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INFINITE TERMS AND A SYSTEM OF NATURAL DEDUCTION

by

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1. Introduction

Tait 1965 shows how the ordinals associated with the terms of Gödel's theory of primitive recursive functionals of finite type can be found in a perspicuous way by expanding them as infinite terms, the reason for this being that, once the formation of infinite terms is allowed, primitive recursion may be reduced to explicit definition. In this paper we propose a simplified formulation of the infinite terms. Besides being simpler, this formulation has the advantage of bringing out more fully the relation to infinitary proof theory which is implicit in Tait's paper. In fact, it turns out that the main theorem, which says that an infinite term can always be reduced to normal form, bears the same relation to the normal form theorem for natural deductions found by Prawitz 1965 as does Tait's 1968 cut elimination theorem for the classical infinitary propositional logic to Gentzen's Hauptsatz.

2. Infinite terms

We start with at least one *atomic type*. An atomic type is a *type*. If σ and τ are types, then

$$\sigma \rightarrow \tau$$

is a type, namely, the type of a function whose arguments are of type σ and whose values are of type τ. If $\tau_0, \tau_1, \cdots, \tau_n, \cdots$ is a countable sequence of types, then

$$\Pi \tau_n$$

is a type, namely, the type of a function whose arguments are the natural numbers and whose value for the argument n is of type τ_n. We use $\tau_1 \rightarrow \cdots \rightarrow \tau_{n-1} \rightarrow \tau_n$ as an abbreviation of $\tau_1 \rightarrow (\cdots \rightarrow (\tau_{n-1} \rightarrow \tau_n) \cdots)$.

For each type we introduce as many *variables*

$$x, y, z, \cdots$$

of that type as we please.

A variable of type τ is a *term* of type τ. If x is a variable of type σ and $t(x)$ is a term of type τ, then

$$\lambda x t(x)$$

is a term of type $\sigma \to \tau$. If t_n is a term of type τ_n for $n = 0, 1, \cdots$, then

$$(t_0, t_1, \cdots, t_n, \cdots)$$

is a term of type $\Pi\tau_n$. If s and t are terms of type σ and $\sigma \to \tau$, respectively, then

$$ts$$

is a term of type τ. If t is a term of type $\Pi\tau_n$ and n is a natural number, then

$$tn$$

is a term of type τ_n. We use $t_1 t_2 \cdots t_n$ as an abbreviation of

$$(\cdots(t_1 t_2)\cdots)t_n.$$

If x and y are variables of type τ_0 and $\Pi(\tau_n \to \tau_{n+1})$, respectively, then

$$\lambda x \lambda y(x, y0x, y1(y0x), \cdots)$$

is an example of a term of type $\tau_0 \to \Pi(\tau_n \to \tau_{n+1}) \to \Pi\tau_n$, which might be called the recursion operator of that type.

The *immediate subterms* of a term are the terms from which it was obtained by means of one of the four inductive clauses that generate the terms. The *subterms* of a term are the subterms of its immediate subterms, which are called *proper subterms*, and the term itself.

An occurrence of a variable x in a term is *bound* if it occurs in a subterm of the form $\lambda x t(x)$. Otherwise it is *free*. We do not distinguish between terms which only differ in the naming of their bound variables. A term is *closed* if it contains no free variables.

We can now state the two *contraction rules*. The first one is the rule of *λcontraction*

$$\lambda x t(x)s \text{ contr } t(s).$$

Here $t(s)$ denotes the result of substituting s for all free occurrences of x in $t(x)$. Before doing this, however, one has to see to it that no free occurrence of a variable in s becomes bound in $t(s)$. This is achieved by renaming the troublemaking bound variables in $t(x)$. The second contraction rule is the rule of *projection*

$$(t_0, t_1, \cdots)n \text{ contr } t_n.$$

The relation s contr t is read s *contracts* into t.

A term is in *normal form* if it has no contractible subterms.

We shall say that s *reduces* to t if, loosely speaking, t can be obtained from s by repeated contractions of subterms. More precisely, the relation s reduces to t, abbreviated s red t, is defined inductively as follows. If x is a variable, then x red x. If s contr t, then s red t. If $s(x)$ red $t(x)$, then $\lambda xs(x)$ red $\lambda xt(x)$. If s_n red t_n for $n = 0, 1, \cdots$, then $(s_0, s_1, \cdots)$ red $(t_0, t_1, \cdots)$. If r red s, then rt red st, and, if s red t, then rs red rt. If s red t, then sn red tn. Finally, if r red s and s red t, then r red t.

We could equally well formulate our system of terms using *combinators* instead of variables and λabstraction. For every pair of types σ and τ we would then have to introduce Schönfinkel's combinator

$$K \text{ of type } \sigma \to \tau \to \sigma$$

and, for every triple of types ρ, σ and τ, his combinator

$$S \text{ of type } (\tau \to \sigma \to \rho) \to (\tau \to \sigma) \to \tau \to \rho.$$

Moreover, for every type $\Pi\tau_n$ and every n, we need a combinator

$$P_n \text{ of type } \Pi\tau_n \to \tau_n$$

and, for every pair of types σ and $\Pi\tau_n$, a combinator

$$Q \text{ of type } \Pi(\sigma \to \tau_n) \to \sigma \to \Pi\tau_n.$$

Combinators and variables are *combinator terms*. If s and t are combinator terms of type σ and $\sigma \to \tau$, respectively, then ts is a combinator term of type τ. If t_n is a combinator term of type τ_n for $n = 0, 1, \cdots$, then $(t_0, t_1, \cdots, t_n, \cdots)$ is a combinator term of type $\Pi\tau_n$.

There are four rules of contraction, one for each of the basic combinators,

$$Kst \text{ contr } s,$$

$$Srst \text{ contr } rt(st),$$

$$P_n(t_0, t_1, \cdots) \text{ contr } t_n,$$

$$Q(t_0, t_1, \cdots)s \text{ contr } (t_0 s, t_1 s, \cdots).$$

The isomorphism between combinator terms and λterms is established in the usual way, only we have a few more cases to consider. When passing from combinator terms to λterms, we replace P_n and Q by $\lambda x(xn)$ and $\lambda x\lambda y(x0y, x1y, \cdots)$, respectively. Conversely, when defining λabstraction by means of the combinators, we let $\lambda x(t_0(x), t_1(x), \cdots)$ be the combinator term $Q(\lambda xt_0(x), \lambda xt_1(x), \cdots)$, assuming that $\lambda xt_n(x)$ has been defined already for $n = 0, 1, \cdots$.

Our main purpose is to show that every term reduces to normal form. But, before doing this, we want to establish the relation between the system of terms and a certain infinitary propositional logic.

3. Relation to infinitary proof theory

We shall now reformulate the system of λterms as a system of natural deduction.

The *formulae* are built up from at least one *atomic formula* by means of the following two inductive clauses. If F and G are formulae, then

$$F \to G$$

is a formula. If $F_0, F_1, \cdots, F_n, \cdots$ is a countable sequence of formulae, then

$$\wedge F_n$$

is a formula. We use $F_1 \to \cdots \to F_{n-1} \to F_n$ as an abbreviation of $F_1 \to (\cdots \to (F_{n-1} \to F_n) \cdots)$.

We start a *deduction* by making some *assumptions* from which we draw conclusions by repeatedly applying the following *deduction rules*.

$$\to\text{introduction} \qquad \frac{\begin{matrix}[F]\\ G\end{matrix}}{F \to G}$$

$$\begin{matrix}\to\text{elimination}\\ \text{modus ponens}\end{matrix} \qquad \frac{F \quad F \to G}{G}$$

$$\wedge\,\text{introduction} \qquad \frac{F_0 \quad F_1 \cdots F_n \cdots}{\wedge F_n}$$

$$\wedge\,\text{elimination} \qquad \frac{\wedge F_n}{F_n}$$

Here the formula F in the $\to$ introduction rule has been enclosed within square brackets in order to indicate that some occurrences of the formula F as assumptions of the deduction of G have been *discharged.* This means that the assumptions of the deduction of $F \to G$ are the assumptions of the deduction of G minus the occurrences of F which are discharged at the inference from G to $F \to G$. When an assumption is discharged, it must be indicated in some unambiguous way at what inference this happens. For example, Gentzen 1934 marks the assumptions that are discharged by a number and writes the same number at the inference by which they are discharged.

A formula is *provable* if there is a deduction of it all of whose assumptions have been discharged.

If a logical sign is introduced only to be immediately eliminated, we shall say that a *cut* occurs and call the formula whose outermost logical sign is at the same time introduced and eliminated a *cut formula.*

Suppose that a deduction contains a cut formula of the form $F \to G$. We can then simplify, the deduction in the following way, the original deduction being pictured to the left and the simplified one to the right.

$$\to\text{reduction}\qquad \dfrac{\begin{array}{c}\vdots\\ F\end{array}\quad \dfrac{\begin{array}{c}[F]\\ \vdots\\ G\end{array}}{F \to G}}{G}\qquad \begin{array}{c}\vdots\\ F\\ \vdots\\ G\end{array}$$

Similarly, if the cut formula is of the form ΛF_n, we have the following method of simplification.

$$\wedge\,\text{reduction}\qquad \dfrac{\dfrac{\begin{array}{cccc}\vdots & \vdots & & \vdots\\ F_0 & F_1 & \cdots & F_n \cdots\end{array}}{\wedge F_n}}{F_n}\qquad \begin{array}{c}\vdots\\ F_n\end{array}$$

We are now prepared to establish the isomorphism between the system of terms and this system of natural deduction. The following dictionary shows the relation.

atomic type	atomic formula
type	formula
variable	assumption
bound variable	discharged assumption
rule of term formation	deduction rule
term	deduction
λcontraction	$\to$reduction
projection	$\wedge$ reduction
normal term	cut free deduction

Curry and Feys 1958 discovered the analogy between their so called theory of functionality and the positive implicational calculus, and Howard 1969 extended it to Heyting arithmetic. I am indebted to William Howard for pointing out this analogy to me.

The combinator formulation of the terms corresponds to having a formal system of Hilbert type instead of a system of natural deduction. There are four *axioms*,

$$F \to G \to F,$$
$$(F \to G \to H) \to (F \to G) \to F \to H,$$
$$\wedge F_n \to F_n,$$
$$\wedge (F \to G_n) \to F \to \wedge G_n,$$

and just two rules of inference, namely, modus ponens and $\wedge$ introduction. The theorem which says that λabstraction is definable by means of the combinators corresponds to the *deduction theorem* for this system of Hilbert type.

Because of the isomorphism we have established, it is merely a question of terminology and notation whether we formulate our results for terms or for deductions.

4. Ordinals associated with terms

The *degree* $d(\tau)$ of a type τ is inductively defined as follows. $d(\tau) = 0$ if τ is atomic. $d(\sigma \to \tau) = \max(d(\sigma)+1, d(\tau))$. $d(\Pi\tau_n) = \max d(\tau_n)$. Here max is used to denote the least upper bound of a set of ordinals so that the degree of a type is an ordinal of the first or second number class.

It is convenient to carry over some of the terminology introduced for deductions to terms. If a term has a convertible subterm, that is, a subterm of the form $\lambda xt(x)s$ or $(t_0, t_1, \cdots)n$, then the type of $\lambda xt(x)$ and $(t_0, t_1, \cdots)$, respectively, is said to be a *cut type*. The *cut degree* of a term is the maximum of the degrees of all its cut types.

A cut type of $t(s)$ is either a cut type of s, the type of s itself or a cut type of $t(x)$. Consequently, the cut degree of $t(s)$ is at most equal to the maximum of the cut degree of s, the degree of the type of s and the cut degree of $t(x)$.

The *length* $l(t)$ of a term t is defined by the following inductive clauses. $l(x) = 0$ if x is a variable. $l(\lambda xt(x)) = l(t(x))+1$. $l(ts) = \max(l(s)+1, l(t))$. $l((t_0, t_1, \cdots)) = \max l(t_n)$. $l(tn) = l(t)$. The length of a term is also an ordinal of the first or second number class. For example, the length of the recursion operator $\lambda x\lambda y(x, y0x, y1(y0x), \cdots)$ is $\omega+2$.

By a straightforward induction on $t(x)$ it is seen that

$$l(t(s)) \leqq l(s)+l(t(x)).$$

This property will be needed in the proof of the normal form theorem.

We shall not only prove that every term reduces to normal form but also estimate the length of the normal term by means of the length and cut degree of the given term. To this end we need the hierarchy of Veblen 1908 based on the normal function 2^α over the domain $\alpha < \Omega$. Thus, we put $\chi_0(\alpha) = 2^\alpha$ and let χ_β enumerate the common fixed points of all χ_γ with $\gamma < \beta$ when $0 < \beta < \Omega$.

Let χ_β^m denote the mth iterate of the function χ_β. The function by means of which we shall estimate the length of the normal form of a term is

$$\varphi_\beta(\alpha) = \chi_{\beta_1}^{m_1}(\chi_{\beta_2}^{m_2}(\cdots \chi_{\beta_k}^{m_k}(\alpha)\cdots))$$

where

$$\beta = \omega^{\beta_1}m_1+\omega^{\beta_2}m_2+\cdots+\omega^{\beta_k}m_k$$

is the Cantor normal form of $\beta > 0$ and $\varphi_0(\alpha) = \alpha$. The functions φ_β form a solution to the functional equation

$$\varphi_\beta(\varphi_\gamma(\alpha)) = \varphi_{\beta+\gamma}(\alpha)$$

under the initial conditions $\varphi_0(\alpha) = \alpha$ and $\varphi_1(\alpha) = 2^\alpha$. To see this, let $\beta = \omega^{\beta_1}m_1+\cdots+\omega^{\beta_k}m_k$ and $\gamma = \omega^{\gamma_1}n_1+\cdots+\omega^{\gamma_l}n_l$ be the Cantor normal forms of β and γ. Then

$$\beta+\gamma = \omega^{\beta_1}m_1+\cdots+\omega^{\beta_j}m_j+\omega^{\gamma_1}n_1+\cdots+\omega^{\gamma_l}n_l$$

where j is the biggest index i such that $\beta_i \geqq \gamma_1$. On the other hand, since the Veblen functions have the fixed point property,

$$\chi_\beta(\chi_\gamma(\alpha)) = \chi_\gamma(\alpha) \text{ if } \beta < \gamma,$$

we get

$$\varphi_\beta(\varphi_\gamma(\alpha)) = \chi_{\beta_1}^{m_1}(\cdots\chi_{\beta_j}^{m_j}(\chi_{\gamma_1}^{n_1}(\cdots\chi_{\gamma_l}^{n_l}(\alpha)\cdots))\cdots)$$

as desired.

There are three properties of the functions φ_β that we need in the proof of the normal form theorem. First, φ_β is strictly increasing for every β. This is obvious since χ_β is strictly increasing for every β. Second, as we have just proved, $\varphi_\beta(\varphi_\gamma(\alpha)) \leqq \varphi_{\beta+\gamma}(\alpha)$. Third, $\varphi_\beta(\alpha)\cdot 2 \leqq \varphi_\beta(\alpha+1)$ for all $\beta > 0$. It is to attain this for $\beta = 1$ that we have chosen $\varphi_1(\alpha) = \chi_0(\alpha) = 2^\alpha$. We then automatically get

$$\begin{aligned}\varphi_\beta(\alpha)\cdot 2 &= \varphi_1(\varphi_{\beta-1}(\alpha))\cdot 2 = \varphi_1(\varphi_{\beta-1}(\alpha)+1)\\ &\leqq \varphi_1(\varphi_{\beta-1}(\alpha+1)) = \varphi_\beta(\alpha+1)\end{aligned}$$

for all $\beta > 0$.

5. Normal form theorem

As a preliminary step we prove the following simple lemma.

All cut types of the form $\Pi\tau_n$ can be eliminated from a term without increasing its length and cut degree.

When a cut type is eliminated by conversion of a subterm, the degrees of the new cut types that may arise do not exceed the degree of the cut type we are eliminating. Thus, when a term is reduced, its cut degree does not increase.

It remains to prove that we can eliminate all cut types of the form $\Pi\tau_n$ from a given term r without increasing its length. This we do by induction on r, that is, assuming it has been proved already for all proper subterms of r, we prove it for r itself. Basis. $r = x$. Then r is normal already. Induction step. Case 1. $r = \lambda x t(x)$. By induction hypothesis, $t(x)$ red $v(x)$ with $l(v(x)) \leqq l(t(x))$ and no cut types of the form $\Pi\tau_n$. But then r red $\lambda x v(x)$ which has the desired properties. Case 2. $r = (t_0, t_1, \cdots)$. By induction hypothesis, t_n red v_n with $l(v_n) \leqq l(t_n)$ and no undesired cut types. But then r red $(v_0, v_1, \cdots)$ which has the desired properties. Case 3. $r = ts$. By induction hypothesis, s red u and t red v where $l(u) \leqq l(s)$ and $l(v) \leqq l(t)$ and u and v have no undesired cut types. But then r red vu which has the desired properties. Note that vu may have become convertible even if ts were not, but, according to the remark above, the degree of the type of v is then no greater than the cut degree of $r = ts$. Case 4. $r = tn$. By induction hypothesis, t red v with $l(v) \leqq l(t)$ and no undesired cut types. If $v = (v_0, v_1, \cdots)$, then r red v_n which has the desired properties. If v is not of the form $(v_0, v_1, \cdots)$, then r red vn which again has the desired properties.

A term of length α and cut degree $\beta+\gamma$ reduces to a term of length $\leqq \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$.

The proof is by induction on γ. Since $\varphi_0(\alpha) = \alpha$, the theorem holds trivially for $\gamma = 0$. So suppose that $\gamma > 0$ and that the theorem has been proved for all $\delta < \gamma$. We prove it for γ by induction on the term r whose length is α. By the lemma we can assume that r has no cut types of the form $\Pi\tau_n$. Basis. $r = x$. Then r is normal already. Induction step. As in the proof of the lemma we distinguish four cases.

Case 1. $r = \lambda x t(x)$. By induction hypothesis, $t(x)$ red $v(x)$ which has length $\leqq \varphi_\gamma(l(t(x)))$ and cut degree $\leqq \beta$. But then r red $\lambda x v(x)$ which has length $\leqq \varphi_\gamma(l(t(x)))+1 \leqq \varphi_\gamma(l(t(x))+1) = \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$.

Case 2. $r = (t_0, t_1, \cdots)$. By induction hypothesis, t_n red v_n where v_n has length $\leqq \varphi_\gamma(l(t_n))$ and cut degree $\leqq \beta$. But then r red $(v_0, v_1, \cdots)$ which has length $\leqq \max \varphi_\gamma(l(t_n)) \leqq \varphi_\gamma(\max l(t_n)) = \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$.

Case 3. $r = ts$. By induction hypothesis, s red u and t red v where $l(u) \leqq \varphi_\gamma(l(s))$ and $l(v) \leqq \varphi_\gamma(l(t))$ and the cut degrees of u and v are $\leqq \beta$. If v is not of the form $\lambda x w(x)$ we are done, because then r red vu which has length $\leqq \max(\varphi_\gamma(l(s))+1, \varphi_\gamma(l(t))) \leqq \varphi_\gamma(\max(l(s)+1, l(t))) = \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$. In the opposite case, r must have been of the form $\lambda x t(x) s_1 \cdots s_n$ where $s_n = s$ and each s_i is either a term or a natural number. Let the maximum of β and the degrees of the types of the s_i that are terms be $\beta+\delta$. Then $\delta < \gamma$ and $\gamma-\delta \leqq \gamma$. By induction hypothesis, $t(x)$ red $v(x)$ which has length $\leqq \varphi_{\gamma-\delta}(l(t(x)))$ and cut degree $\leqq \beta+\delta$. Also, if s_i is a term, then s_i red u_i which has length $\leqq \varphi_{\gamma-\delta}(l(s_i))$ and cut degree $\leqq \beta+\delta$. If s_i is a natural number, put $u_i = s_i$. Then r red

$\lambda xv(x)u_1 \cdots u_n$ and at most n conversions reduce the latter term to a term w of length $\leqq \max l(u_i)+l(v(x)) \leqq \varphi_{\gamma-\delta}(\max l(s_i))+\varphi_{\gamma-\delta}(l(t(x))) \leqq \varphi_{\gamma-\delta}(\max(l(t(x)), \max l(s_i)))\cdot 2 \leqq \varphi_{\gamma-\delta}(\max(l(t(x)), \max l(s_i))+1) = \varphi_{\gamma-\delta}(\alpha)$ and cut degree $\leqq \beta+\delta$. Finally, w reduces to a term of length $\leqq \varphi_\delta(\varphi_{\gamma-\delta}(\alpha)) = \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$.

Case 4. $r = tn$. By induction hypothesis, t red v which has length $\leqq \varphi_\gamma(l(t))$ and cut degree $\leqq \beta$. If $v = (v_0, v_1, \cdots)$ then r red v_n which has length $\leqq l(v) \leqq \varphi_\gamma(l(t)) = \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$. On the other hand, if v is not of the form $(v_0, v_1, \cdots)$, then r red vn which has length $= l(v) \leqq \varphi_\gamma(l(t)) = \varphi_\gamma(\alpha)$ and cut degree $\leqq \beta$. The proof is finished.

We can now deduce the *normal form theorem.*

A term of length α *and cut degree* β *reduces to a normal term of length* $\leqq \varphi_\beta(\alpha)$.

By the previous theorem, a term of length α and cut degree β reduces to a term of length $\leqq \varphi_\beta(\alpha)$ and cut degree 0, and, furthermore, the lemma allows us to assume that the latter term has no cut types of the form $\Pi\tau_n$. But then it must be normal, for if it had a cut type of the form $\sigma \to \tau$ its cut degree would be $\geqq d(\sigma \to \tau) = \max(d(\sigma)+1, d(\tau)) > 0$.

6. Properties of cut free deductions

In this section we carry over some of the terminology and results of Prawitz 1965 to the infinite natural deductions we are considering.

In an application of modus ponens

$$\frac{F \quad F \to G}{G}$$

F is called the *minor premise* and $F \to G$ the *major premise* of the conclusion G. Every premise of an application of any of the other three deduction rules is a major premise of its conclusion. A sequence $F_1, \cdots, F_n$ of formulae in a deduction form a *branch* if F_1 is a top formula, F_i is a major premise of F_{i+1} for every $i < n$ and F_n is either a minor premise of modus ponens or the end formula. In the latter case the branch is said to be a *main branch.*

A branch of a cut free deduction falls into two parts $F_1, \cdots, F_m$ and $F_m, \cdots, F_n$ the first of which consists entirely of elimination inferences and the second of which consists entirely of introduction inferences, the dividing formula F_m being called the *minimum formula* of the branch. In case $m = 1$ or $m = n$ one of the parts is absent. This property of the branches of a cut free deduction makes it very perspicuous. As a simple application we can prove the *consistency theorem.*

No atomic formula is provable.

If an atomic formula were provable, there would be a cut free deduction of it according to the normal form theorem. Since the end formula is atomic, a main branch of the cut free deduction must consist entirely of elimination inferences. But then the assumption in the beginning of a main branch cannot have been discharged, contradicting the supposition that all assumptions were discharged.

The branches of a deduction can be ordered in a natural way as follows. A main branch is assigned the *order* 0. A branch which ends with a minor premise of an application of modus ponens is assigned the order $n+1$ provided the branch to which the corresponding major premise belongs was assigned the order n. Note that, although a deduction may be infinite, every branch of it has finite order. It is convenient to use the notion of order of a branch when proving the *subformula principle.*

Every formula in a cut free deduction is a subformula of either the end formula or an assumption that has not been discharged.

We prove that the assertion holds for all formulae of a certain branch $F_1, \cdots, F_n$ assuming that it has been proved already for all branches of lower order. The assertion holds for F_n because F_n is either the end formula of the deduction or a minor premise of an application of modus ponens. In the latter case F_n is a subformula of the corresponding major premise which occurs on a branch of one lower order, so that the induction hypothesis applies to it. From F_n the assertion immediately carries over to $F_m, \cdots, F_{n-1}$ where F_m is the minimum formula of the branch. $F_1, \cdots, F_m$ are all subformulae of F_1, so if F_1 is not discharged we are done. In the opposite case F_1 must be discharged by an $\rightarrow$introduction, the conclusion of which is of the form $F_1 \rightarrow G$ and either equals one of $F_{m+1}, \cdots, F_n$ or else occurs on a branch of lower order. In either case we reach the desired conclusion.

As an application of the subformula principle we prove that the calculus we are considering is a conservative extension of the positive implicational calculus.

A cut free deduction of a purely implicational formula from purely implicational hypotheses is purely implicational.

This corollary was suggested by William Howard. It is an immediate consequence of the subformula principle.

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