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JOHN GUCKENHEIMER

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HARTMAN'S THEOREM FOR COMPLEX FLOWS IN THE POINCARÉ DOMAIN

by

John Guckenheimer [1]

We are interested in studying the topological behavior of a complex flow near a generic singular point. Recall the classical analytic theories of Poincaré and Siegel [3, 5]: $x = (x_1, \cdots, x_n)$ are standard complex coordinates in $\boldsymbol{C}^n$. Φ is the holomorphic complex flow generated by the vector field

$$X(x) = \sum_\alpha a_\alpha x^\alpha;\ a_\alpha \in \boldsymbol{C}^n,\ \alpha \in (\boldsymbol{Z}^+)^n,\ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We have made the canonical identification of $\boldsymbol{C}^n$ with the tangent space at each of its points in writing this formula. It is assumed that $X(x)$ has an isolated zero at the origin. One then wishes to know when there is a holomorphic change of coordinates defined in some neighbourhood of the origin which 'linearizes' X. Precisely, this means the following: If h is a holomorphic isomorphism, then h acts on the space of holomorphic vector fields by the conjugation γ_h:

$$\gamma_h(X)(x) = Dh_{h^{-1}(x)} X(h^{-1}(x)).$$

If there is an h defined in a neighborhood of the origin so that

$$\gamma_h(X)(x) = \sum_j b_j x_j,\ b_j \in \boldsymbol{C}^n,$$

then we say h *linearizes* X.

The theories of Poincaré and Siegel begin by formally trying to solve recursion formulas for the Taylor coefficients of a linearization of X. Let A be the matrix of linear coefficients of X:

$$A = (a_\alpha)_{\Sigma\alpha_j = 1}.$$

In formally solving for a linearization h, one finds that if the eigenvalues $\xi_1, \cdots, \xi_n$ of A satisfy a relation of the form

$$(*) \qquad \xi_i = \sum_j \alpha_j \xi_j,\ (\alpha_1, \cdots, \alpha_n) \in (\boldsymbol{Z}^+)^n - \{(0 \cdots 0, 1, 0 \cdots 0)\},$$

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then one cannot even formally solve the recursion formulas for the Taylor expansion of h. This corresponds to the geometrical fact that $X_l(x) = Ax$ has a holomorphic first integral while $X(x)$ generally will not.

At this point the theories of Poincaré and Siegel diverge, depending upon the location of the ξ_i in the complex plane. If all of the ξ_i lie in a half plane whose boundary contains the origin, then Poincaré proves without difficulty that the formal linearization of X converges if no relation (*) holds; thus the formal linearization defines a linearization h. The points of $\boldsymbol{C}^n$ which satisfy a relation (*) and whose coordinates lie in a half plane containing the origin in its boundary form an isolated set. Following Arnold [1], we call $\{z \in \boldsymbol{C}^n : 0 \notin \text{convex hull } \{z_1, \cdots, z_n\}\}$ the Poincaré domain $\prod$.

The complement of $\prod$ in $\boldsymbol{C}^n$ is the Siegel domain Σ. The set of points of Σ satisfying a relation (*) is not isolated in Σ. If a formal linearization of $X(x)$ exists with $(\xi_1, \cdots, \xi_n) \in \Sigma$, it is no longer an easy task to determine whether the formal linearization converges. Siegel's theorem asserts that there is a set $T \subset \Sigma$ of measure zero such that if $(\xi_1, \cdots, \xi_n) \in \Sigma - T$, then X does have a linearization.

Analogous theorems have been proved in the real C^∞ category by Sternberg [7]. Sternberg proves that if the linear part of a smooth vector field X with isolated zero at the origin in $\boldsymbol{R}^n$ has eigenvalues which do not satisfy a relation (*), then there is a local C^∞ diffeomorphism h such that X conjugated by h is a linear vector field near the origin.

Our concern is with cruder results which reflect only the topological structure of a flow. Especially, we want to investigate equivalence relations whose equivalence classes contain open sets in a space of vector fields having a zero at the origin. More specifically. consider the following:

HARTMAN'S THEOREM (Pugh [4]). *Let E be a Banach space and L an isomorphism of E with spectrum disjoint from the unit circle. There exists a $\mu > 0$ such that if λ is a uniformly continuous map from E to E, uniformly bounded by μ and Lipschitz with Lipschitz constant bounded by μ, then there exists a unique homeomorphism h of E such that $h \circ (L+\lambda) = L \circ h$.*

Pugh states that if ϕ_t is a linear flow of E and ψ_t is a flow of E such that ψ_1 satisfies the above hypotheses of Hartman's theorem with respect to the isomorphism ϕ_1, then the h given in the conclusion of Hartman's theorem satisfies $h \circ \psi_t = \phi_t \circ h$ for all t. This follows from the uniqueness of h. Pugh also remarks that one obtains a local theorem at the expense of a uniqueness statement for the conjugacy h.

Our goal is to obtain an analogue of Hartman's theorem for complex flows. Throughout X will denote the linear vector field defined on $\boldsymbol{C}^n$ by

$X(z) = Az$; A is an $n \times n$ complex matrix. Φ will denote the complex flow $\Phi : \boldsymbol{C}^n \times \boldsymbol{C} \to \boldsymbol{C}^n$ obtained from integrating X.

$$\Phi(z, t) = e^{tA}z.$$

Pugh's global formulation of Hartman's theorem is *not* suitable as a model for a theorem about complex flows because non-trivial bounded holomorphic perturbations of X do not exist. Notice that Pugh's version of Hartman's theorem does allow perturbation of the linear part of a vector field, and it is a feature which does admit a complex analogue. Thus, the following question about complex flows is more reasonable if X and $\tilde{X}$ are nearby linear holomorphic vector fields (in the sense that the matrices defining X and $\tilde{X}$ are sufficiently close to one another in $\boldsymbol{C}^{n^2}$), when are the singular foliations of the corresponding flows Φ and $\tilde{\Phi}$ topologically conjugate? Our partial answer to this question is contained in the theorem stated below.

Note that we have asked only for a conjugacy mapping Φ orbits to $\tilde{\Phi}$ orbits and *not* for a simultaneous conjugacy of the isomorphisms $\Phi(\cdot, t)$ and $\tilde{\Phi}(\cdot, t)$, for all t. It is not possible to have such a time preserving conjugacy generally. This is evident from the proof of the theorem.

I have succeeded in establishing an analogue to Hartman's theorem only when the eigenvalues of the matrix A defining X lie in the Poincaré domain $\prod$. Our primary results are the following:

THEOREM. *Suppose Φ is the flow defined by $X(z) = Az$ on $\boldsymbol{C}^n$, A an $n \times n$ matrix. If the eigenvalues of A are distinct and do not contain the origin in their convex hull, and if no two eigenvalues of A lie on the same line through the origin, then Φ is globally stable with respect to linear perturbations and locally stable with respect to arbitrary holomorphic perturbation.*

'Stability' in the conclusion of the theorem means precisely that if $\tilde{A}$ is a matrix sufficiently close to A and $\tilde{\Phi}$ is the flow corresponding to the vector field $\tilde{X}(z) = \tilde{A}(z)$, then there is a homeomorphism h of $\boldsymbol{C}^n$ mapping Φ orbits to $\tilde{\Phi}$ orbits. $\hat{X}$ is a holomorphic vector field C^1 close to X in a neighborhood U of the origin, with corresponding flow $\hat{\Phi}$, then there is a local homeomorphism h defined in a neighborhood V of the origin mapping Φ orbits to $\hat{\Phi}$ orbits.

A converse to the theorem is the following proposition:

PROPOSITION. *If $X(z) = Az$ is a linear vector field on $\boldsymbol{C}^n$ and if two eigenvalues of A lie on the same line through the origin in $\boldsymbol{C}$, then X is not stable.*

PROOF. Let ξ_1, ξ_2 be two eigenvalues lying on the same line through the origin. ξ_1 and ξ_2 are real multiples of one another. Let P be the plane corresponding to the eigenvalues ξ_1 and ξ_2. P is invariant under the flow Φ determined by X. If ξ_1/ξ_2 is irrational, then most Φ orbits in P are homeomorphic to $\boldsymbol{C}$. But, if ξ_1/ξ_2 is rational, all Φ orbits in P are not simply connected. Furthermore, there is not another plane near P invariant under Φ. Since we can pass from ξ_1/ξ_2 rational to ξ_1/ξ_2 irrational and vice-versa by arbitrarily small perturbations, it follows that X is not stable. Even the topological type of orbits is not stable under perturbation.

In the theorem, the eigenvalues of A are assumed to be distinct. By a linear change of coordinates, we may assume that A is a diagonal matrix. A vital observation for the proof of the theorem is contained in the following lemma, also observed by Arnold [1]:

LEMMA. *If $X(z) = Az$ is a linear holomorphic vector field on $\boldsymbol{C}^n$ and if A is a diagonal matrix all of whose eigenvalues lie in a half plane bounded by a line through the origin, then the integral curves of X are transverse to each of the spheres S_r defined by*

$$S_r = \{z | \sum_{j=1}^{n} |z_j|^2 = r\}, \, r > 0.$$

PROOF. Let $\{\xi_j\}$ be the eigenvalues of A and $r > 0$. An integral curve of X can fail to be transverse to S_r at $z \in S_r$ only if the complex multiples of $X(z)$ all lie in the tangent space to S_r at z. Let ω be a normal to S_r at z. As a 1-form,

$$\omega(z) = \sum_j (\bar{z}_j dz_j + z_j d\bar{z}_j),$$

up to a real constant factor. If $\alpha \in \boldsymbol{C}$, then the real inner product of αX with ω is

$$\mathrm{Re}\,(\sum_j \alpha\xi_j |z_j|^2).$$

(Here Re denotes 'the real part of'.) If the tangent space to the integral curve of X lies in the tangent space to S_r at z, then

$$\mathrm{Re}\,(\sum_j \alpha\xi_j |z_j|^2) = 0$$

for all $\alpha \in \boldsymbol{C}$. This clearly implies

$$\sum_j \xi_j |z_j|^2 = 0.$$

But if

$$z \neq 0, \sum_j \xi_j |z_j|^2$$

is a positive multiple of a point in the convex hull of $\{\xi_j\}$. Since 0 does not lie in the convex hull of $\{\xi_j\}$, we conclude that

$$\sum_j \xi_j |z_j|^2 \neq 0$$

and S_r is transverse to the integral curves of X.

REMARK. The lemma remains true if the hypothesis that A be a diagonal matrix is omitted.

It follows from this lemma that the intersections of the integral curves of Φ with S_r form a real, orientable 1-dimensional foliation of S_r. This foliation is defined by a real, non-zero vector field X_r on S_r.

LEMMA. *If $X(z) = Az$ is a linear holomorphic vector field such that the eigenvalues of A all lie in a half plane whose boundary contains the origin, and if no two of the eigenvalues of A lie on the same line through the origin, then the real vector field X_r constructed above is Morse-Smale* [6]. *This means that X_r has a finite number of closed orbits, each is generic, and the stable and unstable manifolds of these closed orbits intersect transversely. There are no recurrent points of X_r other than the closed orbits.*

PROOF. Assume the matrix A is diagonal. Then the intersection of each complex coordinate axis with S_r is a closed orbit of X_r. Since no two eigenvalues of A are rational multiples of one another, all of the integral curves of X, except those lying on the coordinate axis are homeomorphic to $\boldsymbol{C}$. Therefore, if γ were a closed orbit of X_r not given as the intersection of S_r with a coordinate axis, then γ bounds a disk D contained in an integral curve of X. The Euclidean distance function of $\boldsymbol{C}^n$ restricted to D is constant on $\partial D = \gamma$ and hence has a critical point in D. This contradicts the previous lemma, so the only closed orbits of X lie in the coordinate axes.

Next we prove that there is no non-trivial recurrence of X_r. Suppose w and $z \in S_r$ lie on the same integral curve of X_r which is not a closed orbit. Choose two indices k, l so that $z_k z_l \neq 0$. These exist because z is not on a coordinate axis. There is a $z \in \boldsymbol{C}^n$ such that $w = e^{At}z$ or $w_j = z_j e^{\xi_j t}$ since A is a diagonal matrix. If w and z are close to each other in $\boldsymbol{C}^n$, t is close to

$$\frac{2\pi n\sqrt{-1}}{\xi_k} \text{ and } \frac{2\pi m\sqrt{-1}}{\xi_l}$$

for some $m, n \in \boldsymbol{Z}$. Since ξ_k and ξ_l are linearly independent over $\boldsymbol{R}$, this implies t is near zero. Therefore, given $z \in S_r$ such that z is not on a closed orbit of X_r, there is a small neighborhood U of z such that the integral curve of X_r through z has connected intersection with U. It follows that there is no non-trivial recurrence of orbits of X_r, and the non-wandering set of X_r is a finite union of closed orbits.

Next we prove that each closed orbit has a Poincaré transformation with no eigenvalues of modulus 1. The flow determined by X is

$$\Phi(z_1, \cdots, z_n; t) = (z_1 e^{\xi_1 t}, \cdots, z_n e^{\xi_n t}).$$

Thus as t runs over the interval from 0 to $2\pi\sqrt{-1}/\xi_1$, the flow traverses the first closed orbit of X_r. The real hypersurface $H = \{z | z_1 \in \boldsymbol{R}\}$ is mapped into itself by $\Phi(\cdot, 2\pi\sqrt{-1}/\xi_1)$. $H \cap S_r$ is a transverse section to the flow X_r, so that the Poincaré transformation Θ of the first closed orbit of X_r at $(r, 0, \cdots, 0)$ on $S_r \cap H$ is computed explicitly to be

$$\Theta(z_1, \cdots, z_n) = r - (\sum_{j=2}^{n} |z_j \eta_j|^2)^{\frac{1}{2}},\ z_2 \eta_2, \cdots, z_n \eta_n)$$

with

$$\eta_j = e^{2\pi\sqrt{-1}\xi_j/\xi_1}.$$

The derivative of Θ at $(r, 0, \cdots, 0)$ is

$$\left(\begin{array}{c|ccc} * & & 0 & \\ \hline & \eta_2 & & \\ * & & \ddots & \\ & & & \eta_n \end{array}\right)$$

In this matrix, η_j represents a real 2×2 matrix obtained from the standard embedding of $\boldsymbol{C}$ into the ring of 2×2 real matrices. Since $\xi_j/\xi_1 \notin \boldsymbol{R}$ if $j \neq 1$, the eigenvalues of $D\Theta$ have modulus different from 1. The first closed orbit of X_r is generic. Similarly, all the closed orbits of X_r are generic.

It remains to check that the stable and unstable manifolds of X_r have transverse intersection. One sees directly that the stable and unstable manifolds of a closed orbit are each the difference of two linear spans of coordinate axes intersected with S_r. The point $z = (z_1, \cdots, z_k, 0, \cdots, 0)$ lies in the stable manifold of the first closed orbit and the unstable manifold of the k^{th} closed orbit if

$$\arg \xi_1 - \arg \xi_j < 0 \text{ for } 1 < j \leqq k, \text{ and}$$

$$\arg \xi_j - \arg \xi_k < 0 \text{ for } 1 \leqq j < k.$$

Since the eigenvalues of A lie on a half plane containing the origin in its boundary, for $j > k$ either $\arg \xi_1 - \arg \xi_j < 0$ or $\arg \xi_j - \arg \xi_k < 0$. Now the stable manifold of the first closed orbit is open and dense in the linear span of those coordinate axes j for which $\arg \xi_1 - \arg \xi_j < 0$. Similarly, the unstable manifold of the k^{th} closed orbit is open and dense in the linear span of those coordinate axes j for which $\arg \xi_j - \arg \xi_k < 0$. It follows that these unstable manifolds intersect transversely at z. This proves the lemma.

PROOF OF THE THEOREM. A theorem of Palis-Smale [2] implies that X_1 is structurally stable. If $\tilde{X}$ is a linear holomorphic vector field close to X, $\tilde{X}_1$ will be C^1 close to X_1. The theorem of Palis-Smale states that there is a topological conjugacy $h_1 : S_1 \to S_1$ from X_1 to $\tilde{X}_1$. Let $\alpha \in \boldsymbol{C}$ be such that the eigenvalues of αA and $\alpha \tilde{A}$ lie in the right half plane bounded by the imaginary axis. Consider the flows Φ and $\tilde{\Phi}$ along the line determined by α. For $t \in \boldsymbol{R}$, define $R_t = \Phi(S_1, t_\alpha)$, $\tilde{R}_t = \tilde{\Phi}(S_1, t\alpha)$. R_t and $\tilde{R}_t$ each form a disjoint family of nested spheres whose union is $\boldsymbol{C}^n - \{0\}$ and which contracts uniformly to 0 as $t \to -\infty$. Define $h : \boldsymbol{C}^n \to \boldsymbol{C}^n$ by

$$h(0) = 0 \text{ and}$$

$$h|R_t = \tilde{\Phi}_{t\alpha} \circ h_1 \circ \Phi_{-t\alpha} : R_t \to \tilde{R}_t.$$

Since $\Phi_{-t\alpha}(R_t) = S_1$, h is well-defined. Clearly, h is a homeomorphism mapping Φ-orbits to $\tilde{\Phi}$-orbits. This proves the global assertion of the theorem.

The local assertion is proved in the same way. A C^1 perturbation $\hat{X}$ of X will be transverse to S_r for all sufficiently small r. For small enough r, there will be a direction $\alpha \in \boldsymbol{C}$ such that as $t \to -\infty$, $\Phi(S_r, t_\alpha)$ and $\Phi(S_r, t_\alpha)$ contract uniformly to the origin. Thus we can apply the above argument on some neighborhood of the origin, starting the argument with X_r (for some sufficiently small r) rather than with X_1.

REMARK. I have been unable to prove the theorem when the eigenvalues of A lie in the Siegel domain. Such a flow corresponds to a real saddle point in the sense that there are orbits which do not contain the origin in their closure. The spheres S_r are no longer transverse to the integral curves of the flow. It is true, however, that the real quadrics

$$V_r = \{z | \sum_{j=1}^{n} \sigma_j |z_j|^2 = r\}$$

are transverse to the integral curves of X if one chooses $\sigma_j = \pm 1$ so that $(\sigma_j \xi_j)$ lies in the Poincaré domain. But now the V_r are no longer compact, so the Palis-Smale theorem does not apply directly. Furthermore, there are continuity difficulties which arise because the V_r do not form a nested family of spheres.

REFERENCES

V. I. ARNOLD

[1] Remarks on Singularities of Finite Codimension in Complex Dynamical Systems, Functional Analysis and its Applications, v. 3 (1969) 1–5.

J. PALIS AND S. SMALE

[2] Structural Stability Theorems, AMS Proceedings of Symposia in Pure Mathematics, v. XIV (1970) 223–232.

H. Poincaré
[3] Oeuvres, v. I.

C. Pugh
[4] On a Theorem of P. Hartman, Am. Journal of Math., v. XCI, (1969) 363–367.

C. L. Siegel
[5] Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgenrichtslösung, Göttinger Nachrichten der Akademie der Wissenschaften, (1952) 21–30.

S. Smale
[6] Differentiable Dynamical Systems, Bulletin of the Am. Math. Soc. 73 (1967) 747–817.

S. Sternberg
[7] On the Structure of Local Homeomorphisms of Euclidean *n*-space II, Am. Journal Math., v. 80 (1958) 623–631.

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Department of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540
U.S.A.