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SIMPLICIAL MAPS WHICH STABILIZE TO NEAR-HOMEOMORPHISMS

by

D. W. Curtis

1. Introduction

Let $U$ be an open cover of a space $Y$. Maps $f, g : X \rightarrow Y$ are $U$-close if for each $x$ in $X$, $f(x)$ and $g(x)$ lie in some member of $U$. A map $f : X \rightarrow Y$ is a near-homeomorphism if it can be uniformly approximated by homeomorphisms – i.e., for every open cover $U$ of $Y$ there exists a homeomorphism $h : X \rightarrow Y$ such that $f$ and $h$ are $U$-close. If $f \times \text{id} : X \times Q \rightarrow Y \times Q$ is a near-homeomorphism, where $Q = \prod_{i=1}^{\infty} [0, 1]_i$ is the Hilbert cube, then $f$ stabilizes to a near-homeomorphism.

The recognition of (stable) near-homeomorphisms, and their application in inverse limit calculations (see below), play an important role in the recent proof by Schori and West [7] that $2^I$ is homeomorphic to $Q$. It seems likely that techniques involving near-homeomorphisms will be useful in further investigations of hyperspaces.

Our main theorem (3.2) characterizes the stable near-homeomorphisms in the simplicial category as the surjections with compact and contractible point-inverses. The proof is by means of $Q$-factor decompositions, discussed in § 2.

Brown showed in [3] that if $(X_i, f_i)$ is an inverse sequence such that each $X_i$ is a copy of a compact metric space $X$ and each $f_i$ is a near-homeomorphism, then $\lim (X_i, f_i)$ is homeomorphic to $X$. In § 4 we note some immediate applications using (3.2), and extend Brown’s theorem to complete metric spaces.

2. $Q$-factor decompositions

A space $X$ is a $Q$-factor if $X \times Q \cong Q$. Note that if $X \times Y \cong Q$, then $X \times Q \cong X \times (X \times Y)^\omega \cong (X \times Y)^\omega \cong Q$, and $X$ is a $Q$-factor. Every $Q$-factor is a compact metric AR; it is not known whether the converse is true. West [8] has shown that every compact contractible polyhedron is a $Q$-factor.

A closed subset $A$ of $X$ is a $Z$-set in $X$ if for every nonempty open
homotopically trivial \((n\text{-}connected \text{ for all } n \geq 0)\) subset \(U\) of \(X\), \(U \setminus A\) is nonempty and homotopically trivial. Z-sets were introduced by Anderson [2], who showed that every homeomorphism between Z-sets in \(Q\) extends to a homeomorphism of \(Q\). The endslice \(W = \{0\} \times \prod_{i=0}^{\infty} [0, 1]_i \subset Q\) is a Z-set; in general, boundaries and collared sets are Z-sets. One useful technique for verifying the Z-set property is the following:

2.1. LEMMA (cf. [8], Lemma 2.2). A closed subset \(A\) of a metric ANR, \(X\) is a Z-set in \(X\) if for each \(\varepsilon > 0\) there exists a map \(f : X \to X \setminus A\) with \(d(f, \text{id}) < \varepsilon\).

PROOF. Clearly \(A\) is nowhere dense. Let \(U\) be open and homotopically trivial, and \(g : S^n \to U \setminus A\) a map of the \(n\)-sphere. There exists an extension \(\tilde{g} : C^{n+1} \to U\) of \(g\) to the \((n+1)\)-cell. As a metric ANR, \(X\) is locally equiconnected, and therefore has the property that for every open cover \(\mathcal{V}\) there exists an open cover \(\mathcal{W}\) such that maps into \(X\) which are \(\mathcal{W}\)-close are \(\mathcal{V}\)-homotopic (paths of the homotopy lie in members of \(\mathcal{V}\)) [6]. By the compactness of \(C^{n+1}\) there exists \(\varepsilon > 0\) such that for any map \(f : X \to X \setminus A\) with \(d(f, \text{id}) < \varepsilon\), \(f\tilde{g}(C^{n+1}) \subset U \setminus A\) and \(g\) is homotopic to \(f \circ g\) in \(U \setminus A\). This homotopy together with the map \(f \circ \tilde{g}\) provides an extension \(\tilde{g} : C^{n+1} \to U \setminus A\) of \(g\).

2.2. DEFINITION. \(\{X_\alpha\}\) is a Q-factor decomposition of a Hausdorff space \(X\) if:

i) \(\{X_\alpha\}\) is a locally finite cover of \(X\) by Q-factors,
ii) \(X_1, X_2 \in \{X_\alpha\}\) and \(X_1 \cap X_2 \neq \phi\) imply \(X_1 \cap X_2 \in \{X_\alpha\}\),
iii) \(X_1, X_2 \in \{X_\alpha\}\) and \(X_1 \sqsubseteq X_2\) imply \(X_1\) is a Z-set in \(X_2\).

The spaces admitting Q-factor decompositions comprise a proper subclass of the class of locally compact metrizable ANR's, and include the locally compact polyhedra.

2.3. DEFINITION. Q-factor decompositions \(\{X_\alpha\}\) and \(\{Y_\beta\}\) indexed by the same set are similar if \(X_1 \cap X_2 \neq \phi\) is equivalent to \(Y_1 \cap Y_2 \neq \phi\). \(\{X_\alpha\}\) and \(\{Y_\beta\}\) are isomorphic if \(X_1 \subset X_2\) is equivalent to \(Y_1 \subset Y_2\).

Isomorphic decompositions are similar: if \(X_1 \cap X_2 \neq \phi\), then \(X_1 \cap X_2 = X_3 \in \{X_\alpha\}\), \(X_3 \subset X_1\) and \(X_3 \subset X_2\), therefore \(Y_3 \subset Y_1\) and \(Y_3 \subset Y_2\), and \(Y_1 \cap Y_2 \cap Y_3 \neq \phi\).

For any space \(X\), \(\tau^n : X \times Q \to I^n\) will denote the projection onto the first \(n\) factors of \(Q\).

2.4. THEOREM. Let \(\{X_\alpha\}\) and \(\{Y_\beta\}\) be isomorphic Q-factor decompositions of \(X\) and \(Y\), respectively, and let a function \(p : A \to Z^+\) from the indexing set into the positive integers be given. Then there exists a homeomorphism \(H : X \times Q \to Y \times Q\) such that \(H(X_\alpha \times Q) = Y_\beta \times Q\) and \(\tau^{p(\alpha)}|X_\alpha \times Q = \tau^{p(\beta)}|H|X_\alpha \times Q\) for each \(\alpha\).
PROOF. Since $X_\alpha = X_\beta$ is equivalent to $Y_\alpha = Y_\beta$, and since \{X_\alpha|X_\alpha = X_\beta\} is a finite collection for each $X_\beta$, there is no loss of generality in assuming that the isomorphic decompositions \{X_\alpha\} and \{Y_\alpha\} are faithfully indexed – i.e., $X_\alpha = X_\beta$ only if $\alpha = \beta$. For any subcollection \{X_\alpha|\alpha \in B \subseteq A\} of \{X_\alpha\}, let $\text{Min} \{X_\alpha|\alpha \in B\} = \{X_\alpha|\alpha \in B; \beta \in B \text{ with } X_\beta \subseteq X_\alpha \text{ implies } \alpha = \beta\}$, the collection of minimal elements. Inductively define $X^{(i)} = X^{(i-1)} \cup \text{Min} \{X_\alpha|X_\alpha \notin X^{(i-1)}\}$, $i \geq 0$, with $X^{(-1)} = \phi$. Then \{X_\alpha\} = \bigcup X^{(i)}; similarly \{Y_\alpha\} = \bigcup Y^{(i)}$. It is easily seen that $X_\alpha \subseteq X^{(i)}$ is equivalent to $Y_\alpha \subseteq Y^{(i)}$. Since the indicator function $p : A \to \mathbb{Z}^+$ can be redefined by setting $p'(\alpha) = \max \{p(\beta)|X_\alpha \subseteq X_\beta\}$, we may assume that $X_\alpha \subseteq X_\beta$ implies $p(\alpha) \leq p(\beta)$.

For each $\alpha$, let $\mathcal{H}_\alpha$ denote the non-empty collection of homeomorphisms of $X_\alpha \times Q$ onto $Y_\alpha \times Q$ of the form $h_\alpha = \tilde{h}_\alpha \times \text{id}_\alpha$, where $\tilde{h}_\alpha : X_\alpha \times \prod \{I_i|i > p(\alpha)\} \to Y_\alpha \times \prod \{I_i|i > p(\alpha)\}$ and $\text{id}_\alpha$ is the identity map on $I^{p(\alpha)} = \prod \{I_i|i > p(\alpha)\}$. Suppose inductively that there exists a homeomorphism $H_i : \bigcup \{X_\alpha|X_\alpha \subseteq X^{(i)}\} \times Q \to \bigcup \{Y_\alpha|Y_\alpha \subseteq Y^{(i)}\} \times Q$ such that $H_i/X_\alpha \times Q$ is in $\mathcal{H}_\alpha$ for each $X_\alpha \subseteq X^{(i)}$. Consider $X_\beta \subseteq X^{(i+1)} \setminus X^{(i)} = \text{Min} \{X_\alpha|X_\alpha \notin X^{(i)}\}$, and set $X_\beta = \bigcup \{X_\alpha|X_\alpha \subseteq X_\beta\}$. Then $X_\beta$, as a finite union of $Z$-sets, is a $Z$-set in $X_\beta$ (it may be empty), and $X_\beta = X_\beta \cap (\bigcup \{X_\alpha|X_\alpha \subseteq X^{(i)}\})$. Similarly for $Y_\beta$; note that $H_i(X_\beta \times Q) = Y_\beta \times Q$. Since $p(\beta) \leq p(\alpha)$ for each $X_\alpha \subseteq X_\beta$, an application of Anderson’s homeomorphism extension theorem to $X_\beta \times \prod \{I_i|i > p(\beta)\}$ and $Y_\beta \times \prod \{I_i|i > p(\beta)\}$ shows there exists $h_\beta \in \mathcal{H}_\beta$ such that $h_\beta/X_\beta \times Q = H_i/X_\beta \times Q$. For distinct elements $X_\alpha$ and $X_\beta$ of $X^{(i+1)} \setminus X^{(i)}$, either $X_\alpha \cap X_\beta = \phi$ or $X_\alpha \cap X_\beta \subseteq X^{(i)}$. Since \{X_\alpha\} is a locally finite closed cover of $X$, we may define $H_{i+1} : \bigcup \{X_\alpha|X_\alpha \subseteq X^{(i+1)}\} \times Q \to \bigcup \{Y_\alpha|Y_\alpha \subseteq Y^{(i+1)}\} \times Q$ by requiring that $H_i$ extend $H_i$ and $H_{i+1}/X_\alpha \times Q = h_\alpha$ for each $X_\alpha \subseteq X^{(i+1)} \setminus X^{(i)}$. Then $H : X \times Q \to Y \times Q$ defined by $H/X_\alpha \times Q = H_i/X_\alpha \times Q$ for $X_\alpha \subseteq X^{(i)}$, $i \geq 0$, is the desired homeomorphism.

In [5] we obtain an extension of (2.4) to similar $Q$-factor decompositions, in which the requirement $H(X_\alpha \times Q) = Y_\alpha \times Q$ is replaced by $H(X_\alpha \times Q) \subseteq \text{St} (Y_\alpha) \times Q$. This result promises to be useful in recognizing stable near-homeomorphisms in situations where Theorem 3.2 (see below) does not apply.

3. Stable near-homeomorphisms

In this section we shall be dealing with simplicial maps between locally finite complexes. A map $f : K \to L$ is compact or contractible if $f^{-1}(x)$ is compact or contractible for each $x$ in $L$.

3.1. Lemma. Let $f : K \to L$ be a compact contractible simplicial surjection, and let $U$ be an open cover of $L$. Then there exist isomorphic $Q$-
factor decompositions \( \{K_a\} \) of \( K \) and \( \{L_a\} \) of \( L \) such that \( \{L_a\} \) refines \( \mathcal{U} \) and \( K_a = f^{-1}(L_a) \) for each \( a \).

**Proof.** It is well-known that there exist subdivisions \( K_\ast \) of \( K \) and \( L_\ast \) of \( L \) such that \( f : K_\ast \to L_\ast \) is simplicial and the cover by vertex stars of \( L_\ast \) refines \( \mathcal{U} \). For notational convenience assume that \( K = K_\ast \) and \( L = L_\ast \). We show that the dual structures on \( K \) and \( L \) described by Cohen [4] are the desired \( Q \)-factor decompositions.

Let \( L' \) be the standard barycentric subdivision of \( L \), and let \( K' \) be a barycentric subdivision of \( K \) chosen so that \( f : K' \to L' \) is simplicial. The barycenter of a simplex \( \sigma \) is denoted by \( \bar{\sigma} \). If \( \sigma_0 \subset \cdots \subset \sigma_q \), then \( \bar{\sigma}_0 \cdots \bar{\sigma}_q \) is the simplex spanned by the barycenters. If \( \alpha \) is a simplex of \( L \), then \( D(\alpha, L) \), the dual to \( \alpha \) in \( L \), and its subcomplex \( \check{D}(\alpha, L) \) are defined by \( D(\alpha, L) = \{\bar{\sigma}_0 \cdots \bar{\sigma}_q | \alpha \subset \sigma_0 \subset \cdots \subset \sigma_q\} \), \( \check{D}(\alpha, L) = \{\bar{\sigma}_0 \cdots \bar{\sigma}_q | \alpha \supset \sigma_0 \subset \cdots \subset \sigma_q\} \). \( D(\alpha, f) \), the dual to \( \alpha \) with respect to \( f \), is a subcomplex of \( K' \) defined by \( D(\alpha, f) = \{\bar{\tau}_0 \cdots \bar{\tau}_q | \alpha \subset f(\tau_0), \tau_0 \subset \cdots \subset \tau_q\} \); similarly for \( \check{D}(\alpha, f) \). Each dual \( D(\alpha, L) \) is a finite subcomplex of \( L' \), and since \( f \) is a compact surjection each \( D(\alpha, f) \) is also finite and non-empty. Clearly \( D(\alpha, L) \) is the join \( \check{D}(\alpha, L) \). It is known [4] that \( D(\alpha, f) = f^{-1}D(\alpha, L) \), \( \check{D}(\alpha, f) = f^{-1}\check{D}(\alpha, L) \), and \( D(\alpha, f) \) collapses to \( f^{-1}(\check{\alpha}) \).

Set \( \{K_a\} = \{D(\alpha, f)\} \) and \( \{L_a\} = \{D(\alpha, L)\} \), where \( \alpha \) runs through all the simplexes of \( L \). Then \( \{K_a\} \) and \( \{L_a\} \) are isomorphic locally finite covers of \( K \) and \( L \), and \( \{L_a\} \) refines \( \mathcal{U} \). Each dual \( D(\alpha, L) \) is contractible, and since \( f \) is contractible each dual \( D(\alpha, f) \) is contractible. It follows from West’s theorem (see § 2) that each \( \delta \) is a \( \mathcal{Q} \)-factor.

3.2. **Theorem.** A simplicial map \( f : K \to L \) stabilizes to a near-homeomorphism if and only if \( f \) is a compact contractible surjection.

**Proof.** Suppose \( f \) is a compact contractible surjection. Let \( \mathcal{W} \) be an open cover of \( L \times \mathcal{Q} \). There exists an open cover \( \mathcal{U} \) of \( L \) and a function \( m : \mathcal{U} \to \mathbb{Z}^+ \) such that for \( (x_1, q_1) \) and \( (x_2, q_2) \) in \( L \times \mathcal{Q} \) with \( \{x_1, x_2\} \subset U \in \mathcal{U} \) and \( \tau^m_U(q_1) = \tau^m_U(q_2) \), \( \{(x_1, q_1), (x_2, q_2)\} \subset W \in \mathcal{W} \). By (3.1)
there exist isomorphic \(Q\)-factor decompositions \(\{K_\alpha\}\) of \(K\) and \(\{L_\alpha\}\) of \(L\) such that \(\{L_\alpha\}\) refines \(\mathcal{U}\) and \(K_\alpha = f^{-1}(L_\alpha)\). Define \(p : A \to \mathbb{Z}^+\) by \(p(a) = \min \{m(U) | L_\alpha \subset U \in \mathcal{U}\}\). By (2.4) there exists a homeomorphism \(H : K \times Q \to L \times Q\) such that \(H(K_\alpha \times Q) = L_\alpha \times Q\) and \(\tau^{p(a)}|K_\alpha \times Q = \tau^{p(a)}H/K_\alpha \times Q\) for each \(a\). Clearly \(H\) and \(f \times \text{id}\) are \(\mathcal{U}\)-close.

Conversely, suppose that \(f \times \text{id}\) is a near-homeomorphism. Since the image of \(f \times \text{id}\) must be dense in \(L \times Q\), \(f\) is surjective. Consider a point \(x\) in \(L\) and the inverse \(f^{-1}(x) \subset K\). Since there exists a homeomorphism of \(K \times Q\) onto \(L \times Q\) taking \(f^{-1}(x) \times Q\) into a compact neighborhood of \(\{x\} \times Q\), \(f^{-1}(x)\) is compact. (The same argument shows that the inverse image of every compact set is compact.) Since \(f\) is simplicial \(f^{-1}(x)\) is polyhedral and therefore a retract of some neighborhood \(U\) in \(K\). Using compactness of the inverse image of a compact neighborhood of \(x\), we obtain a neighborhood \(V\) of \(x\) such that \(f^{-1}(V) \subset U\). Then there exists a contractible neighborhood \(W\) of \(x\) and a homeomorphism \(H : K \times Q \to L \times Q\) such that \(H(f^{-1}(x) \times Q) \subset W \times Q \subset H(U \times Q)\). Thus \(f^{-1}(x) \times Q\) is contractible in the neighborhood \(U \times Q\) which retracts onto it, and therefore \(f^{-1}(x)\) is contractible.

A non-piecewise linear map \(f : K \to L\) which stabilizes to a near-homeomorphism may not be contractible (although it follows from the proof above that point-inverses must have the shape of a point). For example, it is easily seen that there exists a map \(f : I^2 \to I\) such that \(f^{-1}(t)\) is an arc if \(t \neq \frac{1}{2}\), \(f^{-1}(\frac{1}{2})\) is a topologist's sine curve containing \(I \times \{0, 1\}\), and \(f\) is the uniform limit of piecewise-linear maps satisfying the conditions of (3.2). Hence \(f\) itself stabilizes to a near-homeomorphism.

4. Inverse limit applications

Brown's theorem (see § 1) and Theorem (3.2) imply that if \((K_i, f_i)\) is an inverse sequence of finite complexes with simplicial contractible surjections as bonding maps, then \(\text{Lim } (K_i, f_i) \times Q\) is homeomorphic to \(K \times Q\). Since a dendron is an inverse limit of finite trees with elementary collapses as bonding maps, this technique provides a quick proof of the fact, announced in [1] and demonstrated in [8], that every dendron is a \(Q\)-factor.

Let \(J^\infty = \coprod_i [-1, 1]_i\), and let \(J^\infty/R\) be the quotient space obtained by identifying \((x_i)\) with \((-x_i)\). Schori and Barit have recently used the same technique to show that \(J^\infty/R\) is a \(Q\)-factor.

The following extension of Brown's theorem to complete metric spaces permits the application of (3.2) in the non-compact case.

4.1. Theorem. If \((X_i, f_i)\) is an inverse sequence of copies of a complete metric space \(X\) with near-homeomorphisms as bonding maps, then \(\text{Lim } (X_i, f_i)\) is homeomorphic to \(X\).
PROOF. We inductively choose homeomorphisms \( h_i : X_{i+1} \to X_i, i \geq 1 \), such that \( \text{Lim} (X_i, f_i) \) is homeomorphic to \( \text{Lim} (X_i, h_i) \). For \( i < j \) let \( f_{ij} = f_i \circ \cdots \circ f_{j-1} \) and \( h_{ij} = h_i \circ \cdots \circ h_{j-1} \) be compositions of the bonding maps, and let \( f_{i\infty} : \text{Lim} (X_i, f_i) \to X_i \) and \( h_{i\infty} : \text{Lim} (X_i, h_i) \to X_i \) be the projections. Suppose that \( h_1, \ldots, h_{j-1} \) have been chosen. Then there exists an open cover \( \mathcal{U}_j \) of \( X_j \) such that mesh \( f_{ij}(\mathcal{U}_j) < 2^{-j} \) and mesh \( h_{ij}(\mathcal{U}_j) < 2^{-j} \) for \( 1 \leq i < j \). Choose a homeomorphism \( h_j : X_{j+1} \to X_j \) such that \( f_j \) and \( h_j \) are \( \mathcal{U}_j \)-close.

A straight-forward verification shows there exists a map \( F : \text{Lim} (X_i, f_i) \to \text{Lim} (X_i, h_i) \) such that \( h_{i\infty} F(x) = \lim_{n \to \infty} h_{in} f_{i\infty} (x) \) for each \( i \). Likewise there exists a map \( H : \text{Lim} (X_i, h_i) \to \text{Lim} (X_i, f_i) \) such that \( f_{i\infty} H(x) = \lim_{n \to \infty} f_{in} h_{i\infty} (x) \). We show that \( H \circ F \) and \( F \circ H \) are the identity maps. Let \( 1 \leq i < n \) and \( x \in \text{Lim} (X_i, f_i) \) be given. Then \( d(f_{i\infty} H F(x), f_{in} h_{i\infty} F(x)) < 2^{-n+1} \), and for each \( m > n \), \( d(f_{i\infty} (x), f_{in} h_{nm} f_{m\infty} (x)) < 2^{-n+1} \). Since \( h_{n\infty} F(x) = \lim_{m \to \infty} h_{nm} f_{m\infty} (x) \), there exists \( m > n \) such that \( d(f_{in} h_{nm} F(x), f_{in} h_{nm} f_{m\infty} (x)) < 2^{-n} \). Thus \( d(f_{i\infty} H F(x), f_{i\infty} (x)) < 3 \cdot 2^{-n+1} \), and since \( n \) was arbitrary \( H \circ F = \text{id} \). Similarly \( F \circ H = \text{id} \).

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