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## ON FIRST ORDER ELLIPTIC EQUATIONS FOR SECTIONS OF COMPLEX LINE BUNDLES

by

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### Introduction

Let  $M$  be a real 2-dimensional  $C^\infty$  manifold,  $E$  and  $F$  smooth vector-bundles over  $M$  with real 2-dimensional fibres. Then each linear first order elliptic partial differential operator  $L$  from  $C^\infty$  sections of  $E$  to  $C^\infty$  sections of  $F$  can locally be brought into a standard form, as follows.

#### THEOREM 1.

a) For each  $x_0 \in M$  there is a neighborhood  $U$  of  $x_0$ , a local coordinatization  $\gamma$  of  $U$  and local trivializations  $\tau^E$ , resp.  $\tau^F$  of  $E$ , resp.  $F$  over  $U$  in which  $L$  has the form:

$$(1) \quad Lu = \frac{1}{2}(\partial u / \partial x_1 + i \partial u / \partial x_2) + b(x) \cdot \bar{u}.$$

Here  $b(x)$  is a complex valued  $C^\infty$  function and the fiber  $\mathbb{R}^2$  is identified with  $\mathbb{C}$ .

b) If  $\gamma_j, \tau_j^E, \tau_j^F, j = 1, 2$  are local coordinatizations, resp. local trivializations of  $E$  and  $F$  as in a), then either  $\gamma_1 \circ \gamma_2^{-1}$  is holomorphic and  $\tau_1^E \cdot (\tau_2^E)^{-1}, \tau_1^F \cdot (\tau_2^F)^{-1}$  are multiplications with complex numbers in the fibers,  $\tau_1^E \cdot (\tau_2^E)^{-1}$  depending holomorphically on  $x$ , or  $\gamma_1 \circ \gamma_2^{-1}$  is anti-holomorphic and  $\tau_1^E \cdot (\tau_2^E)^{-1}, \tau_1^F \cdot (\tau_2^F)^{-1}$  are multiplications with complex numbers followed by complex conjugation.

c) If  $L$  is a complex linear operator for some given complex structures on  $E$  and  $F$ , then the trivialisations  $\tau^E, \tau^F$  in a) can be chosen complex linear.

This theorem is classical, c.f. Vekua [13] or the supplement to Ch. IV in [4] of Bers. If  $M$  is orientable then this leads to a unique complex analytic structure on  $M$ , and an identification of  $E$  with a holomorphic complex line bundle  $\xi$  on  $M$  and of  $F$  with  $\bar{\kappa} \cdot \xi$ , such that:

$$(2) \quad Lu = \bar{\partial}u + b \cdot \bar{u} \text{ on sections } u \text{ of } \xi.$$

Here  $b \in \Gamma(M, C^\infty(\bar{\kappa} \cdot (\xi)^{-1} \cdot \xi))$  and  $\kappa$  is the canonical bundle of  $M$ . If  $L$  is a complex linear operator then  $M$  is automatically orientable and

$L$  is reduced to  $\bar{\partial}$  acting on  $\xi$ . If  $M$  is not orientable one can study  $L$  by changing to the 2-fold orientable covering of  $M$ .

If  $M$  is not compact then the elliptic theory of Malgrange [9], Ch. 3, combined with the theorem of unique continuation of solutions of  $\bar{\partial}u + a \cdot u + b \cdot \bar{u} = 0$  of Carleman [3], implies that  $L$  is surjective:  $\Gamma(M, C^\infty(E)) \rightarrow \Gamma(M, C^\infty(F))$ . This can be generalized to the case that  $L$  is a first order operator on a higher dimensional manifold  $M$ , acting as an elliptic operator in the direction of the leaves of a 2-dimensional foliation in  $M$ . One obtains semi-global solvability for the equation  $Lu = f$  if no leaf is contained in a compact subset of  $M$ , and global solvability if in addition a convexity condition for the leaves is satisfied as in [5], Theorem 7.1.6. Application to the Hamilton operator  $H_p$  leads to corresponding results for general pseudo-differential operators acting on real 2-dimensional bundles with 2-dimensional bicharacteristic strips. See [5], Ch. 7.

If  $L = L_1 + iL_2$  is a complex vector field acting on a trivial line bundle then semi-global solvability conversely implies that no leaf is contained in a compact subset of  $M$  ([5], Th. 7.1.5). However, in general one can even have global solvability if  $M$  is a compact surface. If more generally  $M$  is fibered by compact surfaces on which  $L$  acts, then global solvability on the fibers leads to global solvability on  $M$ .

So assume from now on that  $M$  is a compact and orientable surface,  $L$  as in (0.2). Then

$$(3) \quad \text{index } L = \text{index } \bar{\partial} = c(\xi) + 1 - g.$$

The first identity follows from general elliptic theory and the second one is the theorem of Riemann-Roch.  $c(\xi)$  is the Chern class of  $\xi$  and  $g$  is the genus of  $M$ . (See Gunning [6] for the theory of compact Riemann surfaces used here.) In particular  $L$  can only be surjective if  $c(\xi) \geq g - 1$ . Using the similarity principle of Bers [2], we obtain for each  $v \in \Gamma(M, C^\infty(\kappa\xi^{-1}))$ ,  ${}^tLv = 0$ ,  $v \neq 0$ , a non-zero holomorphic section  $v'$  of some holomorphic line bundle  $\kappa \cdot (\xi')^{-1}$  with  $c(\xi') = c(\xi)$ . From the results below it therefore follows that  $L$  is surjective if  $c(\xi) > 2(g - 1)$ . So there remains a gap between the necessary and sufficient condition for global solvability if  $g \geq 1$ ,  $g - 1 \leq c(\xi) \leq 2(g - 1)$ .

If  $L$  is complex linear then the reduction to  $\bar{\partial}$  acting on  $\xi$  leads to a much more detailed description. In this case surjectivity is equivalent to the condition that  $\kappa \cdot \xi^{-1} \cdot \zeta_q^{-n}$ , considered as an element of the Jacobi-variety  $J(M)$  of  $M$ , does not belong to the set  $W^n$  defined by:

$$(4) \quad \begin{aligned} W^n &= \emptyset \text{ if } n < 0, W^0 = \{0\}, \text{ and for } n \geq 1: \\ W^n &= \{\zeta_{p_1} \cdot \zeta_{p_2} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} \in J(M); p_1, \dots, p_n \in M\}. \end{aligned}$$

Here  $n = c(\kappa \cdot \xi^{-1}) = 2(g-1) - c(\xi)$ ,  $\xi_p$  is the point bundle of  $p \in M$ . The point  $q \in M$  is arbitrary but fixed. The  $W^n$  are known in algebraic geometry as the varieties of special divisors of  $M$ . They are algebraic subvarieties of  $J(M)$  of complex dimension  $n$  if  $1 \leq n \leq g$ .  $W^n = J(M)$  for  $n \geq g$  because of Riemann-Roch.

In Section 1 we give an elementary proof of Theorem 1, followed by a discussion in more detail of the identification of the operator  $\bar{\partial} + a$  acting on the holomorphic line bundle  $\xi_0$  ( $\xi_0$  fixed,  $a$  varying) with  $\bar{\partial}$  acting on the holomorphic line bundle  $\xi$  depending on  $a$ . In Section 2 we discuss the relation between the surjectivity of  $\bar{\partial}$  and the algebraic varieties  $W^n$  mentioned above. Although this is only a standard application of the classical theory of Riemann surfaces, we like to present this here as an example of an elliptic equation on a compact manifold with a rather intricate global solvability condition on the lower order term  $a$ . We conclude by mentioning what is known about the singularities of the varieties  $W^n$ .

I am indebted to Lars Hörmander for the suggestion that [5], Ch. 7 should be generalized to operators on line bundles, and to Lipman Bers and Frans Oort for helping me with the literature.

### 1. Reduction to $\bar{\partial}$ acting on a holomorphic line bundle

For arbitrary local trivializations of  $E$  and  $F$  over  $U$ , the principal symbol of  $L$  is a  $C^\infty$  mapping:  $(x, \xi) \mapsto A(x, \xi)$  from  $T^*(U)$  to the space of real  $2 \times 2$ -matrices, the mapping is linear in  $\xi$ . Here the principal symbol is defined such that  $L = A(x, \partial/\partial x) + \text{zero order terms}$ , on local coordinates.

Ellipticity means that  $\det A(x, \xi) \neq 0$  for  $\xi \neq 0$ , so  $\det A(x, \xi)$  is the principal symbol of a real second order elliptic operator  $P$  on  $U$ . According to a classical theorem on normal forms of such operators we can find local coordinates such that the second order part of  $P$  is equal to  $c(x) \cdot \Delta$  for a smooth function  $c(x) \neq 0$ . Here  $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$  on  $\mathbf{R}^2$ . (See Courant and Hilbert [4], Ch. III, § 1.) So on these coordinates:

$$(1.1) \quad Lu = A_1(x) \cdot \partial u/\partial x_1 + A_2(x) \cdot \partial u/\partial x_2 + B(x) \cdot u,$$

where  $A_1(x), A_2(x), B(x)$  are real  $2 \times 2$ -matrices depending smoothly on  $x$ , and  $\det (A_1(x)\xi_1 + A_2(x)\xi_2) = c(x) \cdot (\xi_1^2 + \xi_2^2)$ .

Now we re-trivialize  $E$  and  $F$ , that is we write  $u(x) = S(x) \cdot v(x)$ ,  $f(x) = T(x) \cdot g(x)$  for some real  $2 \times 2$ -matrices  $S(x), T(x)$  depending smoothly on  $x$ . Then  $Lu = f$  becomes

$$(1.2) \quad g = T^{-1}A_1S\partial v/\partial x_1 + T^{-1}A_2S\partial v/\partial x_2 + \text{zero order terms.}$$

So we try to choose  $S, T$  such that  $T^{-1}A_1S = \frac{1}{2}I$ ,  $T^{-1}A_2S = \frac{1}{2}i$ , here  $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . This means that  $T = \frac{1}{2}S^{-1}A_1^{-1}$  and

$$(1.3) \quad S^{-1}A_1(x)^{-1}A_2(x)S = i.$$

The equation (1.3) is solvable if and only if  $A_1^{-1}A_2$  has eigenvalues  $\pm i$ . Now

$$\det(A_1^{-1}A_2 - \lambda I) = \det A_1^{-1} \cdot \det(A_2 - \lambda A_1) = \det A_1^{-1} \cdot c(x) \cdot (1 + \lambda^2),$$

so for each  $x \in U$  the equation (1.3) has a solution. The mapping  $S \mapsto SiS^{-1}$  is a smooth fibration of  $GL(2, \mathbb{R})$  over the manifold of real  $2 \times 2$  matrices with eigenvalues  $\pm i$ , so the solution  $S$  can locally be chosen to depend smoothly on  $x$ .

We now have local coordinatizations and trivializations in which  $Lu = \partial u / \partial \bar{z} + a(z) \cdot u + b(z) \cdot \bar{u}$ ,  $z = x_1 + ix_2$ . Here  $a, b$  are complex valued  $C^\infty$  functions of  $z$ . Using Cauchy's integral formula we can find a local solution  $c(z)$  to  $\partial c / \partial \bar{z} = a$ . Writing  $u = e^{-c} \cdot v$ ,  $f = e^{-c} \cdot g$  the equation  $Lu = f$  can be written in the form  $\partial v / \partial \bar{z} + b \cdot e^{c-\bar{c}} \cdot \bar{v} = g$ , which proves part a) of Theorem 1.

For part b) we remark that the equation

$$(1.4) \quad f = \frac{1}{2}(\partial u / \partial x_1 + i \cdot \partial u / \partial x_2) + b \cdot \bar{u}$$

in other local coordinatizations, resp. trivializations as in Theorem 1, a) has the form

$$(1.5) \quad g = \frac{1}{2}(\partial v / \partial y_1 + i \cdot \partial v / \partial y_2) + c \cdot \bar{v}.$$

Here  $y = y(x)$ ,  $u(x) = S(x) \cdot v(y(x))$ ,  $f(x) = T(x) \cdot g(y(x))$ . This leads to  $T^{-1} \cdot (\partial / \partial x_1 + i \cdot \partial / \partial x_2) y_1 \cdot S = I$ ,  $T^{-1} \cdot (\partial / \partial x_1 + i \cdot \partial / \partial x_2) y_2 \cdot S = i$ , so both  $T \circ S^{-1} = (\partial / \partial x_1 + i \partial / \partial x_2) y_1$  and  $T \circ S^{-1} SiS^{-1} = (\partial / \partial x_1 + i \partial / \partial x_2) y_2$  are multiplications with complex numbers. Therefore  $SiS^{-1}$  is a multiplication with a complex number which only can be  $+i$  or  $-i$ .

If  $SiS^{-1} = i$  then we obtain the Cauchy-Riemann equations for  $y_1, y_2$ . Moreover  $S$  and therefore also  $T$  can only be a multiplication with a complex number. Looking at the zero order terms we obtain that  $T^{-1} \partial S / \partial \bar{z} \cdot v + T^{-1} \cdot b \cdot \bar{S} \cdot \bar{v} = c \cdot \bar{v}$  for all  $v$ , so  $\partial S / \partial \bar{z} = 0$ ,  $c = T^{-1} \cdot b \cdot \bar{S}$ . If finally  $SiS^{-1} = -i$  then  $x \mapsto y(x)$  is anti-holomorphic and  $S, T$  are multiplications by complex numbers followed by complex conjugation. This proves b).

For the statement c) in Theorem 1 we observe that  $A_1, A_2, B$  in (1.1) are multiplications by complex numbers if  $L$  is complex linear and we choose  $\tau^E, \tau^F$  complex linear. The formula  $S^{-1}A_1^{-1}A_2S = i$  then implies that  $A_1^{-1}A_2 = \pm i$ . If  $A_1^{-1}A_2 = +i$  it follows that  $S$  and  $T$  are



holomorphic line bundles  $\zeta, \zeta'$  are equal if there exist non-zero meromorphic sections of  $\zeta$ , resp.  $\zeta'$  with equal zeros and poles, the result follows immediately. Defining  $W^n$  as in (4),  $n = c(\zeta) =$  the number of zeros minus the number of poles of meromorphic sections of  $\zeta$ , we obtain that  $\Gamma(M, \mathcal{O}(\zeta)) = 0$  if and only if  $\zeta \cdot \zeta_q^{-n} \notin W^n$ .

If  $\gamma$  is a curve from  $q$  to  $p$  then  $h \mapsto \int_\gamma h, h \in \Gamma(M, \mathcal{O}(\kappa))$ , is an element of  $\Gamma(M, \mathcal{O}(\kappa))^* \cong H^1(M, \mathcal{O})$  (Serre duality), which according to Abel's theorem corresponds to  $\zeta_p \zeta_q^{-1} \in J(M)$ . Therefore  $\Phi : p \mapsto \zeta_p \zeta_q^{-1}$  is an analytic mapping:  $M \rightarrow J(M)$  with image  $W^1$ .  $\Phi$  is injective, hence an analytic embedding of  $M$  into  $J(M)$  if  $g \geq 1$  (the case  $g = 0$  is trivial). Because of Chow's lemma the image  $W^1$  is even an algebraic subvariety (without singularities) of the algebraic variety  $J(M)$ . So  $W^n = W^1 + \dots + W^1$  ( $n$  times) is also an algebraic subvariety of  $J(M)$ . Because of Riemann-Roch,  $\dim \Gamma(M, \mathcal{O}(\zeta)) > 0$  if  $c(\zeta) \geq g$ , hence  $W^n = J(M)$  for  $n \geq g$ . Since  $\dim_{\mathbb{C}} W^{n+1} \leq \dim_{\mathbb{C}} W^n + 1$  it follows that  $\dim_{\mathbb{C}} W^n = n$  for  $1 \leq n \leq g$ .

The possible singularities of  $W^n$  for  $2 \leq n \leq g-1$  are studied quite extensively in algebraic geometry. Define

$$(2.1) \quad G_n^r = \{ \zeta_{p_1} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} \in W^n; \dim \Gamma(M, \mathcal{O}(\zeta_{p_1} \cdots \zeta_{p_n})) \geq r+1 \}.$$

Alternative description: the mapping  $(p_1, \dots, p_n) \mapsto \zeta_{p_1} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} : M^n \rightarrow J(M)$  factors through the symmetric product of  $n$  copies of  $M$ , denoted by  $M^{(n)}$ , thus leading to a mapping  $\Phi^{(n)} : M^{(n)} \rightarrow J(M)$ . The variety  $M^{(n)}$  has no singularities (cf. Andreotti [1]) and the mapping  $\Phi^{(n)}$  is analytic. Then  $\dim \Gamma(M, \mathcal{O}(\zeta_{p_1} \cdots \zeta_{p_n})) = r+1$  if and only if the rank of the differential of  $\Phi^{(n)}$  at  $(p_1, \dots, p_n)$  is equal to  $n-r$  (see Gunning [6], Lemma 17).

Now Weil [14] showed that  $G_n^1$  is equal to the set of singularities of  $W^n$  for all  $n \leq g-1$ . In general  $G_n^{r+1}$  is contained in the set of singularities of  $G_n^r$  (Mayer [12]), but Martens [11] has given examples of singularities of  $G_{g-1}^1$  not coming from  $G_{g-1}^2$ . Martens [10] also proved that

$$(2.2) \quad d = (r+1)(n-r) - rg \leq \dim G_n^r \leq n - 2r \text{ if } 2 \leq n \leq g-1.$$

Kleiman and Laksov [8] proved that  $G_n^r \neq \emptyset$  if the number  $d$  in the left hand side of (2.2) is non-negative. Finally we mention the work of Kempf [7] containing an infinitesimal study of the singularities of the  $W^n$ .

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