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THE LIE ALGEBRA OF ENDMORPHISMS
OF AN INFINITE-DIMENSIONAL VECTOR SPACE

by

Ian Stewart

1. Introduction

The structure of the Lie algebra of all endomorphisms of a finite-dimensional vector space is well known. The purpose of this paper is to investigate the infinite-dimensional case, and in particular to find the lattice of Lie ideals. Rosenberg [6] has carried out the analogous programme for the infinite general linear group.

Notation for Lie algebras will follow that of [9, 10]. Let \( \mathfrak{f} \) be any field. Let \( c \) be any infinite cardinal, with successor \( c^+ \). Let \( V \) be a vector space over \( \mathfrak{f} \) of dimension \( c \), and for any infinite cardinal \( d \leq c^+ \) define \( E(c, d) \) to be the set of all linear transformations \( \alpha : V \to V \) such that the image of \( \alpha \) has dimension \( < d \). Then \( E(c, d) \) is an associative \( \mathfrak{f} \)-algebra. Under commutation \( [\alpha, \beta] = \alpha\beta - \beta\alpha \) \((\alpha, \beta \in E(c, d))\) it becomes a Lie algebra which we shall denote \( L(c, d) \).

Inside \( L(c, c^+) \) we let \( F = L(c, \mathbb{K}_0) \), \( T \) = the set of endomorphisms of trace zero (in the sense of [9] p. 306), \( S \) = the set of scalar multiplications \( v \to vk \) \((v \in V, k \in \mathfrak{f})\). We shall prove:

**Theorem (A).** Let \( L = L(c, c^+) \). Then the ideals of \( L \) are precisely the following:

a) \( L(c, d) \) for \( \mathbb{K}_0 \leq d \leq c^+ \)
b) \( L(c, d) + S \) for \( \mathbb{K}_0 \leq d \leq c \)
c) Any subspace \( X \) of \( L \) such that \( T \leq X \leq F + S \)
d) \( S \)
e) \{0\}.

The lattice of ideals has the form as shown on the next page.

Further, every subideal of \( L \) is an ideal, so that \( L \) lies in the class \( \Xi \) of [9].

An immediate corollary of theorem 3 is that \( L(c, c^+) \) satisfies the minimal condition for subideals, Min-si. We shall use this to show that theorem 3.3 of [9] p. 305 is in a sense best possible.

Finally we apply our results to prove that any Lie algebra can be embedded in a simple Lie algebra.
I am grateful to the referee for many helpful remarks which have simplified and improved the exposition.

2. The endomorphism algebra

We attack the problem through the associative ideal structure of $E(c, d)$, which is easily determined. By Jacobson [5] p. 108 an associative algebra $A$ is simple if and only if it is simple considered as a ring. This remark combines with a theorem of Herstein [3] (see also Baxter [1]) to yield:

**Lemma (1).** If $A$ is a simple associative $\mathcal{T}$-algebra and $[A, A] = A$ then any proper Lie ideal of the Lie algebra associated with $A$ is contained in the centre of $A$, unless $A$ is of dimension 4 over its centre which is a field of characteristic 2.

In the sequel all algebras considered will be infinite-dimensional over their centres, so the exceptional case never arises. By a slight extension of Jacobson [5] p. 93 theorem 1 we have:

**Lemma (2).** Let $c, d$ be infinite cardinals with $d \leq c^+$. Then any non-zero associative ideal of $E(c, d)$ is of the form $E(c, e)$ where $\aleph_0 \leq e \leq d$.

**Corollary.** If $c \geq d$ are infinite cardinals then

$$ E(c, d^+)/E(c, d) $$

is a simple non-commutative associative algebra.

**Lemma (3).** Let $E = E(c, d)$ where $\aleph_0 < d \leq c^+$. Then

$$ [E, E] = E. $$
PROOF. Let \( a \in E \). Decompose \( V \) into a direct sum
\[
V = X \oplus \bigoplus_{i \in \mathbb{Z}} V_i
\]
in such a way that \( \dim V_i = \dim \text{im}(a) \) for all \( i \) and that \( \text{im}(a) \cong W = \bigoplus_{i \in \mathbb{Z}} V_i \). For each \( i \) let \( t_i : V_i \to V_{i+1} \) be an isomorphism. Let the automorphism \( u : W \to W \) be defined by \( u|_{V_i} = t_i \) and let \( t : V \to V \) be defined by \( t|_W = u \) and \( t(X) = \{0\} \). We shall show that there exists \( b \in E \) such that
\[
[b, t] = a.
\]
More precisely we show that there is a unique endomorphism \( b \) of \( V \) satisfying (1) such that
\[
b(V_0) = \{0\}
\]
and
\[
b(V) \leq W
\]
(hence \( b \in E \)).

We set \( a_i = a|_{V_i} \) and \( b_i = b|_{V_i} \). In view of (3) the restrictions of (1) to \( X \), to \( V_{i-1} \) (\( i > 0 \)) and to \( V_i \) (\( i < 0 \)) are respectively equivalent to the following equations:
\[
b|_X = -u^{-1}a|_X
\]
\[
b_i = (a_{i-1} + t_i b_{i-1}) t_{i-1}^{-1} \quad (i > 0)
\]
\[
b_i = t_{i+1}^{-1}(a_i + b_{i+1} t_i) \quad (i < 0)
\]
and now the assertion is obvious since (5) and (6) constitute inductive definitions for the \( b_i \).

Note that if \( d = \mathfrak{a}_0 \) the lemma is false, for then \([E, E]\) is the set of trace zero maps which is smaller than \( E \).

For any associative algebra \( A \) we let \( Z(A) \) denote the centre of \( A \). We then have:

**LEMMA (4).** If \( c \geq d \) are infinite cardinals, then
\[
Z(E(c, d^+)/E(c, d))
\]
is trivial except when \( c = d \). It then has dimension 1 and consists of scalar multiplications (modulo \( E(c, d) \)).

This follows from:

**LEMMA (5).** If \( c \geq d \) are infinite cardinals and \( z \in L(c, c^+) \) satisfies
\[
[z, L(c, d^+)] \leq L(c, d) + S
\]
then \( z \in L(c, d) + S \).
The proof of this lemma is more intricate than one might wish, and will be postponed until later.

Putting together the results so far obtained we have:

**Lemma (6).** If $c \geq d$ are infinite cardinals then the Lie algebra

$$L(c, d^+)/L(c, d)$$

is simple unless $c = d$; when its only nontrivial proper ideal is the centre, which has dimension 1 and consists of scalar multiplications (modulo $L(c, d)$).

The next result is implicit in [9] (p. 310):

**Lemma (7).** Let $L$ be a Lie algebra, $\sigma$ an ordinal, and $(G_\alpha)_{\alpha \leq \sigma}$ an ascending series of ideals such that for all $\alpha < \sigma$

1) $G_{\alpha+1}/G_\alpha$ is simple non-abelian,
2) $C_{L/G_\alpha}(G_{\alpha+1}/G_\alpha) = G_\alpha/G_\alpha$.

Then the only subideals of $L$ are the $G_\alpha$. Consequently $L \in \text{Min-si} \cap \mathfrak{T}$.

**Proof.** Let $M$ be a proper subideal of $L$ and let $\alpha$ be the least ordinal such that $G_\alpha \subseteq M$. It is easy to see that $\alpha$ cannot be a limit ordinal, so $\alpha = \beta + 1$ for some $\beta$. Thus $(M + G_\beta)/G_\beta$ is a subideal of $L/G_\beta$ not containing $G_{\beta+1}/G_\beta$. As the latter is a simple non-abelian ideal of $L/G_\beta$ we have

$$(M + G_\beta)/G_\beta \cap G_{\beta+1}/G_\beta = G_\beta/G_\beta$$

so by [9] lemma 4.6 p. 309 $M$ centralises $G_{\beta+1}/G_\beta$. By part (2) of the hypothesis $M \subseteq G_\beta$, whence $M = G_\beta$.

Obviously $L \in \mathfrak{T}$, and $L \in \text{Min-si}$ since the ordinals are well-ordered.

Now we shall show that $L(c, d) \in \text{Min-si} \cap \mathfrak{T}$. The presence of trace zero and scalar maps causes complications, so we study a suitable quotient algebra. Let $L = L(c, d)$, let $F, S, T$ be as in theorem A, and put $I = F + S$. Then $L^* = L/I$ has an ascending series of ideals

$$O = L^*_0 \leq L^*_1 \leq \cdots \leq L^*_\alpha \leq \cdots \leq L^*_\delta = L^*$$

for a suitable ordinal $\delta$; the $L^*_\alpha$ being the ideals $(L(c, e) + S)/I$ arranged in ascending order.

Now $I$ has a series $O \leq T \leq F \leq I$ of ideals. But $T$ is simple ([9] lemma 4.1 p. 306) and $F/T$ and $I/F$ are 1-dimensional. Therefore $I \in (\text{Min-si})(\mathfrak{T})(\mathfrak{X}) \leq \text{Min-si}$, by [9] lemma 2.2 p. 303. By the same lemma, in order to prove that $L \in \text{Min-si}$, it suffices to show that $L^* \in \text{Min-si}$.

This will follow from lemma 7 provided we can prove that

$$C_{L^*/L_\alpha}(L^*_{\alpha+1}/L^*_\alpha) = L^*_\alpha/L^*_\alpha$$

which is equivalent to the statement of lemma 5.
We now come to the proof of lemma 5. To simplify the notation we let \( L = L(c, c^+), E = L(c, d), G = L(c, d^+) \). To prove lemma 5 we must show that if \( z \in L \) and \([z, G] \subseteq E + S\), then \( z \in E + S\).

If \( V \) is a vector space with basis \((v_\lambda)_{\lambda \in \Lambda}\) and \( a \) is an endomorphism of \( V \), we define \( a_{\alpha\beta}(\alpha, \beta \in \Lambda) \) by:

\[
v_\alpha a = \sum a_{\alpha\beta} v_\beta.
\]

**Lemma (8).** If \( V \) is a vector space with basis \((v_\lambda)_{\lambda \in \Lambda}\) where \( \Lambda \) is infinite, and if \( a \) is an endomorphism of \( V \) such that \( \dim \text{im}(a) = e \) is infinite, then the set

\[
B = \{ \beta : a_{\alpha\beta} \neq 0 \text{ for some } \alpha \in \Lambda \}
\]

has cardinality \( |B| = e \).

**Proof.** Let \( W = \sum_{\lambda \in B} v_\lambda \). By definition \( \dim(W) = |B| \), and since \( \text{im}(a) \subseteq W \) we have \( e \leq |B| \). If \((i_\mu)_{\mu \in M}\) is a basis for \( \text{im}(a) \), then each \( i_\mu \) is a linear combination of finitely many \( v_\lambda (\lambda \in B) \). Therefore \( |B| \leq |Z \times M| = \aleph_0 \cdot e = e \) since \( e \) is infinite.

We now suppose that \( z \) is as above, and that \( V \) is a vector space with basis \((v_\lambda)_{\lambda \in \Lambda}\) where \( |\Lambda| = c \).

**Lemma (9).** There exists \( z' \) such that \( z'_{\alpha\beta} = 0 \) \((\alpha \in A)\), \([z', G] \subseteq E + S\), and \( z - z' \in E + S\).

**Proof.** Let \( \mathcal{M} \) be the set of all pairs \((M, <)\) where \( M \) is a subset of \( \Lambda \) and \(<\) is a well-ordering on \( M \), such that if \( \alpha \in M \) then \( z_{\alpha\beta} \neq z_{\alpha+1, \beta+1} \) (where \( \alpha + 1 \) is the successor to \( \alpha \) in the ordering \(<\)). Then \( \mathcal{M} \) is partially ordered by \( \ll \), where \((M_1, <_1) \ll (M_2, <_2)\) if and only if \( M_1 \) is an initial segment of \( M_2 \). Clearly \( \mathcal{M} \) is not empty and satisfies the hypotheses of Zorn's lemma. Let \((M, <)\) be a maximal element of \( \mathcal{M} \). Suppose for a contradiction that \( |M| \geq d \). Take an initial segment \( I \) of \( M \) with \( |I| = d \), and consider

\[
t = \left[ z, \sum_{\alpha \in I} e_{\alpha, \alpha+1} \right]
\]

where \( e_{\alpha\beta} (\alpha, \beta \in \Lambda) \) is the elementary transformation sending \( v_\alpha \) to \( v_\beta \) and all other basis elements to zero. By hypothesis \( t \in E + S \), yet

\[
t = \sum z_{\alpha\beta} e_{\beta, \beta+1} - \sum z_{\alpha\beta} e_{\alpha-1, \alpha} e_{\alpha\beta}
\]

\[
= \sum (z_{\alpha, \beta-1} - z_{\alpha+1, \beta}) e_{\alpha\beta}
\]

(where terms involving \( \alpha - 1 \) for limit ordinals \( \alpha \) are deemed to be zero).

Now the coefficient of \( e_{\alpha, \alpha+1} \) is \( z_{\alpha\beta} - z_{\alpha+1, \beta+1} \) which is non-zero for \( d \) values of \( \alpha \). By lemma 8 \( t \notin E + S \) which is a contradiction.

Thus after choosing fewer than \( d \) values of \( \alpha \) all the remaining \( z_{\alpha\beta} \) are equal. Thus \( \sum z_{\alpha\beta} e_{\alpha\beta} \in E + S \). Define \( z' = z - \sum z_{\alpha\beta} e_{\alpha\beta} \).
LEMMA (10). Suppose that \( z' \notin E + S \). Then there exist subsets \( A, A' \) of \( \Lambda \) and a bijection \( \phi : A \to A' \) such that
1) \( A \cap A' = \emptyset \)
2) If \( \phi(\alpha) = \alpha' (\alpha \in A) \) then \( z_{\alpha \alpha'} \neq 0 \)
3) \( |A| = |A'| = d \).

PROOF. Let \( \mathcal{S} \) be the collection of all triples \( (A, A', \phi) \) satisfying (1) and (2). Partially order \( \mathcal{S} \) by \( \ll \) where \( (A, A', \phi) \ll (B, B', \Psi) \) if and only if \( A \subseteq B, A' \subseteq B' \), and \( \Psi|_A = \phi \). By Zorn’s lemma there is a maximal element \( (A, A', \phi) \) of \( \mathcal{S} \). For brevity let \( \phi(\alpha) = \alpha' (\alpha \in A) \). We claim that \( |A| = d \).

Suppose not. Then \( |A| = d' < d \). Let
\[
D = \{ \delta : z_{\delta \delta} \neq 0, \gamma \in A \cup A' \}.
\]
Since \( d \) is infinite we have \( |D| < d \). By lemma 8 there must exist \( \gamma' \notin (A \cup A' \cup D) \) such that \( z_{\gamma' \gamma} = 0 \) for some \( \gamma \neq \gamma' \) (since \( z' \notin E + S \)). Then \( \gamma \notin (A \cup A') \) since \( \gamma' \notin D \). Therefore \( \gamma \neq \gamma', \gamma \notin (A \cup A'), \gamma' \notin (A \cup A') \).
Define
\[
B = A \cup \{ \gamma \}
\]
\[
B' = A' \cup \{ \gamma' \}
\]
\[
\Psi(\beta) = \beta' (\beta \in A)
\]
\[
\Psi(\gamma) = \gamma'.
\]
Then \( (B, B', \Psi) \in \mathcal{S} \) and is greater than \( (A, A', \phi) \), a contradiction. Hence \( |A| \geq d \) as claimed.

We may now derive the final contradiction required to prove lemma 5.
Suppose for a contradiction that \( z' \notin E + S \). Then there exists \( (A, A', \phi) \) as in lemma 10. Define \( \pi : V \to V \) by
\[
v_\alpha \pi = v_{\alpha'} \quad (\alpha \in A)
\]
\[
v_{\alpha'} \pi = v_{\alpha'} \quad (\alpha' \in A')
\]
\[
v_\beta \pi = 0 \quad (\beta \in A \setminus (A \cup A')).
\]
By definition \( \pi \in G \). By hypothesis \( u = [z', \pi] \in E + S \). But for \( \alpha \in A \) we have
\[
v_\alpha (z' \pi - \pi z') = \sum z_{\alpha \beta} v_\beta \pi - \sum z_{\alpha' \beta} v_{\alpha'} \cdot
\]
The coefficient of \( v_{\alpha'} \) is
\[
z_{\alpha \alpha'} + z_{\alpha' \alpha'} - z_{\alpha' \alpha'} = z_{\alpha \alpha'} \neq 0
\]
so that \( u_{\alpha \alpha'} \neq 0 \) if \( \alpha \in A \). Since \( |A| = d \) and \( \alpha \neq \alpha' \) we have \( u \notin E + S \), a contradiction.
Hence \( z' \in E + S \), whence \( z \in E + S \), and lemma 5 is proved. By lemma 7 we have:

**Lemma (11).**
1) \( L(c, c^+) \in \text{Min-si} \),
2) **Every subideal of** \( L(c, c^+) \) **which contains** \( F + S \) **is of the form** \( L(c, d) + S \).

**Lemma (12).** \( L(c, c^+) \in \mathcal{E} \).

**Proof.** Suppose \( L = L(c, c^+) \) has a proper ideal \( J \) of finite codimension. Now \( L \) has an ascending series, the finite-dimensional factors of which are abelian, the rest simple. Hence \( L/J \) is soluble, so that \( [L, L] < L \), contrary to lemma 3. Therefore by theorem 3.1 of [9] p. 305 we have \( L \in \mathcal{E} \).

We now proceed to the:

**Proof of Theorem (A).**

All the subalgebras listed are ideals; the only case requiring comment being (c). Since \( L = L(c, c^+) \) has no ideals of finite codimension (proof of lemma 12) the factor \( (F + S)/T \) is central (see [9] p. 305, proof of theorem 3.1). Therefore any subspace \( X \) between \( T \) and \( F + S \) is an ideal.

Suppose now that \( I \) is an ideal of \( L \). If \( I \geq F + S \) then by lemma 11 \( I \) is in the given list. Therefore we may assume \( I \supseteq F + S \). If \( I \cap T = \{0\} \) then \( [I, T] = \{0\} \). But it is easy to see that the only elements of \( L \) centralising every elementary transformation \( e_{\alpha} \) (\( \alpha \neq \beta \)) are the elements of \( S \). Hence \( I \leq S \). Since \( \dim S = 1 \) we have \( I = \{0\} \) or \( S \). But \( T \) is simple ([9] lemma 4.1 p. 306) so if \( I \cap T \neq \{0\} \) then \( T \leq I \). Now \( I + F + S \triangleleft L \), and by lemma 11 \( I + F + S = L(c, d) + S \) for some \( d \). If \( d = \mathfrak{u}_0 \) then \( T \leq I \leq F + S \), which is case (c) of the list. There remains the case \( d > \mathfrak{u}_0 \). Then we have \((I + F + S)/(T + S) = (L(c, d) + S)/(T + S)\) so that \((I + T + S)/(T + S)\) is of codimension \( \leq 1 \) in \((L(c, d) + S)/(T + S) \cong (L(c, d))/T\) which has no proper ideals of finite codimension by the argument of lemma 12. Therefore \( I + T + S = L(c, d) + S \). Now \( T \leq I \) so we have \( I + S = L(c, d) + S \). If \( I \neq L(c, d) + S \) and \( I \neq L(c, d) \) then \( I \cap L(c, d) \) is of codimension 1 in \( L(c, d) \), contradicting lemma 3. Hence \( I = L(c, d) \) or \( I = L(c, d) + S \).

We have already remarked (in lemma 12) that \( L \in \mathcal{E} \); which completes the proof of the theorem.

### 3. Applications

In [9] it is proved that any Lie algebra satisfying \( \text{Min-si} \) and having no ideals of finite codimension has an ascending series of ideals whose factors are either infinite-dimensional simple or 1-dimensional central. The re-
results of theorem A show that the 1-dimensional central factors cannot in general be dispensed with. In [9] this question was left open. The algebras $L(c, d)$ also provide new examples of Lie algebras in $\text{Min-si } \cap \mathfrak{L}$.

Following the general lines of Scott [7] p. 316 section 11.5.4 (for groups) we can prove:

**Theorem (B).** Any Lie algebra can be embedded in a simple Lie algebra.

**Proof.** Let $K$ be a Lie algebra over a field $k$. By Jacobson [4] p. 162 cor. 4 $K$ has a faithful representation by endomorphisms of a vector space $V$ over $k$. By enlarging $V$ if necessary we may embed $K$ in $L(c^+, c^+)$ for some infinite cardinal $c$. If we split $V$ into $c$ subspaces of dimension $c^+$ and copy the $K$-action on each of these we may assume that $K$ is represented by endomorphisms whose image has dimension $\geq c$. Then the composite embedding

$$K \to L(c^+, c^+) \to L(c^+, c^+)/L(c^+, c)$$

maps $K$ into a simple Lie algebra.

One might ask about Lie analogies of other embedding theorems for groups. For example, Dark [2] has proved that every group can be embedded as a subnormal subgroup of a perfect group. Strangely, the analogue of this is false for Lie algebras – an example may be found in [8], p. 98.

**References**

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