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COMPACT ABELIAN GROUP EXTENSIONS OF DYNAMICAL SYSTEMS II

by

Roger Jones and William Parry

0. Introduction

A dynamical system which commutes with a compact abelian group G induces a system on the G orbit space. We say that the former is a G extension of the latter. For a fixed group G we consider the totality of all group extensions of a dynamical system and show that 'most' of the extensions enjoy dynamical properties possessed by the original system, if by dynamical property we mean: ergodicity, weak-mixing, completely positive entropy, minimality or unique ergodicity. The proofs depend on propositions which assert that 'most' cocycles are not coboundaries and the qualification 'most' varies with a metric topology appropriate to the problem being discussed.

This paper is a continuation of the investigations begun in [1] concerning the lifting of dynamical properties. In particular we mention that the technique of perturbations discussed in [1] is broadened here to allow perturbations by nonconstant maps. (The notion of G -stability was further developed in [2].) Perturbations from a rather different point of view were employed previously by Ellis in [3]. Our point of view, like the results in [1], is more akin to Furstenberg's [4]. We wish to mention that this paper found its initial impulse from conversations with H. Furstenberg and R. Ellis concerning an existence proof for minimal non-ergodic dynamical systems. In addition to Ellis and Furstenberg, we acknowledge the help and interest of J. Auslander and P. Walters.

Central to this paper is the proposition that under suitable conditions 'most' cocycles of a dynamical system (with values in the circle group) are not measurable coboundaries. The existence of measurable cocycles which are not coboundaries for the special case of a flow on a compact connected abelian group, is apparently known, and is useful in the construction of simply invariant subspaces for unitary representations of totally ordered discrete groups, cf. [5], [6]. Our existence proof has the advantage of generality. Moreover, the cocycles we prove to exist, which are not coboundaries, are continuous.

1. Preliminaries

Throughout X will denote a compact metric G -space where G is a compact abelian group, i.e., G acts continuously on X in the sense that there is a continuous map $G \times X \rightarrow X$, $(g, x) \rightarrow gx$ where $g(hx) = (gh)x$ and $ex = x$ for all $x \in X$. Suppose T is a homeomorphism of X which commutes with G . Then T induces a homeomorphism T' on $X' = X/G$, (the G -orbit space). In the same way if T_t is a flow (of homeomorphisms) on X (where the map $R \times X \rightarrow X$, $(t, x) \rightarrow T_t x$ is continuous) which commutes with G then T_t induces a flow T'_t on X' . We shall refer to T, T_t as G -extensions of T', T'_t . In most of our discussion there will be no loss of generality in assuming that G acts freely in the sense that if $gx = x$ for some x then $g = e$. If G acts freely then a general G extension of T' (or T'_t) given T (or T_t) takes the form $T^\phi : x \rightarrow \phi(x)Tx$ where $\phi : X \rightarrow G$ and $\phi(gx) = \phi(x)$ for all $g \in G$, $x \in X$ (or $T_t^\phi : x \rightarrow \phi(x, t)T_t x$ where $\phi : X \times R \rightarrow G$ and $\phi(gx, t) = \phi(x, t)$ for all $g \in G$, $x \in X$, $t \in R$ and $\phi(x, t+s) = \phi(T_t x, s)\phi(x, t)$). The maps ϕ are called cocycles (with respect to T or T_t). In fact they are G invariant cocycles with values in G , and they form a group under pointwise multiplication.

It is well-known that for T (or T_t) there always exists a normalised Borel measure m on X such that $mTB = mB$, $mgB = mB$ ($mT_t B = mB$, $mgB = mB$). When speaking of a measure in this paper we shall assume it is invariant in this sense. m' will denote the measure induced by m on X' . m' is invariant for T' (or T'_t).

For definitions of, and criteria for, minimality, unique ergodicity and weak-mixing, cf. [4], [1].

A function of type γ where $\gamma \in \hat{G}$ (the character group of G) means a function mapping X to K (the circle group of complex numbers of absolute value 1) such that $f(gx) = \gamma(g)f(x)$ for $g \in G$, $x \in X$.

(1.1) Minimality. If T' (T'_t) is minimal then $T(T_t)$ is minimal if and only if $fT = f(fT_t = f)$ where f is continuous of type γ implies $\gamma \equiv 1$ (and then f is constant).

(1.2) Unique ergodicity. If T' (T'_t) is uniquely ergodic then $T(T_t)$ is uniquely ergodic if and only if $fT = f$ a.e. $[m]$ ($fT_t = f$ a.e. $[m]$ for each t) where f is a Borel function of type γ implies $\gamma \equiv 1$ (and then f is constant a.e.).

(1.3) Ergodicity. If T' (T'_t) is ergodic with respect to m' then $T(T_t)$ is ergodic with respect to m if and only if $fT = f$ a.e. $[m]$ ($fT_t = f$ a.e. $[m]$ for each t) where f is a Borel function of type γ implies $\gamma \equiv 1$ (and then f is constant a.e.).

- (1.4) Weak-mixing. If T' (T'_t) is weak-mixing with respect to m' then $T(T_t)$ is weak-mixing with respect to m if and only if $fT = e^{2\pi ia}f$ a.e. $[m]$ ($fT_t = e^{2\pi iat}f$ a.e. $[m]$ for each t) where $a \in R$ is constant and where f is a Borel function of type γ implies $\gamma \equiv 1$ (and then f is constant a.e.).

In [7] the latter author showed that weak-mixing G extensions of transformations with completely positive entropy have completely positive entropy, when G is a torus. The corresponding statement for G compact abelian follows from this fact. However, a significantly more general result for G not necessarily abelian has been proved by K. Thomas [8]. Combining these facts with (1.4) we have:

- (1.5) Completely positive entropy. If $T'(T'_t)$ has completely positive entropy with respect to m' then $T(T_t)$ has completely positive entropy with respect to m if and only if $fT = e^{2\pi ia}f$ a.e. $[m]$ ($fT_t = e^{2\pi iat}f$ a.e. $[m]$ for each t) where $a \in R$ is constant and where f is a Borel function of type γ implies $\gamma \equiv 1$ (and then f is constant a.e.).

REMARK. In the above statements, it can be shown that there is no loss in generality if we consider only functions of type γ with range $\gamma(G)$.

If H is a topological abelian group let $C_0(X, H)$ denote the group (under pointwise multiplication) of G invariant continuous cocycles with values in H , i.e.,

$$C_0(X, H) = \{\phi : \phi \text{ maps } X \text{ to } H \text{ continuously such that} \\ \phi(gx) = \phi(x) \text{ for } g \in G, x \in X\}$$

when we are considering a single homeomorphism;

$$C_0(X, H) = \{\phi : \phi \text{ maps } X \times R \text{ to } H \text{ continuously such that} \\ \phi(gx, t) = \phi(x, t) \text{ for } g \in G, x \in X, \\ t \in R \text{ and } \phi(x, t+s) = \phi(T_t x, s)\phi(x, t)\}$$

when we are considering a flow T_t .

We shall be interested in the cases $H = G$ or K . It is clear that G invariant cocycles arise from cocycles defined on X' .

Let $C_\gamma(X, K) = \{f : f \text{ is continuous of type } \gamma\}$ where $\gamma \in \hat{G}$ and $C(X, K) = \bigcup_{\gamma \in \hat{G}} C_\gamma(X, K)$. Obviously $C(X, K)$ is a group under pointwise multiplication and $C_\gamma(X, K) = f_\gamma \cdot C_1(X, K)$ if $f_\gamma \in C_\gamma(X, K)$. (For some $\gamma \in \hat{G}$ it may happen that $C_\gamma(X, K)$ is empty.) When we are considering a single homeomorphism it is clear that $C_1(X, K) = C_0(X, K)$ where 1 denotes the trivial character.

The elements of $C_0(X, K)$ of the form fT/f (fT_t/f) (for $f \in C(X, K)$) are called *continuous coboundaries* with values in K . Clearly the map

$$\begin{aligned} \rho : C(X, K) &\rightarrow C_0(X, K) \\ f &\rightarrow fT/f \text{ (} fT_t/f \text{)} \end{aligned}$$

is a homomorphism.

Now suppose $T, G(T_t, G)$ preserve the normalised Borel measure m . Let

$$\begin{aligned} B_\gamma(X, K) = \{f : f \text{ is Borel of type } \gamma \text{ and } fT/f = h \text{ a.e.} \\ \text{for some } h \in C_0(X, K)\} \end{aligned}$$

when we consider a single homeomorphism and

$$\begin{aligned} B_\gamma(X, K) = \{f : f \text{ is Borel of type } \gamma \text{ and } fT_t/f = h \text{ a.e.} \\ \text{for each } t, \text{ where } h \in C_0(X, K)\} \end{aligned}$$

when we consider a flow.

$B(X, K) = \bigcup_{\gamma \in \hat{G}} B_\gamma(X, K)$ is a group under pointwise multiplication and two elements of $B(X, K)$ shall be identified if they differ only on a set of measure zero.

The cocycles h defined above are unique if we assume the measure m (or m') is positive on non-empty open sets *and this we shall always do*. The map $\rho : f \rightarrow h$ is then a homomorphism of $B(X, K)$ to $C_0(X, K)$. If $f \in B(X, K)$, ρf is called a Borel coboundary.

The groups which play a special role in our investigations are $C_0(X, G)$, $C_0(X, K)$, $C(X, K)$, $C_1(X, K)$, $B(X, K)$, $B_1(X, K)$ and the latter five will frequently be abbreviated to C_0 , C , C_1 , B , B_1 ; $C_0(X, G)$ will not be abbreviated. Notice that $C_1(X, K) \simeq C_0(X', K)$ and $B_1(X, K) \simeq B_0(X', K)$ in obvious senses where G acts trivially on X' . Notice also that C_1, B_1 contain the group of constant functions K and C_0 contains K (for a single homeomorphism) or $\hat{R}$ (for a flow).

Along with the homomorphisms

$$\begin{aligned} \rho : C(X, K) &\rightarrow C_0(X, K) \\ \rho : B(X, K) &\rightarrow C_0(X, K) \end{aligned}$$

we will wish to consider the homomorphisms ρ' obtained by composing ρ with the natural map ' of $C_0(X, K)$ to $C_0(X, K)/K$, or $C_0(X, K)/\hat{R}$. Elements in the K cosets $\rho'C$ ($\rho'B$) will be called *weak* continuous (Borel) coboundaries. (Elements in the $\hat{R}$ cosets of $\rho'C$ ($\rho'B$) will be called *weak* continuous (Borel) coboundaries.)

For each non-trivial character $\gamma \in \hat{G}$ we shall need to consider the homomorphism $\tilde{\gamma}$ of $C_0(X, G)$ to $C_0(X, K)$ defined by $\tilde{\gamma} : \phi \rightarrow \gamma \circ \phi$ when $\phi \in C_0(X, G)$. As before $\tilde{\gamma}'$ will denote the composition of $\tilde{\gamma}$ with the

natural map γ from $C_0(X, K)$ to $C_0(X, K)/K$ or from $C_0(X, K)$ to $C_0(X, K)/\hat{R}$. When we consider a single homeomorphism $C_0(X, G)$, $C_0(X, K)$ shall be endowed with the uniform topologies induced by metrics on G, K . When we consider a flow $C_0(X, G)$, $C_0(X, K)$ shall be endowed with the compact-open topology for maps from $X \times R$ to G or K . In either case $C(X, K)$, $C_1(X, K)$ are given the uniform topology induced by a metric on K . These topologies are all complete separable metric. If d denotes such a metric on $C_0(X, K)$ we give a metric D for $B(X, K)$ ($B_1(X, K)$) as follows:

$$D(f_1, f_2) = \int_X |f_1 - f_2| \, dm + d(\rho f_1, \rho f_2).$$

It is an easy matter to check that D is a metric and $B(X, K)$ is complete in this metric. $B(X, K)$ is, moreover, separable as can be seen by considering the set $\{(f, \rho f) : f \in B\}$ contained in $L^1(X) \times C(X)$.

(1.6) In other words, the above topologies for $C_0(X, G)$, C_0 , C , C_1 , B , B_1 are complete separable metric.

(1.7) The homomorphisms ρ, ρ' are clearly continuous since the topology of the range space has been incorporated into the topology of the domain space where necessary.

(1.8) If G is compact metric connected and abelian then for each non-trivial $\gamma \in \hat{G}$, $\tilde{\gamma} : C_0(X, G) \rightarrow C_0(X, K)$ and $\tilde{\gamma}' : C_0(X, G) \rightarrow C_0(X, K)/K$ or $C_0(X, K)/\hat{R}$ are continuous and open.

If G acts trivially on a space such as X' then $C_0(X', K)$ consists of all continuous cocycles with values in K without reference to G , $B_1(X', K)$ consists of *all* Borel maps f from X' to K such that $\rho f \in C_0(X', K)$ and $C_1(X', K)$ consists of *all* continuous maps from X' to K such that $\rho f \in C_0(X', K)$. In other words, when G acts trivially C_0 , B_1 , C_1 can be defined without reference to G , but reference to the homeomorphism T' or flow T'_t , is, of course, still required.

We shall need the following theorems the proofs of which will be deferred until later:

THEOREM 1. *If T' is a homeomorphism (T'_t is a flow) on the compact metric space X' , with at least one dense aperiodic orbit, then the set ρC_1 of continuous coboundaries is a set of first category in the group C_0 of continuous cocycles.*

THEOREM 2. *If T' is a homeomorphism (T'_t is a flow) on the compact metric space X' , preserving the normalised Borel measure m' which is positive on non-empty open sets and if T' (T'_t) is ergodic, then the set ρB_1 of Borel*

coboundaries is a set of first category in the group C_0 of continuous cocycles.

The proof of Theorem 1 is particularly simple. The proof of Theorem 2 is harder than that of Theorem 1. However, both proofs are easier than, and modifications of, the proof of the next theorem. They will therefore be omitted.

THEOREM 3. *If T' is a non-trivial homeomorphism (T'_t is a non-trivial flow) on the compact metric space X' preserving the normalised Borel measure m' which is positive on non-empty open sets and if T' (T'_t) is weak-mixing then the set $\{\rho f \cdot k : f \in B_1, k \in K\}$ ($\rho f \cdot \eta : f \in B_1, \eta \in \hat{R}$) of weak Borel coboundaries is a set of first category in the group C_0 of continuous cocycles.*

The proof of this theorem will be presented later.

2. Main Theorems

With the aid of Theorems 1, 2, 3 we are now in a position to prove results concerning the lifting of various dynamical properties. For a stronger version of the following theorem cf. [3].

THEOREM 4. *If T is a homeomorphism of the compact metric space X (T_t is a flow on X) and if G is a compact connected abelian group acting freely on X and commuting with T (T_t) then ‘most’ extensions $\phi(x)Tx$ ($\phi(x, t)T_t x$) of T' (T'_t) are minimal if T' (T'_t) is minimal, i.e., $\{\phi \in C_0(X, G) : \phi(x)Tx$ ($\phi(x, t)T_t x$) is not minimal\} is of first category in $C_0(X, G)$.*

PROOF. If T' is minimal then there is an aperiodic orbit unless X' is finite. If T'_t is minimal then there is an aperiodic orbit unless X' is a circle. The latter cases are especially easy to deal with and will not be considered here. We may therefore suppose that the hypothesis of Theorem 1 is satisfied and ρC_1 is of first category in C_0 .

By (1.1), $\phi(x)Tx$ ($\phi(x, t)T_t x$) is not minimal, $\phi \in C(X, G)$, if and only if there exists $1 \neq \gamma \in \hat{G}$ and f continuous of type γ such that $f(\phi(x)Tx) = f(x)$ ($f(\phi(x, t)T_t x) = f(x)$) and therefore $\tilde{\gamma}\phi \in \rho C(X, K)$. Since $\rho C(X, K) = \bigcup_{\gamma \in \hat{G}} \rho f_\gamma \cdot \rho C_1(X, K)$ for a choice of $f_\gamma \in C_\gamma(X, K)$ we have $\rho C(X, K)$ is of first category and $\phi \in \bigcup_{1 \neq \gamma \in \hat{G}} \tilde{\gamma}^{-1} \rho C(X, K)$. The latter is a first category set, however, since $\tilde{\gamma}$ is open when $\gamma \neq 1$. Hence the set of $\phi \in C(X, G)$ such that $\phi(x)Tx$ ($\phi(x, t)T_t x$) is minimal contains a dense G_δ .

THEOREM 5. *If T is a homeomorphism of the compact metric space X (T_t is a flow on X) and if G is a compact connected abelian group acting*

freely on X and commuting with T (T_t) then ‘most’ extensions $\phi(x)Tx$ ($\phi(x, t)T_t x$) of T' (T'_t) are uniquely ergodic or ergodic if T' (T'_t) is uniquely ergodic or ergodic.

PROOF. By (1.2) and (1.3) $\phi(x)Tx$, $(\phi(x, t)T_t)$ is not uniquely ergodic, (assuming T' (T'_t) is uniquely ergodic) if and only if there is a Borel function f of type $\gamma \neq 1$ such that $f(\phi(x)Tx) = f(x)$ a.e. ($f(\phi(x, t)T_t x) = f(x)$ a.e. for each t), i.e., $\bar{\gamma}\phi \in \rho B$. The rest of the proof as for ergodicity, imitates the proof of Theorem 4, except that Theorem 2 needs to be invoked.

THEOREM 6. *If T is a homeomorphism of the compact metric space X (T_t is a flow on X) and if G is a compact connected abelian group acting freely on X and commuting with T (T_t) then ‘most’ extensions $\phi(x)Tx$ ($\phi(x, t)T_t x$) of T' (T'_t) are weak-mixing or have completely positive entropy if T' (T'_t) is weak-mixing or has completely positive entropy and is non-trivial.*

PROOF. In view of (1.5) we need only show that ‘most’ extensions of T (T'_t) are weak-mixing under the assumption that T' (T'_t) is weak-mixing. By (1.4), $\phi(x)Tx$ ($\phi(x, t)T_t x$) is not weak-mixing if and only if there is a Borel function f of type $\gamma \neq 1$ and $k \in K$ ($\eta \in \hat{R}$, $\eta(t) = e^{2\pi iat}$) such that

$$f(\phi(x)Tx) = k \cdot f(x) \text{ a.e. } (f(\phi(x, t)T_t x) = e^{2\pi iat} f(x) \text{ a.e. for each } t)$$

in which case $k\bar{\gamma}\phi \in \rho B$ ($\eta \cdot \bar{\gamma}\phi \in \rho B$). Hence $\phi(x)Tx$ ($\phi(x, t)T_t x$) not weak-mixing implies $\bar{\gamma}\phi \in \rho'B$ for some $\gamma \neq 1$. By Theorem 3 $\rho'B = \bigcup_{\gamma \in \hat{G}} \rho'f_\gamma \cdot \rho'B_1$ has first category in $C_0(X, K)/K$ ($C_0(X, K)/\hat{R}$) where $f_\gamma \in B_\gamma$ is a selection. Hence $\phi \in \bigcup_{1 \neq \gamma \in \hat{G}} \bar{\gamma}^{-1} \rho'B$ which is of first category.

REMARK. A topological analogue of the weak-mixing part of the above theorem has been proved by R. Peleg [9]. Since topological weak-mixing is substantially different from measure theoretic weak-mixing, there seems to be no overlap between Peleg’s and our theorem.

3. Proof of Theorem 3

SINGLE HOMEOMORPHISM CASE.

Assuming T' is a non-trivial weakly mixing homeomorphism of the compact metric space X' with T' invariant normalised Borel measure m' (positive on non-empty open sets) we show that most continuous cocycles are not weak Borel coboundaries, i.e., $\rho'B_1$ is of first category in $C_0(X, K)/K$ where $B_1 = \{f : f \text{ is a Borel map from } X \text{ to } K \text{ and } fT'/f = \text{a.e. a continuous function}\}$ and $C_0(X, K) = \{f : f \text{ is a continuous map from } X \text{ to } K\}$. By non-trivial, in this context we mean any one of the equivalent

conditions, X' is not a single point, m' is non-atomic. In any case T' is aperiodic.

B_1 is a separable, complete, metric group and ρ' is a continuous homomorphism. By the open mapping theorem for such groups (cf. [10]) if $\rho'B_1$ is of second category in $C_0(X, K)/K$ then ρ' maps B_1 openly into $\rho'B_1$. By hypothesis

$$\begin{aligned} \text{kernel } \rho' &= \{f \in B_1 : fT/f \cdot K = K\} \\ &= \{f \in B_1 : fT = kf \text{ for some } k \in K\} \\ &= K \text{ (weak-mixing).} \end{aligned}$$

Hence to show that $\rho'B_1$ is of first category it will suffice to show that the inverse to $\rho' : B_1/K \rightarrow \rho(B_1) \cdot K/K$ is not continuous. To do this we will construct functions $f_n \in B_1$ such that $f_n T'/f_n \rightarrow 1$ yet $f_n K \nrightarrow K$.

In fact let $f_n(x) = \exp 2\pi i r_n(x)$, where r_n maps X' continuously to R . r_n will be constructed so that $|r_n(T'x) - r_n(x)| < 1/n$ ensuring that $f_n T'/f_n \rightarrow 1$ and $R_n(\lambda) = m' \{x \in X' : r_n(x) \leq \lambda\} \rightarrow \lambda$. From this we will have

$$\begin{aligned} \int f_n(x) dm' &= \int \exp 2\pi i r_n(x) dm' \\ &= \int_0^1 \exp 2\pi i \lambda dR_n(\lambda) \rightarrow \int_0^1 \exp 2\pi i \lambda d\lambda = 0 \end{aligned}$$

so that for no sequence $k_n \in K$ can $\int |k_n f_n - 1| \rightarrow 0$.

Since T' is weak-mixing and non-trivial and hence aperiodic, there exists for every positive integer n (cf. [11]) a measurable set A such that $T'^i A \cap T'^j A = \emptyset$ for $i \neq j$, $|i|, |j| \leq n$ and $m'(\bigcup_{i=-n}^n T'^i A) > 1 - 1/n$. Let F be a compact set such that $F \subset A$ and $m'(\bigcup_{i=-n}^n T'^i F) > 1 - 1/n$. Now let U be an open set $U \supset F$ such that $T'^i U \cap T'^j U = \emptyset$ for $i \neq j$, $|i|, |j| \leq n$. Obviously $m'(\bigcup_{i=-n}^n T'^i U) > 1 - 1/n$.

Let $h_n(x)$ be continuous where $0 \leq h_n(x) \leq 1$ and $h_n(x) = 1$ on F , $h_n(x) = 0$ on U^c . Let $r_n(x) = \sum_{i=-n}^n h_n(T'^i x)(1 - (|i|)/n)$. By comparing the distribution $R_n(\lambda)$ of $r_n(x)$ with the distribution function of $\sum_{i=-n}^n \chi_F(T'^i x)(1 - (|i|)/n)$, which converges to the uniform distribution on $[0, 1]$ it is not difficult to see that $R_n(\lambda) \rightarrow \lambda$. Hence for $f_n(x) = \exp 2\pi i r_n(x)$ we have $f_n(T'x)/f_n(x) \rightarrow 1$ and $f_n K \nrightarrow K$.

FLOW CASE.

The proof is similar to the single homeomorphism case but a little more involved. The assumption that T'_t is weak-mixing and non-trivial means that X' is not a single point or m' is non-atomic. Hence T'_t is aperiodic. Consequently (cf. [11]) for every $\tau > 0$ the flow T'_t can be

represented as a flow built under a function with range lying in the interval $[\tau, 2\tau]$. Hence for every positive integer $\tau = n$ there is a measurable set M such that $T'_t M \cap T'_s M = \emptyset$ for $0 \leq s, t \leq n$, $s \neq t$ where $\{T'_t x : x \in M, 0 \leq t \leq n\}$ is measurable and has measure greater than $\frac{1}{2}$. By approximating M by a compact set F' we get $T'_t F' \cap T'_s F' = \emptyset$ for $0 \leq s, t \leq \theta(n)$, $s \neq t$ where $\{T'_t x : x \in F', 0 \leq t \leq \theta(n)\}$ is compact and has measure equal to $\frac{1}{2}$ and $n/2 \leq \theta(n) \leq n$. Put $F = \{T'_t x : x \in F', \varepsilon \leq t \leq 1 - \varepsilon\}$ so that F is compact and $T'_p F \cap T'_q F = \emptyset$ for $p \neq q$, $p, q = 0, 1, \dots, \phi(n) - 1$ where $\phi(n)$ is the largest even integer not exceeding $\theta(n)$. Choose an open set $U \supset F$ such that $T'_p U \cap T'_q U = \emptyset$ for $p \neq q$, $p, q = 0, 1, \dots, \phi(n) - 1$ and $m'(U - F) < \varepsilon/(\phi(n))$. Let $h_n(x)$ be a continuous function on X' such that $0 \leq h_n(x) \leq 1$, $h_n(x) = 1$ on F and $h_n(x) = 0$ on U^c . Let $r_n(x) = \sum_{i=0}^{\phi(n)} h_n(T'^{-i}_- x)(1 - (|i - m|)/m)$, $\phi(n) = 2m$.

As before we have $|r_n(T'_1 x) - r_n(x)| \leq 1/m = 2/(\phi(n))$. Now define $s_n(x) = \int_0^1 r_n(T'_s x) ds$ so that

$$\begin{aligned} |s_n(T'_t x) - s_n(x)| &= \left| \int_t^{1+t} r_n(T'_s x) ds - \int_0^1 r_n(T'_s x) ds \right| \\ &= \left| \int_1^{1+t} r_n(T'_s x) ds - \int_0^t r_n(T'_s x) ds \right| \\ &= \left| \int_0^t (r_n(T'_{s+1} x) - r_n(T'_s x)) ds \right| \leq \frac{2}{\phi(n)} t. \end{aligned}$$

Hence $|s_n(T'_t x) - s_n(x)| \rightarrow 0$ in the compact open topology, and $f_n(T'_t x)/f_n(x) \rightarrow 1$ in the compact open topology on $C_0(X, K)$, where $f_n(x) = \exp 2\pi i s_n(x)$.

If $x \in T'_p F$, $p \leq \phi(n) - 1$, then $\{t \in [0, 1] : T'_t x \in T'_p F \cup T'_{p+1} F\}$ has length greater than $1 - 2\varepsilon$ and

$$r_n(T'_t x) = 1 - \frac{|p - m|}{m} \quad \text{or} \quad 1 - \frac{|p + 1 - m|}{m}$$

for such t , i.e.,

$$|s_n(x) - r_n(x)| \leq \frac{1}{m} + 4\varepsilon.$$

If $x \notin \bigcup_{p=-1}^{\phi(n)} T'_p U$ then $r_n(x) = 0$ and $r_n(T'_t x) = 0$ for $t \in [0, 1]$, i.e., $s_n(x) = 0$. Hence $|s_n(x) - r_n(x)| \leq 1/m + 4\varepsilon$ unless $x \in A_n \equiv \bigcup_{p=-1}^{\phi(n)} T'_p U - \bigcup_{p=0}^{\phi(n)-1} T'_p F$. The latter set has measure $2m'(U) + \phi(n)m(U - F)$ which is less than $2m'(U) + \varepsilon$.

Now let $\varepsilon = 1/(\phi(n))$ so that $|s_n(x) - r_n(x)| \leq 2/(\phi(n)) + 4/(\phi(n)) = 6/(\phi(n))$ for $x \notin A_n$ and $m'A_n \leq 2m'(U) + 1/(\phi(n)) \leq 2m'(F) + 2/(\phi(n)) \leq 1/(\phi(n)) + 2/(\phi(n)) = 3/(\phi(n))$ since $\phi(n)m'(F) < \frac{1}{2}$. From this we see

that the limiting distribution function of $s_n(x)$ is the same as the limiting distribution function of $r_n(x)$, which in turn is the same as the limiting distribution function of $\sum_{i=0}^{\phi(n)} \chi_F(T_{-i}'x)(1 - (|i-m|)/m)$.

The latter is a function which is approximately uniformly distributed on a set of measure $\frac{1}{2}$. In any case it is not difficult to see that the limiting distribution function of $s_n(x)$ is $S(\lambda) = \frac{1}{2} + \lambda/2$. Hence, if $f_n(x) = \exp 2\pi i s_n(x)$ then $\int f_n(x) dm' = \int_0^1 \exp 2\pi i \lambda dS_n(\lambda) \rightarrow \int_0^1 \exp 2\pi i \lambda dS(\lambda) = \frac{1}{2}$, where $S_n(\lambda) = m'\{x: s_n(x) \leq \lambda\}$. Consequently for no sequence $k_n \in K$ can we have $\int |k_n f_n - 1| \rightarrow 0$. In other words, $f_n T_{k_n}/f_n \rightarrow 1$ yet $f_n K \not\rightarrow K$, ρ' is not open and therefore $\rho' B_1$ is of first category in C_0 .

4. One parameter flows on compact connected abelian groups

As we have pointed out before the proof of Theorem 2 proceeds in a similar fashion to that of Theorem 3, with simplifications. We have already referred to a special case of Theorem 2 which seems to be of importance in the theory of representations of totally ordered discrete groups. Perhaps we should point out that ergodicity in Theorem 2 may be relaxed for the special cases which arise in this theory. Without giving the proof, which relies substantially on the methods already presented, we mention:

THEOREM 6. *If X' is a compact connected metric abelian group and if $e_t \in X'$ is a one-parameter subgroup such that $e_t = e$ only if $t = 0$, then the flow $T_t'x = e_t x$ has the property that 'most' continuous cocycles with values in K are not Borel coboundaries, nor even weak Borel coboundaries.*

5. Group extensions of measure preserving transformations

In this section we shall consider single measure preserving transformations of a normalised measure space and dismiss further examination of flows. On the other hand we shall be interested in group extensions by compact metric abelian groups which are not necessarily connected. Had we not insisted on the connectedness of the group in the previous section we should have encountered difficulties arising from the limited number of continuous extensions. This difficulty does not arise, however, if we consider measurable extensions.

Let T be a measure preserving transformation of the normalised separable measure space $(X, \mathfrak{B}, m)$. In this context, the most convenient way of saying that T is a G -extension of T' , where G is a compact abelian metric group acting freely, is to postulate that $X = X' \times G$ where X' is the measure space on which T' acts and $T(x', g) = (T'x', \phi(x')g)$ for some

measurable map $\phi : X' \rightarrow G$. ($m = m' \times dg$ where m' is the normalised measure on X' and dg is Haar measure.)

We shall need the following versions of (1.3), (1.4), (1.5):

- (5.1) If T' is ergodic then T is ergodic if and only if the equation $f(T'x') = \gamma\phi(x')f(x')$ a.e. ($\gamma \neq 1$) has no measurable solution f with range in $\gamma(G)$.
- (5.2) If T' is weak-mixing then T is weak-mixing if and only if the equation $f(T'x) = k\gamma\phi(x')f(x')$ a.e. ($\gamma \neq 1$) has no solution (f, k) with f measurable and range contained in $\gamma(G)$, $k \in \gamma(G)$.
- (5.3) If T' has completely positive entropy then T has completely positive entropy if and only if the equation $f(T'x) = k\gamma(\phi(x'))f(x')$ a.e. ($\gamma \neq 1$) has no solution (f, k) with f measurable and range contained in $\gamma(G)$, $k \in \gamma(G)$.

A cocycle with values in the compact abelian group H is simply a measurable map of X' into H . If h is a cocycle with values in $\gamma(G)$, then h is called a *coboundary* (with respect to T') if there exists a measurable map f from X' to $\gamma(G)$ such that $f(T'x') = h(x')f(x')$ a.e., and a *weak coboundary* if there exists a measurable map f from X' to $\gamma(G)$ and a constant $k \in \gamma(G)$ such that $f(T'x) = kh(x')f(x')$.

Let $B(X', \gamma(G))$ [$B(X', G)$] denote the group, under pointwise multiplication, of cocycles with values in $\gamma(G)$ [G] where two cocycles are identified if they are equal a.e. Define for $f, f' \in B(X', \gamma(G))$ [$B(X', G)$]

$$D(f, f') = \int |f - f'| dm$$

($D(f, f') = \int d(f(x'), f(x')) dm$ where d is a metric on G .) With this metric topology, $B(X', \gamma(G))$ [$B(X', G)$] is a complete separable metric group and the maps,

$$\begin{aligned} B(X', \gamma(G)) &\xrightarrow{\alpha} B(X', \gamma(G)) \xrightarrow{\beta} B(X', \gamma(G))/\gamma(G) \\ f &\rightarrow fT'/f \rightarrow fT'/f \cdot \gamma(G) \end{aligned}$$

and

$$\begin{aligned} B(X', G) &\xrightarrow{\tilde{\gamma}} B(X', \gamma(G)) \xrightarrow{\beta} B(X', \gamma(G))/\gamma(G) \\ \phi &\rightarrow \gamma \circ \phi \rightarrow \gamma \circ \phi \cdot \gamma(G) \end{aligned}$$

are continuous. Moreover $\tilde{\gamma}$ and $\tilde{\gamma}'$ are open.

THEOREM 7. *Let T' be an ergodic (weak-mixing) measure preserving transformation of the non-atomic separable probability space $(X', \mathfrak{B}', m')$. Then 'most' cocycles with values in $\gamma(G)$ ($\gamma \neq 1$) are not coboundaries (weak-coboundaries).*

PROOF. This is similar to the proof of Theorem 3. Let us sketch the proof for the ergodic case only as the proof for the weak-mixing case involves only the kind of difficulties which were surmounted in Theorem 3. We have to show that $\rho B(X', \gamma(G))$ is of first category in $B(X', \gamma(G))$. The kernel of ρ consists of cocycles f such that $fT'/f = 1$. Since T' is ergodic the kernel of ρ consists of the constant maps from X' to $\gamma(G)$. Using the open mapping theorem again, we need only prove that the inverse to the map

$$B(X', \gamma(G))/\gamma(G) \xrightarrow{\rho} B(X', \gamma(G))$$

is not continuous.

Since $(X', \mathfrak{B}', m')$ is non-atomic and T' is ergodic, T' is aperiodic. Therefore for each positive integer n there exists a measurable set M such that $M, T'^{-1}M, \dots, T'^{-n}M$ are mutually disjoint and

$$m'(M \cup \dots \cup T'^{-n}M) > 1 - 1/n.$$

Put $M_n = M \cup T'^{-1}M \cup \dots \cup T'^{-\theta(n)}M$ and let $r_n(x') = a\chi_{M_n}(x')$, where $1 \neq \exp 2\pi ia$ is an element of $\gamma(G)$ independent of n and $\theta(n) \uparrow \infty$ is to be determined later. If $f_n(x') = \exp 2\pi i r_n(x')$ then

$$\int \left| \frac{f_n T'}{f_n} - 1 \right| dm = \int |\exp 2\pi i(r_n T'(x') - r_n(x')) - 1| dm \rightarrow 0$$

i.e., $f_n T'/f_n \rightarrow 1$ in the topology of $B(X', \gamma(G))$. However, $f_n \gamma(G) \not\rightarrow \gamma(G)$ in the topology of $B(X', \gamma(G))/\gamma(G)$ if $\theta(n)/n \rightarrow \frac{1}{2}$, for otherwise we should have $\int f_n \cdot k_n \rightarrow 1$ for some sequence $k_n \in \gamma(G)$. But $\int k_n \exp 2\pi i a \chi_{M_n} dm = k_n [\exp 2\pi i a m'(M_n) + (1 - m'(M_n))] \not\rightarrow 1$ since $m'(M_n) \rightarrow \frac{1}{2}$.

COROLLARY. Let T' be an ergodic measure preserving transformation of the non-atomic separable probability space $(X', \mathfrak{B}', m')$. Then 'most' measurable sets $A \in \mathfrak{B}'$ are not of the form $B \Delta T'^{-1}B$.

PROOF. In this statement we consider $\mathfrak{B}'$ as a complete separable metric group with the metric $d(B_1, B_2) = m'(B_1 \Delta B_2)$ and addition Δ and note that $B(X', Z_2)$ is isomorphic to $\mathfrak{B}'$ as topological groups, where $Z_2 = \{1, -1\}$, via the isomorphism

$$\mathfrak{B}' \rightarrow B(X', Z_2)$$

$$B \rightarrow \exp \pi i \chi_B.$$

Under this isomorphism the homomorphism $\rho : f \rightarrow fT'/f$ becomes $B \rightarrow B \Delta T^{-1}B$.

The following theorem follows from Theorem 7 in much the same way that Theorems 5, 6 follow from Theorems 2, 3. Of course (5.1), (5.2) and (5.3) are needed in place of (1.3), (1.4) and (1.5).

THEOREM 8. *Let T' be an ergodic [weak-mixing] [completely positive entropy] measure preserving transformation of the non-atomic separable probability space $(X', \mathfrak{B}', m')$. Then for a dense G_δ of $\phi \in B(X', G)$, where G is a compact metric abelian group, the transformation $T(x', g) = (T'x', \phi(x')g)$ is an ergodic [weak-mixing] [completely positive entropy] transformation of $X' \times G$.*

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