{x \in L : [x, {}_nL] = 0\}.$$

All $C_n(L)$ are characteristic ideals of L.

Finally we introduce a number of *classes* of Lie algebras which we shall need in the following.

The classes of Lie algebras we consider are

FD_r = the class of Lie algebras of dimension $\leqq r$
FG_s = the class of Lie algebras generated by $\leqq s$ elements
NIL_c = the class of nilpotent Lie algebras of order $\leqq c$
SOL_d = the class of solvable Lie algebras of derived length $\leqq d$
$SI_n : L \in SI_n$ iff $H \leqq L \Rightarrow H \lhd {}^n L$
$NC_m : L \in NC_m$ iff $H \leqq L \Rightarrow \langle H^L \rangle^m \leqq H$.

3. A sharpening of Stewart's theorem

In this chapter we shall derive a result which yields a better bound for the order of nilpotency of a Lie algebra, all of whose subalgebras are n-step subideals. The fundamental step in our argument is theorem 1. All results of Stewart used by us will be called lemmas and proofs are to be found in [1].

LEMMA 1. *If* $H \lhd L$, $H \in FD_n$ *and* L *is locally nilpotent, then* $H \lhd C_n(L)$.

LEMMA 2. *If* $H \lhd L$, $H \in NIL_c$ *and* $L_{/H^2} \in NIL_d$, *then* $L \in NIL_{M_1(c,d)}$ *where* $M_1(c, d) = cd+(c-1)(d-1)$.

We note that this bound is best possible. See [1] p. 318.

LEMMA 3. *If* $L \in FG_r \cap NIL_s$ *then* $L \in FD_{M_2(r,s)}$ *where* $M_2(r, s) = r+r^2+\cdots+r^s$.

We now state and prove an important theorem.

THEOREM 1. $NC_n \subset NIL_{M_3(n)}$ *where* $M_3(1) = 1$ *and* $M_3(n) = n-1+M_2(n, n^2-n)$ for $n = 2, 3, \cdots$.

PROOF. Let $L \in NC_1$. If $H \leqq L$ then $\langle H^L \rangle \leqq H$ and therefore $H = \langle H^L \rangle \lhd L$. Hence $L \in SI_1$. The converse is also true and consequently $NC_1 = SI_1$. We show that L is Abelian if $L \in SI_1$. Suppose $x, y \in L$ then $\boldsymbol{k}x, \boldsymbol{k}y \lhd L$ since $L \in SI_1$. If x and y are linearly independent then $[x, y] \in \boldsymbol{k}x \cap \boldsymbol{k}y = 0$. If x and y are linearly dependent then $[x, y] = 0$ by the definition of the Lie product.

Let now $L \in NC_n$ where $n > 1$.

We assert $x \in L \Rightarrow \langle x^L \rangle \in NIL_{n-1}$.

The proof is as follows. If $x \in L$ then $\boldsymbol{k}x \leqq L$ and consequently $\langle x^L \rangle^n \leqq \boldsymbol{k}x$. Now suppose $\langle x^L \rangle^n \neq 0$, then we have $\boldsymbol{k}x = \langle x^L \rangle^n \lhd L$ and therefore $\boldsymbol{k}x = \langle x^L \rangle = \langle x^L \rangle^n$, but this is impossible. Hence we conclude $\langle x^L \rangle^n = 0$.

Let $x_1, \cdots, x_n \in L$, then $X = \langle x_1, \cdots x_n \rangle \leqq \langle x_1^L \rangle + \cdots + \langle x_n^L \rangle \in NIL_{n^2-n}$ since $\langle x_i^L \rangle \in NIL_{n-1}$. By applying lemma 3 we obtain $X \in FD_r$ where $r = M_2(n, n^2-n)$.

Now we have $\langle [x_1, \cdots, x_n]^L \rangle \leqq \langle X^L \rangle^n \leqq X$ since $L \in NC_n$ and therefore $\langle [x_1, \cdots, x_n]^L \rangle \in FD_r$.

Since X is an arbitrary finitely generated subalgebra of L we have also proved L is locally nilpotent. Hence by lemma 1 $\langle [x_1, \cdots, x_n]^L \rangle \lhd C_r(L)$ and consequently $L^n = \Sigma_{x_i \in L} \langle [x_1, \cdots, x_n]^L \rangle \lhd C_r(L)$; thus $L^{n+r} = [L^n, {}_rL] = 0$ and this concludes the proof of the theorem.

LEMMA 4. $SOL_2 \cap SI_n \subset NC_n$.

THEOREM 2. $SOL_k \cap SI_n \subset NIL_{M_4(k,n)}$ *where* $M_4(1, n) = 1$ and $M_4(k+1, n) = M_1(M_4(k, n), M_3(n))$.

PROOF. This theorem is the same as lemma 3.3.10 of Stewart [1] p. 320, but our bound is something better because in our proof we can refer to theorem 1. For the sake of completeness the proof now follows.

We use induction on k.

$k = 1$: $SOL_1 \cap SI_n \subset NIL_1$

$k = 2$: $SOL_2 \cap SI_n \subset NC_n \subset NIL_{M_3(n)}$ by lemma 4 and theorem 1

$2 \leqq k \Rightarrow k+1$: If $L \in SOL_{k+1} \cap SI_n$ then $H = L^{(k-1)} \in SOL_2 \cap SI_n$ and therefore $H \in NIL_{M_3(n)}$. But $L_{/H^2} \in SOL_k \cap SI_n$ and consequently by induction $L_{/H^2} \in NIL_{M_4(k,n)}$. Finally we apply lemma 2.

Suppose $H \leqq L$.

The series $L = H_0 \trianglerighteq H_1 \trianglerighteq \cdots$, inductively defined by $H_0 = L$, $H_{i+1} = \langle H^{H_i} \rangle$, is called the *ideal closure series* of H.

LEMMA 5. *$H \lhd {}^n L$ iff $H = H_n$.*

LEMMA 6. *$H \leqq L \in SI_n \Rightarrow H_i/H_{i+1} \in SI_{n-i}$ for $i = 0, \cdots, n-1$.*

This lemma is of the first importance for the proof of the main theorem.

THEOREM 3. *$SI_n \subset NIL_{M(n)}$ where $M(1) = 1$ and*

$$M(n+1) = M_3(M_4(n\overline{M(n)+1}, n+1)+1).$$

PROOF. By induction on n.

$n = 1$: $SI_1 = NC_1 \subset NIL_1$

$n \Rightarrow n+1$: Let $L \in SI_{n+1}$. If $H \leqq L$ then because of lemma 6 and by induction $H_i/H_{i+1} \in SI_{n+1-i} \subset SI_n \subset NIL_{M(n)}$ for $i = 1, \cdots, n$. Therefore $H_i/H_{i+1} \in SOL_{\overline{M(n)+1}}$. By lemma 5 $H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_0 = L$ where (H_i) is the ideal closure series of H and it follows now easily that $H_1^{(r)} \leqq H$ where $r = n\overline{M(n)+1}$. Moreover we have

$$H_1/H_1^{(r)} \in SOL_r \cap SI_{n+1}$$

since SI_{n+1} is closed under taking subalgebras and quotient algebras. By applying theorem 2 we now obtain $H_1/H_1^{(r)} \in NIL_s$ where $s = M_4(r, n+1)$ and therefore $H_1^{s+1} \leqq H_1^{(r)} \leqq H$. Thus $\langle H^L \rangle^{s+1} \leqq H$. Hence $L \in NC_{s+1}$. We finish the proof by applying theorem 1.

4. The class SI_2

$M(2) = M_3(M_4(1, 2)+1) = M_3(2) = 1+M_2(2, 2) = 1+2+2^2 = 7$, but this bound can be improved still further.

For the following result I am indebted to my referee.

PROPOSITION. *If the characteristic of the field **k** is not 3, then $SI_2 = NIL_2$ and if the characteristic is 3, then $NIL_2 \subset SI_2 \subset NIL_3$.*

PROOF. Let $x \in L \in SI_2$, then $\mathbf{k}x \lhd K \lhd L$ for some K. Therefore $\mathbf{k}x$ is a minimal ideal of $H = \langle x^L \rangle$, which we know already is nilpotent (theorem 3); so by Lemma 1 $x \in C_1(H)$ which is a characteristic ideal in H and hence $\lhd L$. Therefore $H = C_1(H)$ is Abelian.

If now $y \in L$ it follows that $[[x, y], x] = 0$, so L has Engel 2-condition. By a result of Higgins [3] we now conclude that $L \in NIL_2$, if char. $\neq 3$ and $L \in NIL_3$ if char. $= 3$.

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