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ON FIRST ORDER ELLIPTIC EQUATIONS FOR SECTIONS OF COMPLEX LINE BUNDLES

by

J. J. Duistermaat

Introduction

Let M be a real 2-dimensional C^∞ manifold, E and F smooth vector-bundles over M with real 2-dimensional fibres. Then each linear first order elliptic partial differential operator L from C^∞ sections of E to C^∞ sections of F can locally be brought into a standard form, as follows.

THEOREM 1.

a) For each $x_0 \in M$ there is a neighborhood U of x_0 , a local coordinatization γ of U and local trivializations τ^E , resp. τ^F of E , resp. F over U in which L has the form:

$$(1) \quad Lu = \frac{1}{2}(\partial u / \partial x_1 + i \partial u / \partial x_2) + b(x) \cdot \bar{u}.$$

Here $b(x)$ is a complex valued C^∞ function and the fiber \mathbf{R}^2 is identified with \mathbf{C} .

b) If $\gamma_j, \tau_j^E, \tau_j^F, j = 1, 2$ are local coordinatizations, resp. local trivializations of E and F as in a), then either $\gamma_1 \circ \gamma_2^{-1}$ is holomorphic and $\tau_1^E \cdot (\tau_2^E)^{-1}, \tau_1^F \cdot (\tau_2^F)^{-1}$ are multiplications with complex numbers in the fibers, $\tau_1^E \cdot (\tau_2^E)^{-1}$ depending holomorphically on x , or $\gamma_1 \circ \gamma_2^{-1}$ is anti-holomorphic and $\tau_1^E \cdot (\tau_2^E)^{-1}, \tau_1^F \cdot (\tau_2^F)^{-1}$ are multiplications with complex numbers followed by complex conjugation.

c) If L is a complex linear operator for some given complex structures on E and F , then the trivialisations τ^E, τ^F in a) can be chosen complex linear.

This theorem is classical, c.f. Vekua [13] or the supplement to Ch. IV in [4] of Bers. If M is orientable then this leads to a unique complex analytic structure on M , and an identification of E with a holomorphic complex line bundle ξ on M and of F with $\bar{\kappa} \cdot \xi$, such that:

$$(2) \quad Lu = \bar{\partial}u + b \cdot \bar{u} \text{ on sections } u \text{ of } \xi.$$

Here $b \in \Gamma(M, C^\infty(\bar{\kappa} \cdot (\xi)^{-1} \cdot \xi))$ and κ is the canonical bundle of M . If L is a complex linear operator then M is automatically orientable and

L is reduced to $\bar{\partial}$ acting on ξ . If M is not orientable one can study L by changing to the 2-fold orientable covering of M .

If M is not compact then the elliptic theory of Malgrange [9], Ch. 3, combined with the theorem of unique continuation of solutions of $\bar{\partial}u + a \cdot u + b \cdot \bar{u} = 0$ of Carleman [3], implies that L is surjective: $\Gamma(M, C^\infty(E)) \rightarrow \Gamma(M, C^\infty(F))$. This can be generalized to the case that L is a first order operator on a higher dimensional manifold M , acting as an elliptic operator in the direction of the leaves of a 2-dimensional foliation in M . One obtains semi-global solvability for the equation $Lu = f$ if no leaf is contained in a compact subset of M , and global solvability if in addition a convexity condition for the leaves is satisfied as in [5], Theorem 7.1.6. Application to the Hamilton operator H_p leads to corresponding results for general pseudo-differential operators acting on real 2-dimensional bundles with 2-dimensional bicharacteristic strips. See [5], Ch. 7.

If $L = L_1 + iL_2$ is a complex vector field acting on a trivial line bundle then semi-global solvability conversely implies that no leaf is contained in a compact subset of M ([5], Th. 7.1.5). However, in general one can even have global solvability if M is a compact surface. If more generally M is fibered by compact surfaces on which L acts, then global solvability on the fibers leads to global solvability on M .

So assume from now on that M is a compact and orientable surface, L as in (0.2). Then

$$(3) \quad \text{index } L = \text{index } \bar{\partial} = c(\xi) + 1 - g.$$

The first identity follows from general elliptic theory and the second one is the theorem of Riemann-Roch. $c(\xi)$ is the Chern class of ξ and g is the genus of M . (See Gunning [6] for the theory of compact Riemann surfaces used here.) In particular L can only be surjective if $c(\xi) \geq g - 1$. Using the similarity principle of Bers [2], we obtain for each $v \in \Gamma(M, C^\infty(\kappa\xi^{-1}))$, ${}^tLv = 0$, $v \neq 0$, a non-zero holomorphic section v' of some holomorphic line bundle $\kappa \cdot (\xi')^{-1}$ with $c(\xi') = c(\xi)$. From the results below it therefore follows that L is surjective if $c(\xi) > 2(g - 1)$. So there remains a gap between the necessary and sufficient condition for global solvability if $g \geq 1$, $g - 1 \leq c(\xi) \leq 2(g - 1)$.

If L is complex linear then the reduction to $\bar{\partial}$ acting on ξ leads to a much more detailed description. In this case surjectivity is equivalent to the condition that $\kappa \cdot \xi^{-1} \cdot \zeta_q^{-n}$, considered as an element of the Jacobi-variety $J(M)$ of M , does not belong to the set W^n defined by:

$$(4) \quad \begin{aligned} W^n &= \emptyset \text{ if } n < 0, W^0 = \{0\}, \text{ and for } n \geq 1: \\ W^n &= \{\zeta_{p_1} \cdot \zeta_{p_2} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} \in J(M); p_1, \dots, p_n \in M\}. \end{aligned}$$

Here $n = c(\kappa \cdot \xi^{-1}) = 2(g-1) - c(\xi)$, ξ_p is the point bundle of $p \in M$. The point $q \in M$ is arbitrary but fixed. The W^n are known in algebraic geometry as the varieties of special divisors of M . They are algebraic subvarieties of $J(M)$ of complex dimension n if $1 \leq n \leq g$. $W^n = J(M)$ for $n \geq g$ because of Riemann-Roch.

In Section 1 we give an elementary proof of Theorem 1, followed by a discussion in more detail of the identification of the operator $\bar{\partial} + a$ acting on the holomorphic line bundle ξ_0 (ξ_0 fixed, a varying) with $\bar{\partial}$ acting on the holomorphic line bundle ξ depending on a . In Section 2 we discuss the relation between the surjectivity of $\bar{\partial}$ and the algebraic varieties W^n mentioned above. Although this is only a standard application of the classical theory of Riemann surfaces, we like to present this here as an example of an elliptic equation on a compact manifold with a rather intricate global solvability condition on the lower order term a . We conclude by mentioning what is known about the singularities of the varieties W^n .

I am indebted to Lars Hörmander for the suggestion that [5], Ch. 7 should be generalized to operators on line bundles, and to Lipman Bers and Frans Oort for helping me with the literature.

1. Reduction to $\bar{\partial}$ acting on a holomorphic line bundle

For arbitrary local trivializations of E and F over U , the principal symbol of L is a C^∞ mapping: $(x, \xi) \mapsto A(x, \xi)$ from $T^*(U)$ to the space of real 2×2 -matrices, the mapping is linear in ξ . Here the principal symbol is defined such that $L = A(x, \partial/\partial x) + \text{zero order terms}$, on local coordinates.

Ellipticity means that $\det A(x, \xi) \neq 0$ for $\xi \neq 0$, so $\det A(x, \xi)$ is the principal symbol of a real second order elliptic operator P on U . According to a classical theorem on normal forms of such operators we can find local coordinates such that the second order part of P is equal to $c(x) \cdot \Delta$ for a smooth function $c(x) \neq 0$. Here $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ on \mathbf{R}^2 . (See Courant and Hilbert [4], Ch. III, § 1.) So on these coordinates:

$$(1.1) \quad Lu = A_1(x) \cdot \partial u / \partial x_1 + A_2(x) \cdot \partial u / \partial x_2 + B(x) \cdot u,$$

where $A_1(x), A_2(x), B(x)$ are real 2×2 -matrices depending smoothly on x , and $\det (A_1(x)\xi_1 + A_2(x)\xi_2) = c(x) \cdot (\xi_1^2 + \xi_2^2)$.

Now we re-trivialize E and F , that is we write $u(x) = S(x) \cdot v(x)$, $f(x) = T(x) \cdot g(x)$ for some real 2×2 -matrices $S(x), T(x)$ depending smoothly on x . Then $Lu = f$ becomes

$$(1.2) \quad g = T^{-1}A_1S\partial v/\partial x_1 + T^{-1}A_2S\partial v/\partial x_2 + \text{zero order terms.}$$

So we try to choose S, T such that $T^{-1}A_1S = \frac{1}{2}I$, $T^{-1}A_2S = \frac{1}{2}i$, here $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This means that $T = \frac{1}{2}S^{-1}A_1^{-1}$ and

$$(1.3) \quad S^{-1}A_1(x)^{-1}A_2(x)S = i.$$

The equation (1.3) is solvable if and only if $A_1^{-1}A_2$ has eigenvalues $\pm i$. Now

$$\det(A_1^{-1}A_2 - \lambda I) = \det A_1^{-1} \cdot \det(A_2 - \lambda A_1) = \det A_1^{-1} \cdot c(x) \cdot (1 + \lambda^2),$$

so for each $x \in U$ the equation (1.3) has a solution. The mapping $S \mapsto SiS^{-1}$ is a smooth fibration of $GL(2, \mathbb{R})$ over the manifold of real 2×2 matrices with eigenvalues $\pm i$, so the solution S can locally be chosen to depend smoothly on x .

We now have local coordinatizations and trivializations in which $Lu = \partial u / \partial \bar{z} + a(z) \cdot u + b(z) \cdot \bar{u}$, $z = x_1 + ix_2$. Here a, b are complex valued C^∞ functions of z . Using Cauchy's integral formula we can find a local solution $c(z)$ to $\partial c / \partial \bar{z} = a$. Writing $u = e^{-c} \cdot v$, $f = e^{-c} \cdot g$ the equation $Lu = f$ can be written in the form $\partial v / \partial \bar{z} + b \cdot e^{c-\bar{c}} \cdot \bar{v} = g$, which proves part a) of Theorem 1.

For part b) we remark that the equation

$$(1.4) \quad f = \frac{1}{2}(\partial u / \partial x_1 + i \cdot \partial u / \partial x_2) + b \cdot \bar{u}$$

in other local coordinatizations, resp. trivializations as in Theorem 1, a) has the form

$$(1.5) \quad g = \frac{1}{2}(\partial v / \partial y_1 + i \cdot \partial v / \partial y_2) + c \cdot \bar{v}.$$

Here $y = y(x)$, $u(x) = S(x) \cdot v(y(x))$, $f(x) = T(x) \cdot g(y(x))$. This leads to $T^{-1} \cdot (\partial / \partial x_1 + i \cdot \partial / \partial x_2) y_1 \cdot S = I$, $T^{-1} \cdot (\partial / \partial x_1 + i \cdot \partial / \partial x_2) y_2 \cdot S = i$, so both $T \circ S^{-1} = (\partial / \partial x_1 + i \partial / \partial x_2) y_1$ and $T \circ S^{-1} SiS^{-1} = (\partial / \partial x_1 + i \partial / \partial x_2) y_2$ are multiplications with complex numbers. Therefore SiS^{-1} is a multiplication with a complex number which only can be $+i$ or $-i$.

If $SiS^{-1} = i$ then we obtain the Cauchy-Riemann equations for y_1, y_2 . Moreover S and therefore also T can only be a multiplication with a complex number. Looking at the zero order terms we obtain that $T^{-1} \partial S / \partial \bar{z} \cdot v + T^{-1} \cdot b \cdot \bar{S} \cdot \bar{v} = c \cdot \bar{v}$ for all v , so $\partial S / \partial \bar{z} = 0$, $c = T^{-1} \cdot b \cdot \bar{S}$. If finally $SiS^{-1} = -i$ then $x \mapsto y(x)$ is anti-holomorphic and S, T are multiplications by complex numbers followed by complex conjugation. This proves b).

For the statement c) in Theorem 1 we observe that A_1, A_2, B in (1.1) are multiplications by complex numbers if L is complex linear and we choose τ^E, τ^F complex linear. The formula $S^{-1}A_1^{-1}A_2S = i$ then implies that $A_1^{-1}A_2 = \pm i$. If $A_1^{-1}A_2 = +i$ it follows that S and T are

holomorphic line bundles ζ, ζ' are equal if there exist non-zero meromorphic sections of ζ , resp. ζ' with equal zeros and poles, the result follows immediately. Defining W^n as in (4), $n = c(\zeta) =$ the number of zeros minus the number of poles of meromorphic sections of ζ , we obtain that $\Gamma(M, \mathcal{O}(\zeta)) = 0$ if and only if $\zeta \cdot \zeta_q^{-n} \notin W^n$.

If γ is a curve from q to p then $h \mapsto \int_\gamma h, h \in \Gamma(M, \mathcal{O}(\kappa))$, is an element of $\Gamma(M, \mathcal{O}(\kappa))^* \cong H^1(M, \mathcal{O})$ (Serre duality), which according to Abel's theorem corresponds to $\zeta_p \zeta_q^{-1} \in J(M)$. Therefore $\Phi : p \mapsto \zeta_p \zeta_q^{-1}$ is an analytic mapping: $M \rightarrow J(M)$ with image W^1 . Φ is injective, hence an analytic embedding of M into $J(M)$ if $g \geq 1$ (the case $g = 0$ is trivial). Because of Chow's lemma the image W^1 is even an algebraic subvariety (without singularities) of the algebraic variety $J(M)$. So $W^n = W^1 + \dots + W^1$ (n times) is also an algebraic subvariety of $J(M)$. Because of Riemann-Roch, $\dim \Gamma(M, \mathcal{O}(\zeta)) > 0$ if $c(\zeta) \geq g$, hence $W^n = J(M)$ for $n \geq g$. Since $\dim_{\mathbb{C}} W^{n+1} \leq \dim_{\mathbb{C}} W^n + 1$ it follows that $\dim_{\mathbb{C}} W^n = n$ for $1 \leq n \leq g$.

The possible singularities of W^n for $2 \leq n \leq g-1$ are studied quite extensively in algebraic geometry. Define

$$(2.1) \quad G_n^r = \{ \zeta_{p_1} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} \in W^n; \dim \Gamma(M, \mathcal{O}(\zeta_{p_1} \cdots \zeta_{p_n})) \geq r+1 \}.$$

Alternative description: the mapping $(p_1, \dots, p_n) \mapsto \zeta_{p_1} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} : M^n \rightarrow J(M)$ factors through the symmetric product of n copies of M , denoted by $M^{(n)}$, thus leading to a mapping $\Phi^{(n)} : M^{(n)} \rightarrow J(M)$. The variety $M^{(n)}$ has no singularities (cf. Andreotti [1]) and the mapping $\Phi^{(n)}$ is analytic. Then $\dim \Gamma(M, \mathcal{O}(\zeta_{p_1} \cdots \zeta_{p_n})) = r+1$ if and only if the rank of the differential of $\Phi^{(n)}$ at (p_1, \dots, p_n) is equal to $n-r$ (see Gunning [6], Lemma 17).

Now Weil [14] showed that G_n^1 is equal to the set of singularities of W^n for all $n \leq g-1$. In general G_n^{r+1} is contained in the set of singularities of G_n^r (Mayer [12]), but Martens [11] has given examples of singularities of G_{g-1}^1 not coming from G_{g-1}^2 . Martens [10] also proved that

$$(2.2) \quad d = (r+1)(n-r) - rg \leq \dim G_n^r \leq n - 2r \text{ if } 2 \leq n \leq g-1.$$

Kleiman and Laksov [8] proved that $G_n^r \neq \emptyset$ if the number d in the left hand side of (2.2) is non-negative. Finally we mention the work of Kempf [7] containing an infinitesimal study of the singularities of the W^n .

REFERENCES

A. ANDREOTTI

[1] On a theorem of Torelli, Am. J. Math. 80 (1958), 801-828.

L. BERS

- [2] Article in: Contributions to the Theory of Partial Differential Equations, Princeton 1954.

T. CARLEMAN

- [3] Sur les systèmes linéaires aux dérivées partielles du premier ordre à deux variables, C. R. Acad. Sc. 197 (1933) 471–474.

R. COURANT and D. HILBERT

- [4] Methods of Mathematical Physics, vol. II, Interscience, New York, 1962.

J. J. DUISTERMAAT and L. HÖRMANDER

- [5] Fourier Integral Operators II, Acta Math., 128 (1972), 183–269.

R. C. GUNNING

- [6] Lectures on Riemann Surfaces, Princeton, 1966.

G. KEMPF

- [7] The singularities of certain varieties in the Jacobian of a curve, preprint.

S. L. KLEIMAN and D. LAKSOV

- [8] On the existence of special divisors, preprint.

B. MALGRANGE

- [9] Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, Ann. de l'Inst. Fourier 6 (1955) 271–355.

H. H. MARTENS

- [10] On the varieties of special divisors on a curve, Journal f.d. reine u. angew. Math., 227 (1967), 111–120.

H. H. MARTENS

- [11] II, same journal, 233 (1968), 89–100.

A. L. MAYER

- [12] Special divisors and the Jacobian variety, Math. Ann. 153 (1964), 163–167.

I. N. VEKUA

- [13] Generalized Analytic Functions, Pergamon Press, Oxford–London–New York–Paris, 1962.

A. WEIL

- [14] Zum Beweis des Torellischen Satzes, Nachr. der Akad. der Wiss. in Göttingen 2 (1957), 33–53.

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