COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 24, nº 2 (1972), p. 227-234
<http://www.numdam.org/item?id=CM_1972__24_2_227_0>

COMPOSITIO MATHEMATICA, Vol. 24, Fasc. 2, 1972, pag. 227–234
Wolters-Noordhoff Publishing
Printed in the Netherlands

BARRELEDNESS OF SUBSPACES OF COUNTABLE CODIMENSION AND THE CLOSED GRAPH THEOREM

by

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Introduction

J. Dieudonné [2] showed that a subspace of a barreled space is barreled if its codimension is finite. A simpler proof of this result was given by M. de Wilde [12].

Recently, I. Amemiya and Y. Komura [1] proved that a subspace of a barreled metrizable space is barreled if its codimension is countable. Their proof is based on the following result: a barreled metrizable space is not the union of an increasing sequence of closed absolutely convex nowhere dense sets. By using a generalized version of the closed graph theorem we show that in an ultrabornological space (i.e. a space which is the inductive limit of a family of Fréchet spaces) every subspace of countable codimension is barreled. Thus metrizability is not a necessary condition. The question whether there exists a barreled space with a non-barreled subspace of countable codimension seems to be open. We show that this problem is related to the other open question of whether or not B_r-completeness is preserved by finite products.

I.

In the following 'space' will always mean 'locally convex topological vector space' and 'subspace' will mean 'linear subspace'.

DEFINITION. A *net* (= réseau, cf. [11]) on a space E is a family $\mathscr{R}$ of subsets of E,

$$E_{n_1, n_2, \cdots, n_k}(k, n_1, \cdots, n_k \in N)$$

indexed by a finite but variable set of natural numbers, such that

$$E = \bigcup_{n_1=1}^{\infty} E_{n_1}$$

and more generally,

$$E_{n_1,\cdots,n_{k-1}} = \bigcup_{n_k=1}^{\infty} E_{n_1,\cdots,n_k}$$

for every $k > 1$, $n_1, \cdots, n_{k-1} \in \boldsymbol{N}$.

$\mathscr{R}$ is called *a net of type* $\mathscr{C}$ if it satisfies the following condition. For every sequence of indices n_k, $k \in \boldsymbol{N}$, there exists a sequence of numbers $\lambda_k > 0$ such that for every choice of $f_k \in E_{n_1,\cdots,n_k}$ and $\mu_k \in [0, \lambda_k]$ the series $\sum_{k=1}^{\infty} \mu_k f_k$ converges in E. It is known that many familiar spaces in functional analysis possess nets of type $\mathscr{C}$ and that the permanence properties for such spaces are rather rich (cf. [11]).

The following closed graph theorem is due to M. de Wilde [11].

THEOREM 1. *If E is ultrabornological and if F possesses a net of type* $\mathscr{C}$, *every linear operator with a sequentially closed graph mapping all of E into F is continuous.*

I. Amemiya and Y. Komura [1] proved

THEOREM 2. *If E is a barreled metrizable space and if L is a subspace of countable codimension then L is also barreled.*

Their proof depends on the following result of category type.

THEOREM 3. *A barreled metrizable space is not the union of an increasing sequence of absolutely convex closed nowhere dense subsets.*

We now show that a result analogous to Theorem 2 also holds for a class of spaces which are not necessarily metrizable. Our proof depends on Theorem 1.

THEOREM 4. *Let E be ultrabornological (hence barreled) and let L be a subspace of countable codimension. Then L is barreled.*

PROOF. Let T be a barrel in L, i.e. a closed absolutely convex absorbing subset of L. T_1 being the closure of T in E, let L_1 be the subspace of E generated by T_1. Then T_1 is surely a barrel in L_1. If we can show that T_1 is a 0-neighborhood in L_1, then $T = T_1 \cap L$ is a 0-neighborhood in L, and therefore L is barreled.

We may assume that $\dim E/L_1 = \infty$, for every subspace of E of finite codimension is barreled (Dieudonné [2]).

Let $x_1, x_2, \cdots, x_n, \cdots$ be any linearly independent sequence such that $E = \text{sp}\{x_1, \cdots, x_n, \cdots\} \oplus L_1$. ($\text{sp}\{x_1, \cdots, x_n, \cdots\}$ denotes the linear hull of $x_1, \cdots, x_n, \cdots$). The gauge of T_1 defines a seminorm p on L_1. Let $L_{1,p}$ be the quotient space L_1/N, with $N = \{x \in L_1 : p(x) = 0\}$, equipped with the norm $||\hat{x}|| = p(x)$ ($\hat{x}$ is the coset of $x \in L_1$). $\tilde{L}_{1,p}$ denotes the completion of $L_{1,p}$.

On the subspace $\text{sp}\{x_1, \cdots, x_n, \cdots\}$ we also consider the locally convex direct sum topology. This is the finest locally convex topology rendering the embeddings $\text{sp}\{x_i\} \to \text{sp}\{x_1, \cdots, x_n, \cdots\}$ continuous $(i = 1, 2, \cdots)$.

From now on we denote by

$$\text{sp}\{x_1, \cdots, x_n, \cdots\} \quad \text{and} \quad \bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$$

the linear hull of the sequence (x_n) with the topology inherited from E and with the locally convex direct sum topology, respectively.

We set

$$F = \Big(\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}\Big) \times \tilde{L}_{1,p},$$

where F has the product topology. It now follows from the results of M. de Wilde [11] that F possesses a net of type $\mathscr{C}$.

Our next objective is to show that the linear map $I : E \to F$ defined by

$$\sum_{i=1}^{k} \alpha_{n_i} x_{n_i} + y \xrightarrow{I} \Big(\sum_{i=1}^{k} \alpha_{n_i} x_{n_i}, \hat{y}\Big) \quad (y \in L_1, \alpha_{n_i} \in C)$$

is closed, and therefore sequentially closed.

Let $(x^{(\alpha)})$ be a net in $\text{sp}\{x_1, \cdots, x_n, \cdots\}$ and $(y^{(\alpha)})$ a net in L_1. Suppose that

$$x^{(\alpha)} + y^{(\alpha)} \to x + y \in E, \quad \text{for } \alpha \to \infty \tag{1}$$
$$(x \in \text{sp}\{x_1, \cdots, x_n, \cdots\}, y \in L_1)$$

and

$$(x^{(\alpha)}, \hat{y}^{(\alpha)}) \to (x', z) \in F, \quad \text{for } \alpha \to \infty \tag{2}$$
$$\Big(x' \in \bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}, z \in \tilde{L}_{1,p}\Big).$$

We must show that $x = x'$ and $\hat{y} = z$.

Since

$$F = \Big(\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}\Big) \times \tilde{L}_{1,p}$$

has the product topology, (2) implies that

$$x^{(\alpha)} \to x' \quad \text{in } \bigoplus_{n=1}^{\infty} \text{sp}\{x_n\} \tag{3}$$

Since the topology of $\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$ is finer than that of

$$\text{sp}\{x_1, \cdots, x_n, \cdots\},$$

(3) implies that

(4) $$x^{(\alpha)} \to x' \quad \text{in sp}\,\{x_1, \cdots, x_n, \cdots\}$$

(4) and (1) together yield that

(5) $$y^{(\alpha)} \to x - x' + y \quad \text{in } E$$

By (2), $(\hat{y}^{(\alpha)})$ is a Cauchy net in $L_{1,p}$.

Hence, for every $\varepsilon > 0$ we have

(6) $$y^{(\alpha)} - y^{(\alpha')} \in \varepsilon T_1 \quad \text{for } \alpha, \alpha' \geqq \alpha_0(\varepsilon)$$

Taking the limit $\alpha \to \infty$ and using the fact that T_1 is closed in E, as well as (5), we find that

$$x - x' + y - y^{(\alpha')} \in \varepsilon T_1 \quad \text{for } \alpha' \geqq \alpha_0(\varepsilon).$$

This implies $x = x'$ and $\hat{y}^{(\alpha)} \to \hat{y}$ in $\tilde{L}_{1,p}$.

On the other hand (2) yields that $\hat{y}^{(\alpha)} \to z$. Therefore $z = \hat{y}$ and the closedness of I is proved.

In virtue of Theorem 1 I is continuous. Then also the restriction $I|_{L_1} : L_1 \to \tilde{L}_{1,p}$ is continuous. Since T_1 is the inverse image of the unit ball of $\tilde{L}_{1,p}$ under $I|_{L_1}$, T_1 is a 0-neighborhood in L_1. This completes the proof.

Implicit in the proof of Theorem 4 is the following

Corollary. *The hypotheses being the same as in Theorem* 4, *if L is closed, then any algebraic complement K of L in E is also a topological complement and K has the finest locally convex topology.*

Proof. Observe that the choice of the algebraic complement

$$K = \text{sp}\,\{x_1, \cdots, x_n, \cdots\}$$

was arbitrary.

Take for T an arbitrary closed absolutely convex 0-neighborhood of L. Since L is closed, $T = T_1$ and $L = L_1$. The continuity of I for an arbitrary T means that the topology of E is finer than that of

$$\Big(\bigoplus_{n=1}^{\infty} \text{sp}\,\{x_n\}\Big) \times L.$$

Obviously it is also coarser. Hence E is isomorphic to

$$\Big(\bigoplus_{n=1}^{\infty} \text{sp}\,\{x_n\}\Big) \times L.$$

Remark 1. We shall see later (cf. Theorem 6) that the above Corollary holds for arbitrary barreled E.

REMARK 2. It is known that a metrizable barreled space contains no closed subspaces of countable codimension (cf. [4]). It is easily seen that such subspaces may very well exist in ultrabornological spaces.

II.

We now try to apply another variant of the closed graph theorem to the problem at hand. This version is due to V. Ptak [7] and A. and W. Robertson [8]. For the proof and the definitions involved we refer to H. H. Schaefer [9].

THEOREM 5. *Any closed linear operator mapping all of a barreled space E into a B_r-complete space F is continuous.*

We note that in comparison with Theorem 1 the conditions on E are weaker here, while on the other hand F is required to be B_r-complete. The applicability of Theorem 5 is rather limited, mainly because very little is known about the permanence properties of B_r- and B-completeness. W. H. Summers [10] has recently exhibited two B-complete spaces the product of which is not B-complete. However, it is not known as yet whether or not B_r-completeness is preserved by finite products. This question is intimately related with the question of the existence of a barreled space with a non-barreled subspace of countable codimension.

In the following, let φ denote the locally convex direct sum of countably many copies of $\boldsymbol{C}$.

STATEMENT. If it is true that the topological product of φ with any Banach space is B_r-complete, then any subspace L of countable codimension of a barreled space E is barreled.

PROOF. We proceed as in the proof of Theorem 4. Since

$$F = \Big(\bigoplus_{n=1}^{\infty} \operatorname{sp}\{x_n\}\Big) \times \tilde{L}_{1,p}$$

has the product topology and

$$\bigoplus_{n=1}^{\infty} \operatorname{sp}\{x_n\}$$

is isomorphic to φ and $\tilde{L}_{1,p}$ is a Banach space, F is B_r-complete by assumption. Instead of Theorem 1 we now apply Theorem 5 to the closed linear operator $I : E \to F$. The conclusion is again that I is continuous. The rest of the proof remains unchanged.

REMARK. Note that φ is B-complete, as it is the Mackey dual of the Fréchet space ω, the product of countably many copies of $\boldsymbol{C}$ (cf. G. Köthe

[5], H. H. Schaefer [9]). Therefore, if B_r-completeness is preserved by finite products, the weaker hypothesis in the above statement is certainly fulfilled.

The next theorem shows that the closed subspaces of countable codimension of a barreled space E are barreled and are exactly those subspaces that have a topological complement in E which is isomorphic to φ.

THEOREM 6. *Let E be barreled and let L be a closed subspace of countable codimension. Then L is barreled. Moreover, any algebraic complement K of L in E is a topological complement and K is isomorphic to φ. Conversely, any subspace of E which has a topological complement isomorphic to φ is barreled (and closed).*

PROOF. Suppose that L is a closed subspace of countable codimension. Let $x_1, \cdots, x_n, \cdots$ be any linearly independent sequence such that $E = \text{sp}\{x_1, \cdots, x_n, \cdots\} \oplus L$. We show that E is isomorphic to the topological product

$$(\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}) \times L,$$

where

$$\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$$

has the locally convex direct sum topology and L the relative topology inherited from E.

It is sufficient to prove that the projection P of E onto

$$\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$$

with null space L is continuous, or equivalently, that the associated 1-1 map

$$\hat{P} : E/L \to \bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$$

is continuous. However, E/L is barreled and

$$\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$$

is B-complete, since it is isomorphic to φ. Since the topology of

$$\bigoplus_{n=1}^{\infty} \text{sp}\{x_n\}$$

is the finest possible, $\hat{P}^{-1}$ is continuous, whence $\hat{P}$ is closed. Theorem 5 now yields that $\hat{P}$ is continuous. Hence E is isomorphic to

$$(\bigoplus_{n=1}^{\infty} \operatorname{sp}\{x_n\}) \times L.$$

This in turn implies that L is isomorphic to

$$E / \bigoplus_{n=1}^{\infty} \operatorname{sp}\{x_n\},$$

which is barreled.

Conversely, suppose that L is a subspace of E such that E is isomorphic to the topological product $\varphi \times L$. Then clearly L is closed and also L is barreled since it is isomorphic to the quotient E/φ.

REMARK. Let E be barreled and L a subspace of E. A property equivalent to L being barreled is that for every barrel T_L in L there exists a barrel T_E in E, such that $T_E \cap L \subset T_L$.

The actual construction of T_E, once T_L is given, is easy in case L has finite codimension (cf. de Wilde [12]). If the codimension of L is countable, this construction, even in the metrizable case, seems by no means easy. It would, however, provide a constructive and possibly elementary proof of I. Amemiya and Y. Komura's result [1].

Finally, as an example of how the foregoing theorems might be of some use, e.g. in approximation theory, we prove

THEOREM 7. *Let K be any compact set in the complex plane. Let $R(K)$ be the linear space of functions which are analytic on a (variable) neighborhood of K, two functions being identified if they coincide on a neighborhood of K. Let $B(K)$ denote the linear space of functions continuous on K and analytic on the interior of K. Then* $\dim B(K)/R(K)$ *is uncountable.*

PROOF. $B(K)$ with the sup norm is clearly a Banach space. $R(K)$ can be topologized so as to become a locally convex space which is the strong dual of a reflexive nuclear Fréchet space. This topology is finer than the norm topology inherited from $B(K)$ (cf. Köthe [3], [5]).

Suppose that $\dim B(K)/R(K)$ is countable. Then by Theorem 2, $R(K)$ with the norm topology inherited from $B(K)$ is barreled. $R(K)$ with its original topology is B-complete, as it is the Mackey dual of a Fréchet space and also nuclear since it is the strong dual of a nuclear Fréchet space. Since both topologies are comparable, the identity map is closed and Theorem 5 implies that the topologies coincide. This cannot be, however, since an infinite-dimensional normed space can never be nuclear, by the Dvoretzky-Rogers theorem (cf. [6]).

Added in proof: After this work was completed, two papers have appeared, by M. Valdivia (Ann. Inst. Fourier, Grenoble 21, 2 (1971), 3–13) and S. Saxon and M. Levin (Proc. Amer. Math. Soc. 29, 1 (1971), 91–96) containing similar results, obtained by different methods.

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(Oblatum 16–II–71)

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