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IRREGULARITIES OF DISTRIBUTION. VI

by

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1. Introduction

We are interested in the distribution of an arbitrary sequence of numbers in an interval. We are thus returning to questions investigated in the first part [10] of the present series. However, the present paper can be read independently.

Let U be the unit interval consisting of numbers ξ with $0 < \xi \leqq 1$, and let $\omega = \{\xi_1, \xi_2, \cdots\}$ be a sequence of numbers in this interval. Given an α in U and a positive integer n, we write $Z(n, \alpha)$ for the number of integers i with $1 \leqq i \leqq n$ and $0 \leqq \xi_i < \alpha$. We put

$$D(n, \alpha) = |Z(n, \alpha) - n\alpha|.$$

The sequence ω is called *uniformly distributed* if $D(n) = o(n)$, where $D(n)$ is the supremum of $D(n, \alpha)$ over all numbers α in U. Answering a question of Van der Corput [3], Mrs. Van Aardenne-Ehrenfest [1] showed that $D(n)$ cannot remain bounded. Later [2] she proved that there are infinitely many integers n with $D(n) > c_1 \log\log n/\log\log\log n$ where c_1 is a positive absolute constant, and K. F. Roth [9] improved this to $D(n) > c_2(\log n)^{\frac{1}{2}}$.

For $\kappa \geqq 0$ let $S(\kappa)$ be the set of all numbers α in U with

$$D(n, \alpha) \leqq \kappa \qquad (n = 1, 2, \cdots).$$

Further let $S(\infty)$ be the union of the sets $S(\kappa)$, i.e. the set of numbers α in U for which $D(n, \alpha)$ remains bounded as a function of n. Erdös [4, 5] asked whether $S(\infty)$ was necessarily a proper subset of U. This question was answered in the affirmative by the author in the first paper [10] of this series, where among other things it was shown that $S(\infty)$ has Lebesgue measure zero. In the present paper we shall show that $S(\infty)$ is at most a countable set.

Recall that a number γ is a *limit point* of a set S if there is a sequence of distinct elements of S which converge to γ. The *derivative* $S^{(1)}$ of S

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consists of all the limit points of S. The higher derivatives are defined inductively by $S^{(d)} = (S^{(d-1)})^{(1)}$ $(d = 2, 3, \cdots)$. Our main theorem is as follows.

THEOREM. *Suppose $d > 4\kappa$. Then $S^{(d)}(\kappa)$ is empty.*

No special importance attaches to the quantity 4κ which could be somewhat reduced at the cost of further complications. But at the end of this paper we shall exhibit a sequence[2] for which $S^{(d)}(d)$ is not empty for $d = 1, 2, \cdots$, and hence 4κ may not be replaced by $\kappa-\varepsilon$ where $\varepsilon > 0$. One shows easily by induction on d that a set S of real numbers for which $S^{(d)}$ is empty is at most countable and is nowhere dense. We therefore obtain the

COROLLARY. *The sets $S(\kappa)$ are at most countable and they are nowhere dense. The set $S(\infty)$ is at most countable.*

Let θ be irrational and let $\omega = \omega(\theta)$ be the sequence $\{\theta\}, \{2\theta\}, \cdots$ where $\{\ \}$ denotes fractional parts. One can easily show (see Hecke [6], § 6) that the numbers $\{k\theta\}$ where k is an integer belong to $S(\infty)$. In answer to a question by Erdös and Szüsz, it was shown by Kesten [7] that the numbers $\{k\theta\}$ are the only elements of $S(\infty)$. Hence in this case the set $S(\infty)$ is known and is countable.

Now let $\boldsymbol{I}$ be a subinterval of U of the type $\alpha < \xi \leqq \beta$ and put $D(n, \boldsymbol{I}) = |Z(n, \beta)-Z(n, \alpha)-n(\beta-\alpha)|$. If $\omega = \omega(\theta)$, then $D(n, \boldsymbol{I})$ is bounded as a function of n if (Ostrowski [8]) and only if (Kesten [7]) $\boldsymbol{I}$ has length $l(\boldsymbol{I}) = \beta-\alpha = \{k\theta\}$ where k is an integer. Hence in this example there are *continuum many intervals* $\boldsymbol{I}$ for which $D(n, \boldsymbol{I})$ remains bounded.

2. A proposition which implies the theorem

In what follows, U^0 will be the open interval $0 < \xi < 1$. All the numbers $\alpha, \beta, \gamma, \delta, \theta, \eta, \lambda, \mu, \alpha_i, \beta_i, \cdots$ will be in U^0. A *neighborhood* of a number α will by definition be an open interval containing α which is contained in U^0. It will be convenient to extend the definition of the derivatives of a set S by putting $S^{(0)} = S$. By $I, J, \cdots$ we shall denote intervals of the type $a < n \leqq b$ where the end points are integers with $0 \leqq a < b$. Such an interval of length $l(I) = b-a$ contains precisely $l(I)$ integers.

The sequence ω will be fixed throughout. For α in U^0 we put

$$f(n, \alpha) = Z(n, \alpha)-n\alpha,$$

[2] In fact it is Van der Corput's sequence, as constructed in [3].

so that $D(n, \alpha) = |f(n, \alpha)|$. We write

$$g^+(I, \alpha) = \max_{n \in I} f(n, \alpha), \quad g^-(I, \alpha) = \min_{n \in I} f(n, \alpha)$$

and

$$h(I, \alpha) = g^+(I, \alpha) - g^-(I, \alpha).$$

PROPOSITION. *Suppose $d \geqq 0$ and $\varepsilon > 0$. Let R be a set whose d-th derivative $R^{(d)}$ has a non-empty intersection with U^0.*

Then there are $w = 2^d$ elements $\lambda_1, \cdots, \lambda_w$ of R with neighborhoods $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_w$ and a number p such that

$$w^{-1} \sum_{j=1}^{w} h(I, \mu_j) > \tfrac{1}{2}(d+1) - \varepsilon \tag{1}$$

for every interval I with $l(I) \geqq p$ and every $\mu_1 \in \boldsymbol{L}_1, \cdots, \mu_w \in \boldsymbol{L}_w$.

Applying this with $\mu_1 = \lambda_1, \cdots, \mu_t = \lambda_t$ we see that there is a λ_j with $h(I, \lambda_j) > \frac{1}{2}(d+1) - \varepsilon$. There are integers m, n with $f(m, \lambda_j) - f(n, \lambda_j) > \frac{1}{2}(d+1) - \varepsilon$, hence with

$$\max(D(m, \lambda_j), D(n, \lambda_j)) = \max(|f(m, \lambda_j)|, |f(n, \lambda_j)|) > \tfrac{1}{4}(d+1) - \tfrac{1}{2}\varepsilon.$$

This shows that $\lambda_j \notin S(\frac{1}{4}(d+1) - \varepsilon)$.

Now take $R = S(\frac{1}{4}(d+1) - \varepsilon)$. The assumption that an element α of U^0 lies in $R^{(d)}$ leads to the contradiction that $\lambda_j \in R$ and $\lambda_j \notin R$. Hence $R^{(d)} = S^{(d)}(\frac{1}{4}(d+1) - \varepsilon)$ is empty except for the possible elements 0 and 1. At any rate $S^{(d+1)}(\frac{1}{4}(d+1) - \varepsilon)$ is empty for $d \geqq 0$, and hence $S^{(d)}(\frac{1}{4}d - \varepsilon)$ is empty for $d \geqq 1$. It follows that $S^{(d)}(\kappa)$ is empty for $d > 4\kappa$.

Hence our proposition implies the theorem. The proposition will be proved by induction on d. Its generality is necessary to carry out this inductive proof.

3. The case $d = 0$

When $d = 0$ the hypotheses of the proposition are satisfied if R consists of a single element α in U^0. In this case the conclusion must hold with $w = 2^0 = 1$ and with $\lambda_1 = \alpha$. Hence when $d = 0$ the proposition may be reformulated as follows.

LEMMA 1. *Suppose α is in U^0 and $\varepsilon > 0$. There is a neighborhood $\boldsymbol{A}$ of α and a number p such that*

$$h(I, \beta) > \tfrac{1}{2} - \varepsilon \tag{2}$$

for every β in $\boldsymbol{A}$ and every interval I with $l(I) \geqq p$.

For α in U^0 put

$$c(\alpha) = \begin{cases} 0 \text{ if } \alpha \text{ is irrational,} \\ 1/z \text{ if } \alpha = y/z \text{ with coprime positive integers } y, z. \end{cases}$$

Since $0 \leqq c(\alpha) \leqq \frac{1}{2}$, Lemma 1 is a consequence of

LEMMA 2. *The inequality* (2) *in Lemma* 1 *may be replaced by*

$$h(I, \beta) > 1-c(\alpha)-\varepsilon.$$

PROOF. If $\alpha = y/z$, then given any real ψ there are integers m, n with $1 \leqq n \leqq z$ and $|n\alpha - m - \psi| \leqq c(\alpha)/2$. Now suppose that α is irrational. Kronecker's Theorem implies that for every ψ there are positive integers m, n with $|n\alpha - m - \psi| < \varepsilon/8$. Find particular solutions m, n for $\psi = 0$, $\varepsilon/4$, $2\varepsilon/4, \cdots, [4\varepsilon^{-1}]\varepsilon/4$ (where [ ] denotes the integer part), and denote the maximum of the numbers n so obtained by p. Then for every ψ there will be integers m, n with $1 \leqq n \leqq p$ and with $|n\alpha - m - \psi| < \varepsilon/4$. Hence for *every* α in U^0 there is a $p = p(\alpha, \varepsilon)$ such that for every ψ there are integers m, n with

$$1 \leqq n \leqq p \text{ and } |n\alpha - m - \psi| < \tfrac{1}{2}c(\alpha) + \tfrac{1}{4}\varepsilon.$$

Let A be the neighborhood of α consisting of numbers β in U^0 with $|\beta - \alpha| p < \varepsilon/4$. For every β in A and every ψ there are integers m, n with $1 \leqq n \leqq p$ and $|n\beta - m - \psi| < \frac{1}{2}c(\alpha) + \frac{1}{2}\varepsilon$. Since this is true for every ψ, there will also be integers m, n with $1 \leqq n \leqq p$ and $0 < n\beta - m - \psi < c(\alpha) + \varepsilon$. It is clear that the interval $1 \leqq n \leqq p$ may be replaced by any interval I with $l(I) \geqq p$. Thus for every such interval I and every ψ and for every β in A there are integers m, n with

$$n \in I \text{ and } 0 < n\beta - m - \psi < c(\alpha) + \varepsilon.$$

Now choose integers m, n with

$$n \in I \text{ and } 0 < n\beta - m + g^-(I, \beta) < c(\alpha) + \varepsilon.$$

We have $Z(n, \beta) = f(n, \beta) + n\beta \geqq g^-(I, \beta) + n\beta > m$, whence $Z(n, \beta) \geqq m+1$. This implies that

$$\begin{aligned} h(I, \beta) &= g^+(I, \beta) - g^-(I, \beta) \\ &\geqq Z(n, \beta) - n\beta - g^-(I, \beta) \\ &> m + 1 - n\beta + (n\beta - m - c(\alpha) - \varepsilon) \\ &= 1 - c(\alpha) - \varepsilon. \end{aligned}$$

4. Variations on Lemma 2

Write

$$f(n, \alpha, \beta) = f(n, \beta) - f(n, \alpha) = Z(n, \beta) - Z(n, \alpha) - n(\beta - \alpha).$$

LEMMA 3. *Suppose* $\varepsilon > 0$, $q \geqq 1$ *and*

(3) $$0 < |\alpha-\beta| < \varepsilon/(8q).$$

Then there is a p and there are neighborhoods $\boldsymbol{A}$ *of* α *and* $\boldsymbol{B}$ *of* β *such that for every* $\gamma \in \boldsymbol{A}$ *and* $\delta \in \boldsymbol{B}$ *and for every interval I with* $l(I) \geqq p$ *there are two subintervals J and J' with* $l(J) = l(J') = q$ *such that*

(4) $$f(n, \gamma, \delta)-f(n', \gamma, \delta) > 1-\varepsilon$$

for every $n \in J$ *and every* $n' \in J'$.

PROOF. We may assume that $\alpha < \beta$. Put $p_0 = [(\beta-\alpha)^{-1}]$. Every number in U has a distance less than $\varepsilon/8$ from at least one of the numbers $\beta-\alpha$, $2(\beta-\alpha)$, $\cdots$, $p_0(\beta-\alpha)$. Thus for every ψ there are integers m, n with $1 \leqq n \leqq p_0$ and $|n(\beta-\alpha)-m-\psi| < \varepsilon/8$. Let $\boldsymbol{A}, \boldsymbol{B}$ be disjoint neighborhoods of α, β such that elements γ of $\boldsymbol{A}$ and δ of $\boldsymbol{B}$ satisfy

(5) $$16|\gamma-\alpha| \max(q, p_0) < \varepsilon \text{ and } 16|\delta-\beta| \max(q, p_0) < \varepsilon,$$

respectively. For every $\gamma \in \boldsymbol{A}$ and $\delta \in \boldsymbol{B}$ and for every ψ there are integers m, n with $1 \leqq n \leqq p_0$ and $|n(\delta-\gamma)-m-\psi| < \varepsilon/4$, and similarly there are numbers m, n with $1 \leqq n \leqq p_0$ and $0 < n(\delta-\gamma)-m-\psi < \varepsilon/2$. Here the interval $1 \leqq n \leqq p_0$ may be replaced by any interval I_0 with $l(I_0) \geqq p_0$.

Now suppose that $\gamma \in \boldsymbol{A}$, $\delta \in \boldsymbol{B}$ and $l(I_0) \geqq p_0$. Let n_0' be the integer in I_0 with

$$f(n_0', \gamma, \delta) = \min_{n \in I_0} f(n, \gamma, \delta).$$

Choose integers m, n_0 with

$$n_0 \in I_0 \text{ and } 0 < n_0(\delta-\gamma)-m+f(n_0', \gamma, \delta) < \varepsilon/2.$$

We have

$$Z(n_0, \delta)-Z(n_0, \gamma) = f(n_0, \gamma, \delta)+n_0(\delta-\gamma) \geqq f(n_0', \gamma, \delta)+n_0(\delta-\gamma) > m,$$

whence $Z(n_0, \delta)-Z(n_0, \gamma) \geqq m+1$. This implies that

(6) $$\begin{aligned} f(n_0, \gamma, \delta)-f(n_0', \gamma, \delta) &\geqq m+1-n_0(\delta-\gamma)-f(n_0', \gamma, \delta) \\ &> m+1-m-\tfrac{1}{2}\varepsilon \\ &= 1-\tfrac{1}{2}\varepsilon. \end{aligned}$$

Since $\alpha < \beta$ and since $\boldsymbol{A}, \boldsymbol{B}$ are disjoint, any elements $\gamma \in \boldsymbol{A}$ and $\delta \in \boldsymbol{B}$ have $\gamma < \delta$. Moreover by (3) and (5) they satisfy

(7) $$0 < q(\delta-\gamma) < \varepsilon/4.$$

Put $p = p_0+2q$ and let I be an interval with $l(I) \geqq p$. The interval I_0 obtained from I by removing intervals of length q from both ends has

$l(I_0) \geqq p_0$. Hence for every $\gamma \in \boldsymbol{A}$ and $\delta \in \boldsymbol{B}$ there are integers n_0, n'_0 in I_0 with (6). Let J and J' be the intervals

$$n_0 < n \leqq n_0+q \text{ and } n'_0-q < n' \leqq n'_0,$$

respectively. For every n in J and every n' in J' one has

$$f(n, \gamma, \delta)-f(n_0, \gamma, \delta) \geqq -(\delta-\gamma)(n-n_0) \geqq -q(\delta-\gamma) > -\varepsilon/4,$$
$$f(n', \gamma, \delta)-f(n'_0, \gamma, \delta) < \varepsilon/4$$

by (7). These inequalities in conjunction with (6) yield (4). Since J and J' have length q and are contained in I, the lemma follows.

Write

$$g^+(J, \alpha, \beta) = \max_{n \in J} f(n, \alpha, \beta), g^-(J, \alpha, \beta) = \min_{n \in J} f(n, \alpha, \beta).$$

The statement in Lemma 3 that (4) holds for $n \in J$ and $n' \in J'$ may now be expressed by

$$g^-(J, \gamma, \delta)-g^+(J', \gamma, \delta) > 1-\varepsilon.$$

We shall need the function

$$h(J, J', \alpha, \beta) = \max(g^-(J, \alpha, \beta)-g^+(J', \alpha, \beta), g^-(J', \alpha, \beta)-g^+(J, \alpha, \beta)).$$

LEMMA 4. *Suppose $\theta_1, \cdots, \theta_t$ belong to the derivative $R^{(1)}$ of some set R. Let $\boldsymbol{D}_1, \cdots, \boldsymbol{D}_t$ be neighborhoods of $\theta_1, \cdots, \theta_t$, respectively. Suppose $\varepsilon > 0$ and $q \geqq 1$.*

Then there is an r and there are elements $\alpha_1, \beta_1, \cdots, \alpha_t, \beta_t$ of R with neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_t, \boldsymbol{B}_t$ satisfying $\boldsymbol{A}_i \subseteq \boldsymbol{D}_i$, $\boldsymbol{B}_i \subseteq \boldsymbol{D}_i$ $(i = 1, \cdots, t)$ and with the following property. For $\gamma_1 \in \boldsymbol{A}_1$, $\delta_1 \in \boldsymbol{B}_1, \cdots, \gamma_t \in \boldsymbol{A}_t$, $\delta_t \in \boldsymbol{B}_t$ and for intervals I, I' with $l(I) \geqq r$, $l(I') \geqq r$ there are subintervals $J \subseteq I$ and $J' \subseteq I'$ with

$$l(J) = l(J') = q$$

and with

$$h(J, J', \gamma_i, \delta_i) > 1-\varepsilon \qquad (i = 1, \cdots, t). \tag{8}$$

We shall apply this lemma only in the special case when $I = I'$. The general formulation is necessary to carry out a proof by induction on t.

PROOF. Suppose at first that $t = 1$. Since θ_1 is a limit point of R, there are elements α_1, β_1 of R which belong to $\boldsymbol{D}_1$ and which have

$$0 < |\alpha_1-\beta_1| < \varepsilon/(8q).$$

By Lemma 3 there is a p and there are neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1$ of α_1, β_1 such that for every $\gamma_1 \in \boldsymbol{A}_1$, $\delta_1 \in \boldsymbol{B}_1$ and for every I with $l(I) \geqq p$ there are subintervals J_1, J_2 of length q with

(9) $$g^{-}(J_1, \gamma_1, \delta_1) - g^{+}(J_2, \gamma_1, \delta_1) > 1-\varepsilon.$$

We may shrink the neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1$, if necessary, to get $\boldsymbol{A}_1 \subseteq \boldsymbol{D}_1$, $\boldsymbol{B}_1 \subseteq \boldsymbol{D}_1$. If an interval I' also has length $l(I') \geqq p$, then I' has subintervals J_1', J_2' of length q with

(10) $$g^{-}(J_1', \gamma_1, \delta_1) - g^{+}(J_2', \gamma_1, \delta_1) > 1-\varepsilon.$$

By adding (9) and (10) we see that either

$$g^{-}(J_1, \gamma_1, \delta_1) - g^{+}(J_2', \gamma_1, \delta_1) > 1-\varepsilon$$

or

$$g^{-}(J_1', \gamma_1, \delta_1) - g^{+}(J_2, \gamma_1, \delta_1) > 1-\varepsilon.$$

In the first case we take $J = J_1$, $J' = J_2'$, and in the second case we take $J = J_2$, $J' = J_1'$. The inequality (8) is then true for $i = 1$. Hence when $t = 1$, Lemma 4 is true with $r = r^{(1)} = p$.

The induction from $t-1$ to t goes as follows. Construct $\alpha_1, \beta_1, \cdots, \alpha_{t-1}, \beta_{t-1}, \boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_{t-1}, \boldsymbol{B}_{t-1}$ and $r^{(t-1)}$ such that (8) holds (under the conditions stated in the lemma) for $i = 1, \cdots, t-1$. By the case $t = 1$ we can find α_t, β_t in $\boldsymbol{R}$ with neighborhoods $\boldsymbol{A}_t, \boldsymbol{B}_t$ contained in $\boldsymbol{D}_t$ and a number $\bar{r}^{(1)}$ such that for every $\gamma_t \in \boldsymbol{A}_t$, $\delta_t \in \boldsymbol{B}_t$ and for intervals I, I' with $l(I) \geqq \bar{r}^{(1)}$, $l(I') \geqq \bar{r}^{(1)}$ there are subintervals $I_0 \subseteq I$, $I_0' \subseteq I'$ with $l(I_0) = l(I_0') = r^{(t-1)}$ such that

(11) $$h(I_0, I_0', \gamma_t, \delta_t) > 1-\varepsilon.$$

By our construction of $\alpha_1, \beta_1, \cdots, \alpha_{t-1}, \beta_{t-1}, \boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_{t-1}, \boldsymbol{B}_{t-1}$ and $r^{(t-1)}$ there are subintervals $J \subseteq I_0$, $J' \subseteq I_0'$ with $l(J) = l(J') = q$ such that (8) holds for $i = 1, \cdots, t-1$. Now in view of (11) and since $h(J, J', \gamma_t, \delta_t) \geqq h(I_0, I_0', \gamma_t, \delta_t)$, the inequality (8) holds for $i = 1, \cdots, t$. This shows that Lemma 4 is true with $r = \bar{r}^{(1)}$.

5. An inequality

LEMMA 5. *Suppose α, β belong to U^0 and suppose that J, J' are subintervals of an interval I. Then*

(12) $$h(I, \alpha) + h(I, \beta) \geqq h(J, J', \alpha, \beta) + \tfrac{1}{2}(h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta)).$$

PROOF. We may assume without loss of generality that

$$h(J, J', \alpha, \beta) = g^{-}(J, \alpha, \beta) - g^{+}(J', \alpha, \beta).$$

Then we have $f(n, \alpha, \beta) - f(n', \alpha, \beta) \geqq h(J, J', \alpha, \beta)$, i.e.

(13) $$f(n, \beta) - f(n, \alpha) - f(n', \beta) + f(n', \alpha) \geqq h(J, J', \alpha, \beta)$$

for every $n \in J$ and every $n' \in J'$. Let $m_\alpha, n_\alpha, m_\beta, n_\beta$ be integers in J with

$$f(m_\alpha, \alpha) = g^+(J, \alpha), \quad f(n_\alpha, \alpha) = g^-(J, \alpha),$$
$$f(m_\beta, \beta) = g^+(J, \beta), \quad f(n_\beta, \beta) = g^-(J, \beta).$$

Then

$$f(m_\alpha, \alpha) - f(n_\alpha, \alpha) = h(J, \alpha), \tag{14}$$

$$f(m_\beta, \beta) - f(n_\beta, \beta) = h(J, \beta). \tag{15}$$

Similarly, there are elements $m'_\alpha, n'_\alpha, m'_\beta, n'_\beta$ of J' such that

$$f(m'_\alpha, \alpha) - f(n'_\alpha, \alpha) = h(J', \alpha), \tag{16}$$

$$f(m'_\beta, \beta) - f(n'_\beta, \beta) = h(J', \beta). \tag{17}$$

Applying (13) with $n = m_\alpha, n' = m'_\beta$ we obtain

$$f(m_\alpha, \beta) - f(m_\alpha, \alpha) - f(m'_\beta, \beta) + f(m'_\beta, \alpha) \geqq h(J, J', \alpha, \beta).$$

Applying (13) with $n = n_\beta, n' = n'_\alpha$ we obtain

$$f(n_\beta, \beta) - f(n_\beta, \alpha) - f(n'_\alpha, \beta) + f(n'_\alpha, \alpha) \geqq h(J, J', \alpha, \beta).$$

Adding these two inequalities and the four equations (14), (15), (16), (17) we get

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \geqq 2h(J, J', \alpha, \beta) + h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta),$$

where

$$\varphi_1 = f(m'_\alpha, \alpha) - f(n_\alpha, \alpha), \quad \varphi_2 = f(m'_\beta, \alpha) - f(n_\beta, \alpha),$$
$$\varphi_3 = f(m_\beta, \beta) - f(n'_\beta, \beta), \quad \varphi_4 = f(m_\alpha, \beta) - f(n'_\alpha, \beta).$$

Since $h(I, \alpha) \geqq \varphi_1$, $h(I, \alpha) \geqq \varphi_2$, $h(I, \beta) \geqq \varphi_3$, $h(I, \beta) \geqq \varphi_4$, the lemma follows.

6. Proof of the proposition

Lemma 1 shows the truth of the proposition when $d = 0$. From here on we shall have $d \geqq 1$, and we shall assume the truth of the proposition for $d-1$ and proceed to prove it for d.

By this assumption we see that if $\varepsilon > 0$ and if $R^{(d)}$ and U^0 have a non-empty intersection, then there are $t = 2^{d-1}$ elements $\theta_1, \cdots, \theta_t$ of $R^{(1)}$ with neighborhoods $\boldsymbol{D}_1, \cdots, \boldsymbol{D}_t$ and a number $p^{(d-1)}$ such that

$$t^{-1} \sum_{j=1}^{t} h(I, \eta_j) > \tfrac{1}{2}d - \tfrac{1}{2}\varepsilon \tag{18}$$

for $\eta_1 \in \boldsymbol{D}_1, \cdots, \eta_t \in \boldsymbol{D}_t$ and every interval I with $l(I) \geqq p^{(d-1)}$. We now apply Lemma 4 with these particular $\theta_1, \cdots, \theta_t, \boldsymbol{D}_1, \cdots, \boldsymbol{D}_t$ and with

$q = p^{(d-1)}$. We construct elements $\alpha_1, \beta_1, \cdots, \alpha_t, \beta_t$ of R with neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_t, \boldsymbol{B}_t$ and

$$r = r(\theta_1, \cdots, \theta_t; \boldsymbol{D}_1, \cdots, \boldsymbol{D}_t; p^{(d-1)}) \tag{19}$$

with the properties enunciated in that lemma.

Now suppose that $l(I) = r$ and let $\gamma_1, \delta_1, \cdots, \gamma_t, \delta_t$ be elements of $\boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_t, \boldsymbol{B}_t$, respectively. There are subintervals J, J' of I with $l(J) = l(J') = p^{(d-1)}$ such that

$$h(J, J', \gamma_i, \delta_i) > 1-\varepsilon \qquad (i = 1, \cdots, t).$$

Hence by Lemma 5 we have

$$h(I, \gamma_i)+h(I, \delta_i) > (1-\varepsilon)+\tfrac{1}{2}(h(J, \gamma_i)+h(J, \delta_i) + h(J', \gamma_i)+h(J', \delta_i)) \qquad (i = 1, \cdots, t). \tag{20}$$

Now γ_j lies in $\boldsymbol{D}_j$ since $\gamma_j \in \boldsymbol{A}_j$ and $\boldsymbol{A}_j \subseteq \boldsymbol{D}_j$ $(j = 1, \cdots, t)$. We therefore may apply (18) with $\eta_1 = \gamma_1, \cdots, \eta_t = \gamma_t$, and we obtain

$$\sum_{j=1}^{t} h(J, \gamma_j) > t(\tfrac{1}{2}d-\tfrac{1}{2}\varepsilon).$$

More generally, each of the four quantities

$$\chi_1 = \sum_{j=1}^{t} h(J, \gamma_j), \chi_2 = \sum_{j=1}^{t} h(J, \delta_j), \chi_3 = \sum_{j=1}^{t} h(J', \gamma_j), \chi_4 = \sum_{j=1}^{t} h(J', \delta_j)$$

exceeds $t(\frac{1}{2}d-\frac{1}{2}\varepsilon)$. Taking the sum of the inequalities (20) with $i = 1, \cdots, t$ and dividing by $2t$ we obtain

$$(2t)^{-1}\Big(\sum_{i=1}^{t} h(I, \gamma_i)+\sum_{i=1}^{t} h(I, \delta_i)\Big) > (\tfrac{1}{2}-\tfrac{1}{2}\varepsilon)+(4t)^{-1}(\chi_1+\chi_2+\chi_3+\chi_4) > \tfrac{1}{2}(d+1)-\varepsilon. \tag{21}$$

The $w = 2t = 2 \cdot 2^{d-1} = 2^d$ quantities $\lambda_1 = \alpha_1, \cdots, \lambda_t = \alpha_t$, $\lambda_{t+1} = \beta_1, \cdots, \lambda_{2t} = \beta_t$ and their respective neighborhoods $\boldsymbol{L}_1 = \boldsymbol{A}_1, \cdots, \boldsymbol{L}_t = \boldsymbol{A}_t, \boldsymbol{L}_{t+1} = \boldsymbol{B}_1, \cdots, \boldsymbol{L}_{2t} = \boldsymbol{B}_t$ and $p = r$ where r is given by (19) have the desired properties stated in the proposition. Namely, (21) shows that (1) is true for every interval I with $l(I) \geqq p$ and arbitrary elements $\mu_1, \cdots, \mu_w$ in $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_w$.

7. An example

Let R_0 be the set consisting of 0, and for integers $d \geqq 1$ let R_d be the set consisting of 0 and of the numbers

$$2^{-g_1}+ \cdots +2^{-g_t} \tag{22}$$

where $t, g_1, \cdots, g_t$ are integers with

$$1 \leqq t \leqq d \text{ and } 1 \leqq g_1 < g_2 < \cdots < g_t. \tag{23}$$

LEMMA 6. *For every $d \geqq 1$,*

$$R_d^{(1)} = R_{d-1}.$$

PROOF. It is clear that $R_1^{(1)} = R_0$. We now proceed by induction on d and assume that $d \geqq 2$ and that $R_{d-1}^{(1)} = R_{d-2}$. Since the relation $R_{d-1} \subseteq R_d^{(1)}$ is rather obvious, it will remain for us to show that $R_d^{(1)} \subseteq R_{d-1}$.

Let ξ be the limit of a sequence of distinct numbers $\eta(1), \eta(2), \cdots$ of R_d; we have to show that ξ lies in R_{d-1}. We clearly may assume that none of the numbers $\eta(n)$ is 0. Let $t(n), g_1(n), \cdots, g_{t(n)}(n)$ be the numbers $t, g_1, \cdots, g_t$ in (22) which belong to $\eta(n)$. In view of (23) there are only finitely many numbers in R_d for which g_t lies under a given upper bound, and hence $g_{t(n)}(n)$ must tend to infinity. Therefore ξ is also the limit of the sequence $\hat{\eta}(1), \hat{\eta}(2), \cdots$ where

$$\hat{\eta}(n) = \eta(n) - 2^{-g_{t(n)}(n)}.$$

The numbers $\hat{\eta}(n)$ lie in R_{d-1}. If infinitely many of them are equal, then their limit ξ is in R_{d-1}. If infinitely many among them are distinct, then we know by induction that their limit ξ is in R_{d-2}, hence a fortiori in R_{d-1}.

We now construct a sequence $\omega_0 = \{\xi_1, \xi_2, \cdots\}$ as follows. We put $\xi_1 = 0$, and if $k \geqq 0$ and if $\xi_1, \cdots, \xi_{2^k}$ have already been constructed, then we define $\xi_{2^k+1}, \cdots, \xi_{2^{k+1}}$ by

$$\xi_{2^k+t} = \xi_t + \frac{1}{2^{k+1}} \qquad (t = 1, \cdots, 2^k). \tag{24}$$

Thus $\omega_0 = \{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \cdots\}$. In what follows, *the sets $S(\kappa)$ will be defined in terms of this sequence ω_0.*

LEMMA 7. *For every integer $d \geqq 0$,*

$$R_d \subseteq S(d).$$

Repeated application of Lemma 6 shows that $R_d^{(d)}$ consists of 0, and we obtain the

COROLLARY. *The sets $S^{(d)}(d)$ are non-empty for $d = 0, 1, 2, \cdots$.*

PROOF OF LEMMA 7. The assertion is true for $d = 0$ since $S(0)$ contains 0. Assuming the truth of the lemma for $d-1$ we now proceed to prove it for d. It will suffice to show that every element η of R_d of the type

$$\eta = 2^{-g_1} + \cdots + 2^{-g_d}$$

lies in $S(d)$. Put $\hat{\eta} = 2^{-g_1} + \cdots + 2^{-g_{d-1}}$. We know by our inductive hypothesis that

$$|Z(n, \hat{\eta}) - n\hat{\eta}| \leqq d-1 \qquad (n = 1, 2, \cdots). \tag{25}$$

The first 2^{g_d} elements of ω_0 are the numbers $j2^{-g_d}$ $(j = 0, 1, \cdots, 2^{g_d}-1)$ in some order. Hence there is precisely one t_0 with $1 \leqq t_0 \leqq 2^{g_d}$ and $\xi_{t_0} = \hat{\eta}$. The other elements ξ_t with $1 \leqq t \leqq 2^{g_d}$ lie outside the interval $\boldsymbol{I}$ given by

$$\hat{\eta} \leqq \xi < \eta = \hat{\eta} + 2^{-g_d}.$$

Now if $t' = t + m2^{g_d}$ where $1 \leqq t \leqq 2^{g_d}$ and where m is a nonnegative integer, then

$$\xi_t \leqq \xi_{t'} < \xi_t + 2^{-g_d - 1} + 2^{-g_d - 2} + \cdots = \xi_t + 2^{-g_d}$$

by repeated application of (24). Therefore ξ_t lies in $\boldsymbol{I}$ precisely if $t \equiv t_0 \pmod{2^{g_d}}$. This implies that

$$n2^{-g_d} - 1 < Z(n, \eta) - Z(n, \hat{\eta}) < n2^{-g_d} + 1 \qquad (n = 1, 2, \cdots),$$

hence that

$$|Z(n, \eta) - Z(n, \hat{\eta}) - n(\eta - \hat{\eta})| = |Z(n, \eta) - Z(n, \hat{\eta}) - n2^{-g_d}| < 1 \qquad (n = 1, 2, \cdots).$$

Combining this inequality with (25) we obtain $|Z(n, \eta) - n\eta| < d$, which shows that η lies in $S(d)$.

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