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CHOICE SEQUENCES AND MARKOV'S PRINCIPLE [1]

by

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1. Introduction

Bishop has proposed in [1] to avoid some difficulties in the interpretation of intuitionistic implication by adopting instead for his constructive mathematics an interpretation of implication suggested by Gödel's [2].

If quantifiers of all appropriate types and corresponding intuitionistic rules are added to Gödel's theory T of functionals of finite type, three additional principles can be seen to be necessary and sufficient to allow a proof that any formula is equivalent to its normal form for the functional interpretation $\exists\varphi^{\sigma}\forall\chi^{\tau}A$ (A logic free). The first of these is the axiom of choice:

$$*AC: \quad \forall x\exists yA(x,y) \supset \exists\alpha\forall xA(x,\alpha(x)),$$

for x, y variables of any type and α of the appropriate type. The others, presumably non-intuitionistic, are

$$*M: \quad \forall x(A(x)\vee\neg A(x))\;\&\;\neg\forall xA(x)\supset\exists x\neg A(x)$$

(which for x a number variable is just what is usually called Markov's principle), and

$$*C: \quad \forall x(A(x)\vee\neg A(x))\;\&\;(\forall xA(x)\supset\exists yB(y))\supset\exists y(\forall xA(x)\supset B(y)),$$

each to be available for variables of any type. Cf. Yasugi [10].

A detailed investigation of theories extending T to formalize Bishop's mathematics has been carried out by Myhill, who calls the system finally proposed in [9] DQ^{+}.

Viewed formally, Bishop's proposal appears to call for the adoption of a new concept of constructive implication for which $*M$ and $*C$ are valid. (Of course Bishop, like Brouwer, seems to wish to avoid commitment to any specific set of postulates for his logic.)

We shall explore the consequences of adopting this new constructive

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logic (for two types only) for a theory of free choice sequences. Specifically, to Kleene's formal system I of intuitionistic analysis ([6], hereafter cited as *FIM*, and which should be consulted for all otherwise unexplained notations or concepts), we shall add (in effect) as axioms $*M$ (which is then Markov's principle for choice sequences) and a (probably) strengthened $*C$, to obtain a new system I^+. We shall show that I^+ satisfies (Theorem 1) a realizability property, and (Theorem 2) a normal form property for formulas like that obtained in the functional interpretation (but now staying in only two types).

The realizability result is established intuitionistically except for use of $*M$; so in this sense, the extended $*C$ is justified from $*M$.

The theory I^+ cannot claim to be intuitionistic, by reason of Brouwer's explicit denial of Markov's principle for choice sequences. But to refute it the intuitionist, if he follows Brouwer, must use that strongest weapon (Kripke's schema) of whose other proper uses he seems unsure. So the margin of difference is not yet clear.

On the other hand, I^+ is the only known extension of intuitionistic analysis with a normal form property for formulas, no parallel to which exists in present intuitionistic theories. (There is of course no prenex normal form theorem for intuitionistic predicate logic.)

2. Representing higher types in I

In I we cannot deal directly with objects of higher type. Instead we consider species C^h of one-place number-theoretic functions, where each C^h or h corresponds to a level in the finite type structure for the one-place functionals. We call the indices h *C-indices* and define the species of such C-indices inductively as follows.

DEFINITION 1.
1. 0 is a C-index.
2. If i and j are C-indices then so is (i, j).
3. If i and j are C-indices then so is $(i : j)$.
4. An object is a C-index only as required by 1-3.

We abbreviate indices made up by iterated applications of clause 2 by omitting parentheses under a convention of associating to the left. So $(0, 0, 0, 0)$ abbreviates $(((0, 0), 0), 0)$.

For each C-index h a species C^h of number-theoretic functions is now defined. Actually, we want more: we want for each C^h a formula, abbreviated $\alpha \in C^h$, in I which is to express the property $\alpha \in C^h$. We shall define such formulas $\alpha \in C^h$ by induction on h (h and other C-indices i, j, etc. being of course not symbols of I but of the metalanguage).

The corresponding informal species C^h are then just those such that $\alpha \in C^h$ is expressed by the formula $\alpha \in C^h$. Each such species consists of Kleene-style representing functions (cf. [4]) of a certain class of functionals, described below.

We make only slight modification of Kleene's representing functions. We do not insist that for τ computing β from $\alpha : (t)(E!y)\tau(2^{t+1} * \bar{\alpha}(y)) > 0$, but only that $(t)(Ey)\tau(2^{t+1} * \bar{\alpha}(y)) > 0$ (then we can effectively (from τ, α) pick out for each t, $\mu y\tau(2^{t+1} * \bar{\alpha}(y)) > 0$, which will of course be unique). Only this modification enables us to show (Lemma 5 below) that for every h and every sequence number z, z can be extended to give $\alpha \in C^h$.

We shall abbreviate $\forall\alpha(\alpha \in C^h \supset A(\alpha))$ and $\exists\alpha(\alpha \in C^h \,\&\, A(\alpha))$ as $\forall\alpha^h A(\alpha)$ and $\exists\alpha^h A(\alpha)$, and similarly in the informal case. We use $\{\tau\}[\alpha] = \beta$ to abbreviate

$$\forall t \exists y[\tau(2^{t+1} * \bar{\alpha}(y)) = \beta(t)+1 \,\&\, \forall z_{z<y}\tau(2^{t+1} * \bar{\alpha}(z)) = 0].$$

DEFINITION 2.

1. $\tau \in C^0$ is (abbreviates) $\tau = \tau$.

2. If i and j are C-indices and $h = (i, j)$, then $\tau \in C^h$ is $(\tau)_0 \in C^i$ & $(\tau)_1 \in C^j$.

3. If i and j are C-indices and $h = (i : j)$, then $\tau \in C^h$ is

$$\forall\alpha^j \forall t \exists y \tau(2^{t+1} * \bar{\alpha}(y)) > 0 \,\&\, \forall\alpha^j \forall\beta[\{\tau\}[\alpha] = \beta \supset \beta \in C^i].$$

The species C^h is the species of the representing functions of the intensional countable (or continuous) functionals of type corresponding to h. The countable functionals were introduced by Kleene [4] and Kreisel [7]; the intensional ones, by Kreisel [8] p. 154. Both kinds of functionals can be treated in I via representing functions. But for the countable functionals one must insure extensionality by introducing for every h both a species $\bar{C}^h$ and an identity relation I^h such that e.g. when $h = (i : j)$:

$$\forall\alpha^h \forall\beta^h[\forall\gamma^j \forall\delta\forall\zeta[\{\alpha\}[\gamma] = \delta \,\&\, \{\beta\}[\gamma] = \zeta \supset I^i(\delta, \zeta)] \sim I^h(\alpha, \beta)].$$

We are using representing functions without extensionality conditions and this is just the way Kreisel obtains the intensional countable functionals. Thus our formal results translate directly into higher types in the theory of the intensional countable functionals.

3. The system I^+

The system I^+ is obtained from I by adding first as axiom Markov's principle for choice sequences:

$$M : \neg\forall x\alpha(x) = 0 \supset \exists x\alpha(x) \neq 0.$$

(M can be derived in I from the case of $^*M$ above in which x is a number variable, while conversely with M and C below we can derive every instance of $^*M$ in which x is replaced by α^j.)

Secondly, we complete I^+ by adding a principle corresponding to $^*C$. The evident form in which to take $^*C$ in this context would seem to be

$$\forall\alpha^j(A(\alpha)\vee\neg A(\alpha))\ \&\ (\forall\alpha^j A(\alpha)\supset\exists\beta^i B(\beta))\supset\exists\beta^i(\forall\alpha^j A(\alpha)\supset B(\beta)).$$

As will emerge from the work to follow, in the presence of M, this is no stronger than the case:

$$^{**}C \qquad \forall x(A(x)\vee\neg A(x))\ \&\ (\forall x A(x)\supset\exists\beta^i B(\beta))\supset\exists\beta^i(\forall x A(x)\supset B(\beta)).$$

We do not adopt $^{**}C$ but instead an apparently stronger principle. (We have no proof that it is an actual strengthening.) For, it seems appropriate to extend the continuity property (Brouwer's principle of *FIM* § 7) to cover now functionals of arbitrary type. This can be combined conveniently with $^{**}C$ to give us our new axiom schema C. It should be noted that C includes AC.

In stating C, we replace the hypothesis of excluded middle on A by a structural condition which will insure excluded middle, but is more convenient for our purpose, namely that $A(\alpha, x)$ should be constructed from prime formulas by use of propositional connectives and bounded number quantifiers

$$\exists x_{x<s},\ \exists x_{x\leqq s},\ \forall x_{x<s},\ \forall x_{x\leqq s},$$

s a term not containing x (cf. *FIM* Remark 4.1). For short, we say A has no quantifiers except bounded ones. It can be shown that this is equivalent to the principle with the hypothesis of excluded middle on A.

So, for $A(\alpha, x)$ containing no quantifiers except bounded ones, for $B(\beta)$ an arbitrary formula, i and j arbitrary C-indices and $h = (i:j)$, we have as axiom:

$$C: \qquad \forall\alpha^j(\forall x A(\alpha, x)\supset\exists\beta^i B(\beta))\supset$$
$$\exists\tau^h\forall\alpha^j\forall\beta\forall\varphi\{\forall t[\tau(2^{t+1}*\bar{\alpha}(\varphi(t)))=\beta(t)+1\ \&\ \forall z_{z<\varphi(t)}\tau(2^{t+1}*\bar{\alpha}(z))=0]\supset$$
$$\supset[\forall x A(\alpha, x)\supset B(\beta)]\}.$$

Using *FIM* p. 30 $^*D$, $^*E$, etc., we can find a prime formula $G(\tau, \alpha, \beta, \varphi(t), t)$ equivalent to the formula in the scope of $\forall t$, so that C is equivalent in I to

$$C': \qquad \forall\alpha^j(\forall x A(\alpha, x)\supset\exists\beta^i B(\beta))\supset$$
$$\exists\tau^h\forall\alpha^j\forall\beta\forall\varphi\{\forall t G(\tau, \alpha, \beta, \varphi(t), t)\supset[\forall x A(\alpha, x)\supset B(\beta)]\},$$

where:

$\vdash G(\tau, \alpha, \beta, y, t) \sim \tau(2^{t+1} * \bar{\alpha}(y)) = \beta(t)+1 \;\&\; \forall z_{z<y}\tau(2^{t+1} * \bar{\alpha}(z)) = 0.$

To indicate derivability in I^+ we use $\vdash^+$.

4. Realizability in I^+

LEMMA 1. (a) *Let E be a formula of I containing free only the variables Ψ and containing no quantifiers except bounded ones. There is a primitive recursive function $\varepsilon_E[\Psi]$ such that for each Ψ:*

(i) *If* $(E\varepsilon)$ *(ε realizes-Ψ E), then E is true-Ψ.*

(ii) *If E is true-Ψ, then $\varepsilon_E[\Psi]$ realizes-Ψ E.*

PROOF. By induction. Cf. *FIM* Lemma 8.4a. As there, if E is of form $P, A \;\&\; B, A \supset B, \neg A$, let $\varepsilon_E[\Psi] = \lambda t0, \langle\varepsilon_A[\Psi], \varepsilon_B[\Psi]\rangle, \Lambda\alpha\varepsilon_B[\Psi], \lambda t0$.

CASE 5. E is $A \vee B$. Let $\varphi_A(\Psi)$ be the primitive recursive characteristic function of the predicate A expressed by A (*FIM* Lemma 3.3 and Remark 3.4). Let

$$\varepsilon_E[\Psi] = \langle\lambda t\varphi_A(\Psi), \lambda t(sg(\varphi_A(\Psi)) \cdot \varepsilon_A[\Psi](t) + sg(\varphi_A(\Psi)) \cdot \varepsilon_B[\Psi](t))\rangle.$$

CASE 6. E is $\forall x_{x<s}A(x)$. Let $\varepsilon_E[\Psi] = \Lambda x \Lambda \pi \varepsilon_{A(x)}[\Psi]$.

CASE 7. E is $\exists x_{x<s}A(x)$. Let $\varepsilon_E(\Psi)$ be

$$\langle\lambda t\mu x_{x<s(\Psi)}A(\Psi, x), \varepsilon_{A(x)}[\Psi, \mu x_{x<s(\Psi)}A(\Psi, x)]\rangle,$$

where $s(\Psi)$ and $A(\Psi, x)$ are the primitive recursive function and predicate expressed by s and A, respectively.

LEMMA 1. (b) *For A as E in part* (a) *and $\Psi = a, \Psi_1$, there is a primitive recursive function $\varepsilon_{\forall aA}[\Psi]$ such that for every Ψ_1:*

(i) $(E\varepsilon)$ *(ε realizes-$\Psi_1 \forall aA$) $\rightarrow \forall aA$ is true-Ψ_1.*

(ii) $\forall aA$ *is true-$\Psi_1 \rightarrow \varepsilon_{\forall aA}[\Psi_1]$ realizes-$\Psi_1 \forall aA$.*

(c) *Similarly, for $\Psi = \alpha, \Psi_1$ there is a primitive recursive function $\varepsilon_{\forall\alpha A}[\Psi]$ such that for every Ψ_1:*

(i) $(E\varepsilon)$ *(ε realizes-$\Psi_1 \forall\alpha A$) $\rightarrow \forall\alpha A$ is true-Ψ_1.*

(ii) $\forall\alpha A$ *is true-$\Psi_1 \rightarrow \varepsilon_{\forall\alpha A}$ realizes-$\Psi_1 \forall\alpha A$.*

PROOFS. (b) Let $\varepsilon_{\forall aA}[\Psi_1] = \Lambda a\varepsilon_A[a, \Psi_1]$, for $\varepsilon_A[a, \Psi_1]$ obtained by part (a) of the lemma.

(c) $\varepsilon_{\forall\alpha A}[\Psi_1] = \Lambda\alpha\varepsilon_A[\alpha, \Psi_1]$.

LEMMA 2. *For every C-index h there is a partial recursive function $\varepsilon_h[\tau]$ such that*

(i) $(E\varepsilon)$ *(ε realizes-τ $\tau \in C^h$) $\rightarrow \tau \in C^h$.*

(ii) $\tau \in C^h \rightarrow \varepsilon_h[\tau]$ *realizes-τ $\tau \in C^h$.*

PROOF. By induction on h we show: the formula $\tau \in C^h$ contains no $\vee$ and all occurrences of $\exists$ are in parts of the form $\exists x P(x)$ with $P(x)$ containing no quantifiers except bounded ones, and hence expressing a primitive recursive predicate. The proof of *FIM* Lemma 8.4b applies.

LEMMA 3. *For every C-index h*:

$$\vdash^{+} \neg\neg\, \alpha \in C^h \sim \alpha \in C^h.$$

PROOF. By ind. on h. IND. STEP.

CASE 1. $h = (i,j)$.
Then

$$\begin{aligned} \neg\neg\, \alpha \in C^h &\sim \neg\neg\, ((\alpha)_0 \in C^i \,\&\, (\alpha)_1 \in C^j) \sim \\ &\neg\neg(\neg\neg(\alpha)_0 \in C^i \,\&\, \neg\neg(\alpha)_1 \in C^j) \text{ [hyp. ind.]} \sim \\ &\neg\neg(\alpha)_0 \in C^i \,\&\, \neg\neg(\alpha)_1 \in C^j \text{ [}^*\text{60d-f]} \sim \\ &(\alpha)_0 \in C^i \,\&\, (\alpha)_1 \in C^j \text{ [hyp. ind.]} \sim \alpha \in C^h. \end{aligned}$$

CASE 2. $h = (i:j)$. We need show only $\supset$. Let the conjunction $\alpha \in C^h$ be abbreviated $E(\alpha) \,\&\, F(\alpha)$. Assume $\neg\neg\alpha \in C^h$. By *25, (i) $\neg\neg E(\alpha) \,\&\, \neg\neg F(\alpha)$.

We shall deduce first $E(\alpha)$. Assume (ii) $\exists\beta^j \exists t \forall y \neg\, \alpha(2^{t+1} * \bar{\beta}(y)) > 0$. Assume $\beta \in C^j \,\&\, \forall y \neg\, \alpha(2^{t+1} * \bar{\beta}(y)) > 0$. Then $\neg\forall\beta^j \forall t \exists y \alpha(2^{t+1} * \bar{\beta}(y)) > 0$, i.e. $\neg E(\alpha)$, contradicting (i). So, rejecting (ii), (iii) $\neg\exists\beta^j \exists t \forall y \neg\, \alpha(2^{t+1} * \bar{\beta}(y)) > 0$. Now assume (iv) $\beta \in C^j$. Using (iii), $\neg\exists t \forall y \neg\, \alpha(2^{t+1} * \bar{\beta}(y)) > 0$, whence using $M: \forall t \exists y \alpha(2^{t+1} * \bar{\beta}(y)) > 0$. By $\supset$-introd. from (iv), etc.: (v) $E(\alpha)$.

Next we shall deduce $F(\alpha)$. Assume (vi) $\exists\beta^j \exists\gamma \exists\varphi[\forall t G(\alpha, \beta, \gamma, \varphi(t), t) \,\&\, \neg\, \gamma \in C^i]$. (Cf. end § 3.) Then $\neg\forall\beta^j \forall\gamma \forall\varphi[\forall t G(\alpha, \beta, \gamma, \varphi(t), t) \supset \gamma \in C^i]$, i.e. $\neg F(\alpha)$, contradicting (i). So rejecting (vi), (vii) $\neg\exists\beta^j \exists\gamma \exists\varphi[\forall t G(\alpha, \beta, \gamma, \varphi(t), t) \,\&\, \neg\, \gamma \in C^i]$. Now assume (viii) $\beta \in C^j$ and (ix) $\forall t G(\alpha, \beta, \gamma, \varphi(t), t)$. Then using (vii), $\neg\neg\, \gamma \in C^i$, whence by ind. hyp., $\gamma \in C^i$. By $\supset$-introd. from (ix), then $\forall$-introds., $\supset$-introd. from (viii), $\forall$-introd.: (x) $F(\alpha)$.

Combining (v) and (x): $\alpha \in C^h$.

In Lemma 5 below we establish that for every C-index h, every finite sequence of natural numbers can be continued to give a function in C^h. We need this (a) as a formal result in I^+ and (b) as an informal result with in this case the additional information that the continuation can be given primitive recursively.

Lemma 4 provides two results needed in the proof of Lemma 5. (Proofs in these lemmas hold in Kleene's basic system B; cf. *FIM* p. 8.)

LEMMA 4. (a) $\forall s \forall z \forall w[B(s, z) \,\&\, w \geqq z \supset B(s, w)]$
$\vdash_B \forall s_{s<b} \exists z B(s, z) \supset \exists z \forall s_{s<b} B(s, z)$

(b) $\vdash_B \mathrm{Seq}(x) \,\&\, m < (x)_0 \,\&\, y < lh(x) \dot{-} 1 \supset$
$2^{m+1} * \prod_{k<y} p_k^{(x)_{k+1}} < x.$

PROOFS. (a) Like *FIM* *26.5.

(b) Let the hyps. be (i)–(iii), respectively.

By (ii), $m < (x)_0$; so (iv) $2^{m+1} \leqq 2^{(x)_0}$.

By (iii), $y \leqq lh(x) \dot{-} 2 < lh(x) \dot{-} 1$.

So (v) $\prod_{k<y} p_{k+1}^{(x)_{k+1}} \leqq \prod_{k<lh(x)\dot{-}2} p_{k+1}^{(x)_{k+1}} <$
$\prod_{k<lh(x)\dot{-}1} p_{k+1}^{(x)_{k+1}}$ [using also (i)].

Thus $2^{m+1} * \prod_{k<y} p_k^{(x)_{k+1}} \leqq 2^{(x)_0} \cdot \prod_{k<y} p_{k+1}^{(x)_{k+1}}$[(iv), $^{\mathrm{x}}$21.1] $<$
$2^{(x)_0} \cdot \prod_{k<lh(x)\dot{-}1} p_{k+1}^{(x)_{k+1}}$[(v)] $= x$.

LEMMA 5. (a) *For every C-index h*

$$\vdash_B \exists\tau\forall z_{\mathrm{Seq}(z)} \mathrm{Ext}^h(\tau, z),$$

where $\mathrm{Ext}^h(\tau, z)$ *abbreviates* $\lambda s\tau(\langle z, s\rangle) \in C^h \,\&\, \forall s_{s<lh(z)} \tau(\langle z, s\rangle)+1 = (z)_s$.

(b) *For every C-index h there is a primitive recursive function τ such that* $(z)_{\mathrm{Seq}(z)} \mathrm{Ext}^h(\tau, z)$, *where* $\mathrm{Ext}^h(\tau, z)$ *is expressed by the formula* $\mathrm{Ext}^h(\tau, z)$ *of* (a).

PROOF. We give a detailed formal proof of (a). The corresponding informal argument establishes (b), when supplemented by the observations that the definitions of τ in Case 1 (iii) and Case 2 (iii) below are primitive recursive (using ind. hyp. in Case 2), and that this remains true if we consider these definitions as yielding functions $\tau_0(z, x)$ of two variables z and x. Then in the final paragraph let $\tau(\langle z, x\rangle) = \tau_0(z, x)$ and avoid the application of $^{\mathrm{x}}$2.1. (Alternatively, we could establish (b) with a general recursive, but not primitive recursive, τ by using (a), *FIM* Theorem 9.3(a) with Lemma 8.4b (i), etc.)

PROOF of (a). By induction on h.

IND. STEP. CASE 1: $h = (i, j)$. Assume from ind. hyp.

(i) $\forall z_{\mathrm{Seq}(z)} \mathrm{Ext}^i(\tau_i, z)$ and (ii) $\forall z_{\mathrm{Seq}(z)} \mathrm{Ext}^j(\tau_j, z)$.

Introduce τ:

$$\text{(iii)} \qquad \forall x \tau(x) = \begin{cases} ((x)_0)_{(x)_1} \dot{-} 1 \text{ if } (x)_1 < lh((x)_0), \\ < \tau_i(\langle \prod_{k<lh((x)_0)} p_k \exp((((x)_0)_k \dot{-} 1)_0 + 1), (x)_1 \rangle), \\ \tau_j(\langle \prod_{k<lh((x)_0)} p_k \exp((((x)_0)_k \dot{-} 1)_1 + 1), (x)_1 \rangle\rangle \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Assume (iv) Seq(z). We easily deduce from (iii) and (iv):

$$\text{(v)} \qquad \forall s_{s<lh(z)} \tau(\langle z, s \rangle) + 1 = (z)_s.$$

Next we shall deduce

(a) $(\lambda s \tau(\langle z, s \rangle))_0 \in C^i$. A parallel deduction would give

(b) $(\lambda s \tau(\langle z, s \rangle))_1 \in C^j$. Let (vi) $w = \prod_{k<lh(z)} p_k \exp(((z)_k \dot{-} 1)_0 + 1)$; so $lh(w) = lh(z)$. From (i): (vii) $\text{Ext}^i(\tau_i, w)$ whence (viii) $\lambda s \tau_i(\langle w, s \rangle) \in C^i$. We shall deduce (ix) $(\tau(\langle z, s \rangle))_0 = \tau_i(\langle w, s \rangle)$, whence by $\forall$-introd. and (viii): (a). For (ix) use cases ($s < lh(w)$, $s \geqq lh(w)$).

Case A: $s < lh(w) = lh(z)$. Then $(\tau(\langle z, s \rangle))_0 = ((z)_s \dot{-} 1)_0$ [case A hyp., (iii)] $= (w)_s \dot{-} 1$ [(vi)] $= \tau_i(\langle w, s \rangle)$ [(vii)].

Case B: $s \geqq lh(w) = lh(z)$. Then $(\tau(\langle z, s \rangle))_0 = \tau_i(\langle \prod_{k<lh(z)} p_k \exp(((z)_k \dot{-} 1)_0 + 1), s \rangle)$[(iii)] $= \tau_i(\langle w, s \rangle)$[(vi)].

Now from (a) and (b): $\lambda s \tau(\langle z, s \rangle) \in C^h$. Then with (v): $\text{Ext}^h(\tau, z)$.

Case 2: $h = (i : j)$. From ind. hyp.: (i) $\forall z_{\text{Seq}(z)} \text{Ext}^i(\tau, z)$. (We do not need the part of the ind. hyp. for j.)

First we shall deduce (a) $\forall z_{\text{Seq}(z)} \exists \tau^h \bar{\tau}(lh(z)) = z$. So, assume (ii) Seq($z$). We use the following abbreviations for terms in introducing τ in (iii) below.

$$q(x, z) = \mu m_{m<x} 2^{m+1} \geqq lh(z)$$

$$k(\tau, x, m) = \mu y_{y<lh(x) \dot{-} 1} \tau(2^{m+1} * \prod_{n<y} p_n^{(x)_{n+1}}) > 0$$

$$u(\tau, x, z) = \prod_{m<q((x)_0, z)} p_m \exp(\tau(2^{m+1} * \prod_{n<k(\tau, x, m)} p_n^{(x)_{n+1}})).$$

Now we propose to introduce τ under Lemma 5.5(c). Verification that the cases in (iii) are exhaustive and exclusive is routine. To show that the fourth case could be brought under (c) of the lemma by appropriate use of *B, etc., use Lemma 4(b) to justify the case hyp. To justify the corresponding definiens, assume $m < q((x)_0, z)$. Under the case hyp.,

$$\text{Seq}(x) \;\&\; \exists y_{y<lh(x) \dot{-} 1} \tau(2^{m+1} * \prod_{k<y} p_k^{(x)_{k+1}}) > 0.$$

So by *E6, $k(\tau, x, m) < lh(x) \dot{-} 1$. Then by Lemma 4(b)

$2^{m+1} * \prod_{n<k(\tau,x,m)} p_n^{x_n+1} < x$. So $u(\tau, x, z)$ can be expressed from $\bar{\tau}(x)$ and z.

$$\text{(iii)} \qquad \forall x \tau(x) = \begin{cases} (z)_x \dot{-} 1 & \text{if } x < lh(z), \\ \tau_i(\langle z, (x)_0 \dot{-} 1 \rangle) + 1 & \text{if } x \geqq lh(z) \;\&\; lh(z) \leqq 1, \\ 1 & \text{if Seq}(x) \;\&\; x \geqq lh(z) \;\&\; 2^{(x)_0} < lh(z), \\ \tau_i(\langle u(\tau, x, z), (x)_0 \dot{-} 1 \rangle) + 1 & \text{if Seq}(x) \;\&\; 2^{(x)_0} \geqq lh(z) \;\&\; lh(z) > 1 \;\&\; \\ & \forall m_{m<q((x)_0, z)} \exists y_{y<lh(x) \dot{-} 1} \tau(2^{m+1} * \prod_{n<y} p_n^{(x)_{n+1}}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then: (b) $\bar{\tau}(lh(z)) = z$.
To obtain $\tau \in C^h$ we must deduce

(c) $\forall \alpha^j \forall t \exists y \tau(2^{t+1} * \bar{\alpha}(y)) > 0,$

(d) $\forall \alpha^j \forall \beta [\forall t \exists y G(\tau, \alpha, \beta, y, t) \supset \beta \in C^i]$.(Cf. end § 3.)

Towards (c), we shall deduce first:

(iv) $\forall t (2^{t+1} < lh(z) \supset \exists y \tau(2^{t+1} * \alpha(y)) > 0)$. Assume $2^{t+1} < lh(z)$.

Letting $x = 2^{t+1} * \bar{\alpha}(lh(z))$, we can deduce

$$\text{Seq}(x),\ x > lh(x) = lh(z)+1 > lh(z), \text{ and } 2^{(x)_0} = 2^{t+1} < lh(z).$$

So by (iii): $\tau(x) = 1$, whence (iv). Next we deduce

(v) $\exists y \tau(2^{t+1} * \bar{\alpha}(y)) > 0$, by cases $(2^{t+1} < lh(z), 2^{t+1} \geqq lh(z))$.

Case A: $2^{t+1} < lh(z)$. Use (iv).

Case B: $2^{t+1} \geqq lh(z)$.

Subcase B.1: $lh(z) \leqq 1$. Then $2^{t+1} * \bar{\alpha}(0) = 2^{t+1} \geqq lh(z)$. So

$$\tau(2^{t+1} * \bar{\alpha}(0)) = \tau_i(\langle z, t \rangle) + 1 \quad [\text{(iii)}] > 0.$$

Subcase B.2: $lh(z) > 1$. Using case hyp., *149a, etc., we can assume prior to $\exists$-elim.:

(vi) $m_1 \leqq t+1 \;\&\; 2^{m_1} \geqq lh(z) \;\&\; \forall m (m < m_1 \supset \neg(m \leqq t+1 \;\&\; 2^m \geqq lh(z))$.

Using subcase hyp. and

(vi): $2^{m_1} > 1$, whence $m_1 > 0$. So (vii) $m_1 \dot{-} 1 < m_1 \leqq t+1$[(vi)].

Then $\neg(m_1 \dot{-} 1 \leqq t+1 \;\&\; 2^{m_1 \dot{-} 1} \geqq lh(z))$ [(vi), (vii)], whence with (vii) again: (viii) $2^{m_1 \dot{-} 1} < lh(z)$. From (vii), (viii) and (vi), etc., and $\exists$-introd.:

(ix) $\exists m(m < t+1 \,\&\, 2^m < lh(z) \,\&\, 2^{m+1} \geqq lh(z))$. Assuming $m < q(t+1, z)$, we can deduce $\neg 2^{m+1} \geqq lh(z)$ [(ix), *E5], whence $2^{m+1} < lh(z)$ and then by (iv), $\exists y \tau(2^{m+1} * \bar{\alpha}(y)) > 0$; and so easily: $\exists w \exists y_{y \leqq w} \tau(2^{m+1} * \bar{\alpha}(y)) > 0$. Letting $B(s, w)$ be $\exists y_{y \leqq w} \tau(2^{s+1} * \bar{\alpha}(y)) > 0$ we have thus by $\supset$- and $\forall$-introd.: (x) $\forall m(m < q(t+1, z) \supset \exists w B(m, w))$. Also we can deduce: (xi) $\forall s \forall z \forall w(B(s, z) \,\&\, w \geqq z \supset B(s, w))$. Now using (x) and (xi) in Lemma 4(a), assume (xii) $\forall m(m < q(t+1, z) \supset B(m, w))$. Let $x = 2^{t+1} * \bar{\alpha}(w+1)$. Then (xiii) Seq$(x)$, (xiv) $w+1 = lh(x) \dot{-} 1$, (xv) $2^{(x)_0} \geqq lh(z)$ [case hyp.]. Assuming $m < q((x)_0, z) = q(t+1, z)$ we can deduce from (xii) a formula from which we may assume: $y \leqq w \,\&\, \tau(2^{m+1} * \bar{\alpha}(y)) > 0$, whence with (xiv), etc.: $y < lh(x) \dot{-} 1 \,\&\, \tau(2^{m+1} * \prod_{n<y} p_n^{(x)_{n+1}}) > 0$. Thus by $\exists$-, $\supset$-introd. etc.:

$$\text{(xvi)} \quad \forall m(m < q((x)_0, z) \supset \exists y_{y < lh(x) \dot{-} 1} \tau(2^{m+1} * \textstyle\prod_{n<y} p_n^{(x)_{n+1}}) > 0.$$

Using (xiii), (xv), subcase hyp. and (xvi) in (iii): $\tau(x) = \tau_i(\langle u(\tau, x, z), (x)_0 \dot{-} 1\rangle) + 1 > 0$. By $\exists$-, $\forall$-introds., (v). From (v) easily: (c).

Now towards (d), assume (xvii) $\alpha \in C^j$ and (xviii) $\forall t \exists y G(\tau, \alpha, \beta, y, t)$. We shall deduce (e) $\beta \in C^i$ by cases ($lh(z) \leqq 1$, $lh(z) > 1$).

Case 1: $lh(z) \leqq 1$. Assume from (xviii), $\tau(2^{t+1} * \bar{\alpha}(y)) = \beta(t) + 1$. Now $2^{t+1} * \bar{\alpha}(y) > 1 \geqq lh(z)$ [case hyp.]; so $\beta(t) + 1 = \tau(2^{t+1} * \bar{\alpha}(y)) = \tau_i(\langle z, t\rangle) + 1$ [(iii)], whence $\beta(t) = \tau_i(\langle z, t\rangle)$. By ^{x}0.1 and $\forall$-introd. and using (ii) and (i): $\beta \in C^i$.

Case 2: $lh(z) > 1$. Since $lh(z) < lh(z)+1 \,\&\, 2^{lh(z)+1} \geqq lh(z)$, we have (xix) $\exists t_{t < lh(z)+1} 2^{t+1} \geqq lh(z)$. Let (xx) $M = \mu t_{t < lh(z)+1} 2^{t+1} \geqq lh(z)$. Using *2.2 with (xviii), assume (xxi) $\forall t G(\tau, \alpha, \beta, \varphi(t), t)$. Let (xxii) $w = \prod_{t<M} p_t \exp(\tau(2^{t+1} * \bar{\alpha}(\varphi(t))))$. Then (xxiii) Seq$(w)$ and (xxiv) $lh(w) = M$. We shall deduce (f) $\beta(t) + 1 = \tau_i(\langle w, t\rangle) + 1$ by cases ($t < lh(w)$, $t \geqq lh(w)$). Case A: $t < lh(w)$. Then $\beta(t) + 1 = \tau(2^{t+1} * \bar{\alpha}(\varphi(t)))$ [(xxi)] $= (w)_t$ [case hyp., (xxii)] $= \tau_i(\langle w, t\rangle) + 1$ [(xxiii), (i), case hyp.]. Case B: $t \geqq lh(w)$. Now (xxv) $2^{t+1} = 2 \cdot 2^t \geqq 2 \cdot 2^{lh(w)} = 2^{M+1}$ [(xxiv)] $\geqq lh(z)$ [(xix), (xx)]. So (xxvi) $2^{t+1} * \bar{\alpha}(\varphi(t)) \geqq lh(z)$. Also, from (xxi): $\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) > 0$; so the last case in (iii) is not the one used in evaluating $\tau(2^{t+1} * \bar{\alpha}(\varphi(t)))$. But similarly the first three cases are ruled out by (xxvi), case 2 hyp. and (xxv), respectively. So the fourth case applies and thus: (xxvii) $\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) = \tau_i(\langle u(\tau, 2^{t+1} * \bar{\alpha}(\varphi(t)), z), t\rangle) + 1$, and (xxviii) $\forall m_{m < q(t+1, z)} \exists y_{y < \varphi(t)} \tau(2^{m+1} * \prod_{n<y} p_n^{\alpha(n)+1}) > 0$. We have (xxix) $M = lh(w)$ [(xxiv)] $\leqq t$ [case B hyp.] $< t+1$ and (xxx) $2^{M+1} \geqq lh(z)$ [(xxv)]; so $\exists m_{m<t+1} 2^{m+1} \geqq lh(z)$. Now $q(t+1, z) = \mu m_{m<t+1} 2^{m+1} \geqq lh(z)$ and by *E5 if $M < q(t+1, z)$ then $\neg 2^{M+1} \geqq lh(z)$, contradicting (xxx), but also if $M > q(t+1, z)$ then we contradict (xix)–(xx). So (xxxi) $q(t+1, z) = M$. Let $K(m) = k(\tau, 2^{t+1} * \bar{\alpha}(\varphi(t)), m)$. We deduce (xxxii)

$\forall m(m < q(t+1, z) \supset K(m) = \varphi(m))$, as follows. Assume $m < q(t+1, z)$. By (xxviii): (xxxiii) $\exists y_{y<\varphi(t)} \tau(2^{m+1} * \bar{\alpha}(y)) > 0$. By (xxi): (xxxiv) $\tau(2^{m+1} * \bar{\alpha}(\varphi(m))) > 0$ & $\forall v_{v<\varphi(m)} \tau(2^{m+1} * \bar{\alpha}(v)) = 0$. Since

$$K(m) = \mu y_{y<\varphi(t)} \tau(2^{m+1} * \bar{\alpha}(y)) > 0,$$

we can use (xxxiii) and *E5 with (xxxiv) to show $K(m) \not< \varphi(m)$ and $K(m) \not> \varphi(m)$, whence $K(m) = \varphi(m)$. Thus by $\supset$-introd. etc.: (xxxii). Thence

$$u(\tau, 2^{t+1} * \bar{\alpha}(\varphi(t)), z) = \prod_{m<q(t+1,z)} p_m \exp(\tau(2^{m+1} * \prod_{n<\varphi(m)} p_n^{(\alpha(\varphi(t)))_n}))$$
$$= \prod_{m<M} p_m \exp(\tau(2^{m+1} * \bar{\alpha}(\varphi(m)))) \quad [\text{(xxxi), (xxxiii)}] = w \quad [\text{(xxii)}].$$

So $\beta(t)+1 = \tau(2^{t+1} * \bar{\alpha}(\varphi(t)))$ [(xxi)] $= \tau_i(\langle w, t\rangle)+1$ [(xxvii)]. Thence $\beta = \lambda t \tau_i(\langle w, t\rangle)$ and by (xxiii) and (i): $\beta \in C^i$.

Now by $\supset$-introd. from (xviii), (xvii), etc.: (d).

Combining (c) and (d): $\tau \in C^h$. Then with (b): (a).

Next, from (a) by $^{\times}2.1$, etc. (with $\mathrm{Seq}(z) \vee \neg\mathrm{Seq}(z)$):

$$\exists\tau\forall z_{\mathrm{Seq}(z)}(\lambda s\tau(\langle z, s\rangle) \in C^h \;\&\; \overline{\lambda s\tau(\langle z, s\rangle)}(lh(z)) = z).$$

Assume prior to $\exists$-elim., $\forall z_{\mathrm{Seq}(z)}(\lambda s\tau(\langle z, s\rangle) \in C^h \;\&\; \overline{\lambda s\tau(\langle z, s\rangle)}(lh(z)) = z)$. Then if $\mathrm{Seq}(z)$, $\lambda s\tau(\langle z, s\rangle) \in C^h$ and if $s < lh(z) : \tau(\langle z, s\rangle)+1 = (\overline{\lambda s\tau(z, s)}(lh(z)))_s = (z)_s$. Thus $\mathrm{Ext}^h(\tau, z)$. So finally, $\exists\tau\forall z_{\mathrm{Seq}(z)} \mathrm{Ext}^h(\tau, z)$.

THEOREM 1. *Using Markov's schema M: if $\Gamma \vdash^+ E$ and the formulas Γ are realizable, then E is realizable.*

PROOF. The proof of Theorem 9.3(a) in *FIM* pp. 105–109 provides all that is needed except for the cases in which E is M or an axiom by C.

CASE M. Use Theorem 11.7(a)^C of *FIM* pp. 129–130, with $\alpha(y) \neq 0$ as $A(x, y)$. The realizing function can be taken to be simply $\Lambda\pi\langle\mu y\alpha(y) \neq 0, \varepsilon_{\alpha(y)\neq 0}\rangle$, using *FIM* Lemma 8.4a. The argument is classical, as indicated by the superscript C on the theorem number, but the only non-intuitionistic principle is M (used at line 3 of the proof on p. 130).

CASE C. Suppose (1) ε realizes-Ψ the hyp. of C. Then for every α, π, η: if π realizes-α $\alpha \in C^j$ and η realizes-Ψ, α $\forall x A(\alpha, x)$ then $\{\{\{\varepsilon\}[\alpha]\}[\pi]\}[\eta]$ realizes-Ψ $\exists\beta^i B(\beta)$. Thus, using Lemmas 2 and 1(b): (2) if π realizes-α $\alpha \in C^j$ and η realizes-Ψ, α $\forall x A(\alpha, x)$, then

$$\varphi[\varepsilon, \alpha, \Psi] = \{\{\{\varepsilon\}[\alpha]\}[\varepsilon_j[\alpha]]\}[\varepsilon_{\forall x A(\alpha, x)}[\alpha, \Psi]]$$

realizes-Ψ, α $\exists\beta^i B(\beta)$.

Now define the function φ of ε, α, Ψ by $\varphi[\varepsilon, \alpha, \Psi] = \{\{\{\varepsilon\}[\alpha]\}[\varepsilon_j[\alpha]]\}$ $[\varepsilon_{\forall x A(\alpha,x)}[\alpha, \Psi]]$. Since $\varphi[\varepsilon, \alpha, \Psi] = \lambda t\varphi(\varepsilon, \alpha, \Psi, t)$ is a partial recursive function there is by the normal form theorem (*IM* Theorem XIX p. 330) a Gödel number e of φ such that

(3) $$\varphi(\varepsilon, \alpha, \Psi, t) \sim U(\mu y T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)),$$

where

$$(Ey)T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t) \rightarrow (E!\, y)T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t).$$

Also, $A(\alpha, x)$ expresses a primitive recursive predicate $A(\alpha, x)$ [*FIM* Remark 3.4 p. 13]. So there is by the normal form theorem [*IM* Theorem IX p. 288] a Gödel number f (of the representing function of $A(\alpha, x)$ considered as predicate of α, Ψ, x) such that:

(4) $$A(\alpha, x) \equiv U(\mu y T_n^m(f, \bar{\alpha}(y), \bar{\Psi}(y), x)) = 0,$$

(5) $$\bar{A}(\alpha, x) \equiv U(\mu y T_n^m(f, \bar{\alpha}(y), \bar{\Psi}(y), x)) = 1,$$

where

$$(\alpha)(\Psi)(x)(E!\, y)T_n^m(f, \bar{\alpha}(y), \bar{\Psi}(y), x).$$

I. We want to define (primitive recursively from ε with Ψ as parameters) a function τ with the properties expressed in the conclusion of C. Our aim is to make τ represent the following algorithm $\mathscr{T}$, which works on a given t and a given initial segment $\bar{\alpha}(x)$ of a function α to give effectively either a value $\beta(t)$ or the answer that no value $\beta(t)$ can be computed.

(From now through the definition of τ in (21) below, except for Remark 1, we let ε, Ψ be arbitrary (not necessarily satisfying (1)) as we describe $\mathscr{T}$ and τ uniformly in ε, Ψ.)

Given $\bar{\alpha}(x)$ and asked to determine $\beta(t)$ the algorithm $\mathscr{T}$ first determines whether the following are satisfied.

(a) $x > t$,

(b) $(s)_{s<t}(Ey)_{y<x}(\beta(s)$ is determined by $\mathscr{T}$ from $\bar{\alpha}(y))$.

If not (a) or not (b), $\mathscr{T}$ signals that it can compute no value for $\beta(t)$ from $\bar{\alpha}(x)$.

If both (a) and (b), $\mathscr{T}$ next determines if there is for this $\bar{\alpha}(x)$ a refuting pair of numbers $(s, \bar{\alpha}(y))$ – i.e. a pair of numbers $s \leqq x$ and $\bar{\alpha}(y)$, $y \leqq x$, such that $T_n^m(f, \bar{\alpha}(y), \bar{\Psi}(y), s)$ & $U(y) = 1$; so that by (5): $\bar{A}(\alpha, s)$.

If there is such a pair, $\mathscr{T}$ computes $\beta(t)$ to insure $\beta \in C^i$ as follows. First $\mathscr{T}$ looks back (using (b)) to determine if there is $t_1 < t$ such that the computation of $\beta(t_1)$ by $\mathscr{T}$ requires a segment $\bar{\alpha}(x_1)$, $x_1 \leqq x$, such that there exists for $\bar{\alpha}(x_1)$ a refuting pair. Setting t_0 = least such t_1 if one exists ($= t$ otherwise), $\mathscr{T}$ then computes $\beta(t)$ as $\tau_i(\langle\bar{\beta}(t_0), t\rangle)$, using the primitive recursive function τ_i given by Lemma 5(b).

If there is no refuting pair for $\bar{\alpha}(x)$, $\mathscr{T}$ determines if for any $y < x$ $T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)$. If so, $\mathscr{T}$ computes $\beta(t)$ as $U(y)+1 = \varphi(\varepsilon, \alpha, \Psi, t)+1$ [(3)]. If not, $\mathscr{T}$ signals that it can make no computation of $\beta(t)$ from $\bar{\alpha}(x)$.

(REMARK 1. That $\mathscr{T}$ associates a function $\beta \in C^i$ with every $\alpha \in C^j$ then follows, using M, provided that ε satisfies (1) above (details below)).

We shall represent $\mathscr{T}$ by a function τ so that in computing $\beta(t)$ we apply τ to arguments $2^{t+1} * \bar{\alpha}(x)$ for increasing x. We shall arrange that $\tau(u) = 0$ unless $\mathrm{Seq}(u)$ & $lh(u) > 1$.

Towards defining τ by primitive (course of values) recursion, we first observe that if we put $z = \bar{\alpha}(x)$, the condition (a) & (b) becomes

$$(6) \quad P(\tau, z, t) \equiv lh(z) > t \,\&\, (s)_{s<t}(Ek)_{k<lh(z)}\tau(2^{s+1} * \prod_{i<k} p_i^{(z)_i}) > 0.$$

Then easily

$$(7) \qquad P(\tau, \bar{\alpha}(x), t) \,\&\, x_1 \geqq x \rightarrow P(\tau, \bar{\alpha}(x_1), t).$$

Further, it is easy to find a primitive recursive predicate $P'(w)$ such that letting

$$(8) \quad B(u, t, z) \equiv \mathrm{Seq}\,(u) \,\&\, lh(u) > 1 \\ \&\, t = (u)_0 \dot{-} 1 \,\&\, z = \prod_{i<lh(u)\dot{-}1} p_i^{(u)_{i+1}},$$

we can show

$$(9) \qquad B(u, t, z) \rightarrow [P(\tau, z, t) \equiv P'(\bar{\tau}(u))].$$

Next the condition that there is for $z = \bar{\alpha}(x)$ a refuting pair is expressed primitive recursively by

$$(10) \qquad \mathrm{Ref}\,(\Psi, z) \equiv (Es)_{s<lh(z)}(Ek)_{k<lh(z)} \\ (T_n^m(f, \prod_{i<k} p_i^{(z)_i}, \bar{\Psi}(k), s) \,\&\, U(k) = 1).$$

Clearly:

$$(11) \qquad \mathrm{Ref}(\Psi, \bar{\alpha}(x)) \,\&\, x_1 \geqq x \rightarrow \mathrm{Ref}(\Psi, \bar{\alpha}(x_1)).$$

Now, if τ computes a value $\beta(s)$ for $s < t$ from some proper initial segment $\bar{\alpha}(y)$ of $z = \bar{\alpha}(x)$, the length y of that segment is given by

$$(12) \qquad Y(\tau, z, s, t) = \begin{cases} \mu y_{y<lh(z)}\tau(2^{s+1} * \prod_{i<y} p_i^{(z)_i}) > 0 \text{ if } s < t, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly:

$$(13) \qquad Y(\tau, z, s, t) \leqq lh(z).$$

Again, it is easy to define a primitive recursive function $Y'(s, w)$ such that for $B(u, t, z)$ as in (8):

$$(14) \qquad B(u, t, z) \rightarrow [Y(\tau, z, s, t) = Y'(s, \bar{\tau}(u))].$$

For $s \leqq t$ the sequence number $\bar{\beta}(s)$ giving the accumulated values of β determined by $z = \bar{\alpha}(x)$ (if any) is given by

$$(15) \qquad Z(\tau, z, s, t) = \begin{cases} \prod_{i<s} p_i \exp(\tau(2^{i+1} * \prod_{j<Y(\tau,z,i,t)} p_j^{(z)_j})) \text{ if } s \leqq t, \\ 1 \text{ otherwise.} \end{cases}$$

Using (13) and (14), we can find a primitive recursive function $Z'(s, w)$ such that for $B(u, t, z)$ as in (8):

$$(16) \qquad B(u, t, z) \rightarrow [Z(\tau, z, s, t) = Z'(s, \bar{\tau}(u))].$$

The least $s < t$ (if any) such that the computation of $\beta(s)$ by τ requires a z for which there is a refuting pair is

$$(17) \qquad W(\Psi, \tau, z, t) = \mu s_{s<t} \operatorname{Ref}(\Psi, \prod_{j<Y(\tau,z,s,t)} p_j^{(z)_j}).$$

Again, using (14), we can find a primitive recursive $W'(\Psi, w)$ such that

$$(18) \qquad B(u, t, z) \rightarrow [W(\Psi, \tau, z, t) = W'(\Psi, \bar{\tau}(u))].$$

Following $\mathscr{T}$ we shall need, if $P(\tau, z, t)$ & $\operatorname{Ref}(\Psi, z)$, the sequence number $\bar{\beta}(W(\Psi, \tau, z, t))$. Using (15) and (17) this is given by

$$(19) \qquad V(\Psi, \tau, z, t) = Z(\tau, z, W(\Psi, \tau, z, t), t).$$

Using (16) and (18)

$$(20) \qquad B(u, t, z) \rightarrow [V(\Psi, \tau, z, t) = Z'(W(\Psi, \tau, z, t), \bar{\tau}(u)) = Z'(W'(\Psi, \bar{\tau}(u)), \bar{\tau}(u))].$$

Now we can define τ uniformly from Ψ, ε, writing for short $t = (u)_0 \dot{-} 1$ and $z = \prod_{i<lh(u) \dot{-} 1} p_i^{(u)_i}$ and using τ_i provided by Lemma 5(b) so that

$$(21) \qquad (s)_{\operatorname{Seq}(z)} \operatorname{Ext}^i(\tau_i, z).$$

$$(22) \qquad \tau(u) = \begin{cases} 1 + \tau_i(V(\Psi, \tau, z, t), t) \text{ if } \operatorname{Seq}(u) \mathbin{\&} lh(u) > 1 \mathbin{\&} \\ \qquad P(\tau, z, t) \mathbin{\&} \operatorname{Ref}(\Psi, z), \\ 1 + (U(\mu y_{y \leqq lh(z)} T_n^{m+1}(e, \bar{\varepsilon}(y), \prod_{i<y} p_i^{(z)_i}, \bar{\Psi}(y), t))_0 \\ \qquad \text{if } \operatorname{Seq}(u) \mathbin{\&} lh(u) > 1 \mathbin{\&} P(\tau, z, t) \mathbin{\&} \overline{\operatorname{Ref}}(\Psi, z) \mathbin{\&} \\ \qquad (Ey)_{y<lh(z)} T_n^{m+1}(e, \bar{\varepsilon}(y), \prod_{i<y} p_i^{(z)_i}, \bar{\Psi}(y), t), \\ 0 \text{ otherwise.} \end{cases}$$

Using (9) and (20), we see that τ is defined by primitive (course of values) recursion.

Assuming $\tau(2^{t+1} * \bar{\alpha}(y)) > 0$, we have $P(\tau, \bar{\alpha}(y), t)$, whence $y = lh(\bar{\alpha}(y)) > t$ and $(s)_{s<t}(Ek)_{k<y}\tau(2^{s+1} * \bar{\alpha}(k)) > 0$. So

$$(23) \qquad (\alpha)(y)(t)(\tau(2^{t+1} * \bar{\alpha}(y)) > 0 \rightarrow y > t \;\&\; (s)_{s<t}(Ek)_{k<y}\tau(2^{s+1} * \bar{\alpha}(k)) > 0).$$

II. For the τ just defined we next want to show $\tau \in C^h$. We need:

(i) $(\alpha)^j(t)(Ey)\tau(2^{t+1} * \bar{\alpha}(y)) > 0$, and

(ii) $(\alpha)^j(\beta)[(t)(Ey)(\tau(2^{t+1} * \bar{\alpha}(y)) = \beta(t)+1 \;\&\; (z)_{z<y}\tau(2^{t+1} * \bar{\alpha}(z)) = 0) \rightarrow \beta \in C^i]$.

Towards (i), assume (24) $\alpha \in C^j$. We shall deduce

$$(\text{iii}) \qquad (s)_{s<t}(Ey)\tau(2^{s+1} * \bar{\alpha}(y)) > 0 \rightarrow (Ey)\tau(2^{t+1} * \bar{\alpha}(y)) > 0.$$

Assume (25) $(s)_{s<t}(Ey)\tau(2^{s+1} * \bar{\alpha}(y)) > 0$. Assume further

$$(26) \qquad (\overline{Ey})\tau(2^{t+1} * \bar{\alpha}(y)) > 0.$$

Suppose (27) $\mathrm{Ref}(\Psi, \bar{\alpha}(y))$. Using (25) choose y_1 large enough so that $y_1 > \max(y, t, 1) \;\&\; (s)_{s<t}(Ey)_{y<y_1}\tau(2^{s+1} * \bar{\alpha}(y)) > 0$. Then letting $u = 2^{t+1} * \bar{\alpha}(y_1)$, we have $\mathrm{Seq}(u), lh(u) = y_1+1 > 1, P(\tau, \bar{\alpha}(y_1), t)$ [(6)], $\mathrm{Ref}(\Psi, \bar{\alpha}(y_1))$ [(27), (11)]. So by the first case of (22), $\tau(2^{t+1} * \bar{\alpha}(y_1)) = \tau(u) > 0$, contradicting (26). So, rejecting (27): $\overline{\mathrm{Ref}}(\Psi, \bar{\alpha}(y))$, whence (28) $(y)\,\overline{\mathrm{Ref}}(\Psi, \bar{\alpha}(y))$, and by (10): (29) $(y)(\overline{Es})_{s<y}(Ek)_{k<y}(T_n^m(f, \bar{\alpha}(k), \bar{\Psi}(k), s) \;\&\; U(k) = 1)$. Now if for any s $\bar{A}(\alpha, s)$ then by (5) for some y $T_n^m(f, \bar{\alpha}(y), \bar{\Psi}(y), s) \;\&\; U(y) = 1$, and then letting $y_1 = 1+\max(y, s)$: $(Es)_{s<y_1}(Ek)_{k<y_1}(T_n^m(f, \bar{\alpha}(k), \bar{\Psi}(k), s) \;\&\; U(k) = 1)$, contradicting (29). So $\bar{\bar{A}}(\alpha, s)$, whence (30) $(s)A(\alpha, s)$. Then by Lemma 1(b), $\varepsilon_{\forall sA(\alpha,s)}$ realizes-Ψ, α $\forall sA(\alpha, s)$. Also by (24) and Lemma 2, $\varepsilon_j[\alpha]$ realizes-α $\alpha \in C^j$. So by (2), $\varphi[\varepsilon, \alpha, \Psi]$ realizes-Ψ, α $\exists\beta^i B(\beta)$. So in particular $\varphi(\varepsilon, \alpha, \Psi, t)$ is defined. By (3) $(Ey)T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)$. Assume (30) $T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)$. Using (25) choose y_1 large enough so that $y_1 > \max(y, t, 1) \;\&\; (s)_{s<t}(Ey)_{y<y_1}\tau(2^{t+1} * \bar{\alpha}(y)) > 0$. Then letting $u = 2^{t+1} * \bar{\alpha}(y_1)$: $\mathrm{Seq}(u), lh(u) = y_1+1 > 1, P(\tau, \bar{\alpha}(y_1), t)$ [(6)], $\overline{\mathrm{Ref}}(\Psi, \alpha(y_1))$ [(28)], $(Ey)_{y<y_1} T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)$. [(30)]. So by the second case of (22): $\tau(2^{t+1} * \bar{\alpha}(y)) = \tau(u) > 0$, contradicting (26). So rejecting (26): $(\overline{\overline{Ey}})\tau(2^{t+1} * \bar{\alpha}(y)) > 0$. By M, $(Ey)\tau(2^{t+1} * \bar{\alpha}(y)) > 0$. Finally from (iii) by course of values induction, $(t)(Ey)\tau(2^{t+1} * \bar{\alpha}(y)) > 0$, and then, discharging (24): (i).

Towards (ii), assume (31) $\alpha \in C^j$ and

$$(32) \qquad (t)(Ey)[\tau(2^{t+1} * \alpha(y)) = \beta(t)+1 \;\&\; (z)_{z<y}\tau(2^{t+1} * \bar{\alpha}(z)) = 0].$$

Using (31) and (i): (33) $(t)\tau(2^{t+1} * \bar{\alpha}(y_t)) > 0$ where $y_t = \mu y \tau(2^{t+1} * \bar{\alpha}(y)) > 0$. Then using (32): (34) $(t)\beta(t)+1 = \tau(2^{t+1} * \alpha(y_t))$. Using (23): (35) $(s)(t)(s \leqq t \rightarrow y_s \leqq y_t)$. By (12): (36) $(s)(t)(s < t \rightarrow Y(\tau, \bar{\alpha}(y_t), s, t) = y_s)$.

We shall show

(iii) $(Ey) \operatorname{Ref}(\Psi, \bar{\alpha}(y)) \rightarrow \beta \in C^i$.

and

(iv) $(\overline{Ey}) \operatorname{Ref}(\Psi, \bar{\alpha}(y)) \rightarrow \beta \in C^i$.

Towards (iii), suppose (37) $(Ey) \operatorname{Ref}(\Psi, \bar{\alpha}(y))$.

Let (38) $r = \mu y \operatorname{Ref}(\Psi, \bar{\alpha}(y))$. By (33), $\tau(2^{r+1} * \bar{\alpha}(y)_r) > 0$. By (23), $y_r > r$; so $(Es) y_s \geqq r$. Let (39) $R = \mu s y_s \geqq r$. Then (40) $(\overline{Es})_{s<R} \operatorname{Ref}(\Psi, \bar{\alpha}(y_s))$, for if $s < R \,\&\, \operatorname{Ref}(\Psi, \bar{\alpha}(y_s))$ then $s < R \,\&\, y_s \geqq r$ [(38)], contradicting (39).

We shall show (41) $\beta(t) = \tau_i(\prod_{s<R} p_s^{\beta(s)+1}, t)$ by cases ($t < R, t \geqq R$).

Case 1: $t < R$. Use (21).

Case 2: $t \geqq R$. Then $y_t \geqq y_R$ [(35)] $\geqq r$ [(39)]. So by (38) and (11): (42) $\operatorname{Ref}(\Psi, \bar{\alpha}(y_t))$. We shall show (43) $W(\Psi, \tau, \bar{\alpha}(y_t), t) = R$ by cases ($t = R, t > R$).

Case A: $t = R$. Now if (1_A) $(Es)_{s<t} \operatorname{Ref}(\Psi, \prod_{j<Y(\tau, \bar{\alpha}(y_t), s, t)} p_j \exp(\bar{\alpha}(y_t))_j)$, then $(Es)_{s<t} \operatorname{Ref}(\Psi, \prod_{j<y_s} p_j \exp(\bar{\alpha}(y_t))_j)$ [(36)], whence $(Es)_{s<t} \operatorname{Ref}(\Psi, \bar{\alpha}(y_s))$ [(35)]. But this contradicts Case A hyp. and (40). So, rejecting (1_A), we obtain (43) (cf. *IM* pp. 225, 229).

Case B: $t > R$. Then $W(\Psi, \tau, \bar{\alpha}(y_t), t) = \mu s_{s<t} \operatorname{Ref}(\Psi, \alpha(y_s))$ [(17), (35), (36)] $= R$ [(39), (38), (11)]. – This shows (43). Then

$$
\begin{aligned}
(44)\ V(\Psi, \tau, \bar{\alpha}(y_t)t), &= Z(\tau, \bar{\alpha}(y_t), R, t) \ [(19), (43)] \\
&= \prod_{s<R} p_s \exp(\tau(2^{s+1} * \prod_{j<Y(\tau, \alpha(y_t), s, t)} p_j \exp(\bar{\alpha}(y_t))_j)) \ [(15), \text{Case 2 hyp.}] \\
&= \prod_{s<R} p_s \exp(\tau(2^{s+1} * \bar{\alpha}(y_s))) \ [\text{Case 2 hyp., (36), (35)}].
\end{aligned}
$$

Now $\beta(t)+1 = \tau(2^{t+1} * \bar{\alpha}(y_t))$ [(34)] $= 1 + \tau_i(V(\Psi, \tau, \bar{\alpha}(y_t)), t)$ [(22), (42)] $= 1 + \tau_i(\prod_{s<R} p_s \exp(\tau(2^{s+1} * \bar{\alpha}(y_s))), t)$ [(44)] $= 1 + \tau_i(\prod_{s<R} p_s \exp(\beta(s)+1), t)$ [(34)]. Thence, (41).

Now from (41) and (21): $\beta \in C^i$.

Next, towards (iv), assume (45) $(\overline{Ey}) \operatorname{Ref}(\Psi, \bar{\alpha}(y))$. Then $\tau(2^{t+1} * \bar{\alpha}(y_t)) = \beta(t)+1$ [(34)] > 0. So by (22) with (45): (46) $(Ey)_{y<y_t} T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \overline{\Psi}(y), t)$. Then (47) $\beta(t)+1 = \tau(2^{t+1} * \bar{\alpha}(y_t))$ [(34)] $= 1 + (U(\mu y_{y \leqq y_t} T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \overline{\Psi}(y), t)))_0$. Also from (45), $(\overline{Ey})(Es)_{s<y}(Ek)_{k<y}(T_n^m(f, \bar{\alpha}(k), \overline{\Psi}(k), s) \,\&\, U(k) = 1)$. Thence $(\overline{Es})(Ek)(T_n^m(f, \bar{\alpha}(k), \overline{\Psi}(k), s)$

& $U(k) = 1$). So by (5), $(\overline{Es})\bar{A}(\alpha, s)$. Then $(s)\bar{\bar{A}}(\alpha, s)$, whence $(s)A(\alpha, s)$. Then $\varepsilon_{\forall sA(\alpha,s)}[\Psi, \alpha]$ realizes-Ψ, α $\forall sA(\alpha, s)$ [Lemma 1b]. Also by (31), $\varepsilon_j[\alpha]$ realizes-α $\alpha \in C^j$ [Lemma 2]. So by (2), $\varphi[\varepsilon, \alpha, \Psi]$ realizes-Ψ, α $\exists\beta^i B(\beta)$, and thus $(\varphi[\varepsilon, \alpha, \Psi])_1$ realizes-Ψ, α, $(\varphi[\varepsilon, \alpha, \Psi])_0$ $\beta \in C^i$ & $B(\beta)$. So $(\varphi[\varepsilon, \alpha, \Psi])_{1,0}$ realizes-Ψ, α, $(\varphi[\varepsilon, \alpha, \Psi])_0$ $\beta \in C^i$. Then $\beta \in C^i$ is true-$(\varphi[\varepsilon, \alpha, \Psi])_0$ [Lemma 2]. But by (3) with (46)–(47): $(\varphi[\varepsilon, \alpha, \Psi])_0 = \beta$. So $\beta \in C^i$.

Now from (iii) and (iv), (48) $\overline{\overline{\beta \in C^i}}$, for if $\overline{\beta \in C^i}$, then from (iii) $(\overline{Ey})$ Ref$(\Psi, \alpha(y))$, whence from (iv), $\beta \in C^i$. And from (48) and (the informal analogue of) Lemma 3; $\beta \in C^i$.

Finally from (i)–(ii), we have (49) $\tau \in C^h$ is true-τ and so by Lemma 2: (49) $\varepsilon_h[\tau]$ realizes-τ $\tau \in C^h$.

III. Next we must find a function to realize-Ψ, τ the formula

$$(*)\quad \forall\alpha^j\forall\beta\forall\varphi[\forall t[(\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) = \beta(t)+1 \\ \& \forall z_{z<\varphi(t)}\tau(2^{t+1} * \alpha(z)) = 0] \supset (\forall x A(\alpha, x) \supset B(\beta))].$$

Suppose for some α (v) π realizes-α $\alpha \in C^j$, and for some β, φ, ζ:

(vi) ζ realizes-α, τ, β, φ $\forall t[\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) = \beta(t)+1$ & $\forall z_{z<\varphi(t)}\tau(2^{t+1} * \bar{\alpha}(z)) = 0]$, and for some η:

(vii) η realizes-Ψ, α $\forall x A(\alpha, x)$.

Then by *FIM* Lemma 8.4a(i) and (vi):

$$(51)\quad (t)(\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) = \beta(t)+1 \;\&\; (z)_{z<\varphi(t)}\tau(2^{t+1} * \bar{\alpha}(z)) = 0).$$

By Lemma 2 and (v):

$$(52)\quad \alpha \in C^j.$$

By (i) of II with (52): (53) $(t)\tau(2^{t+1} * \bar{\alpha}(y_t)) > 0$, where $y_t = \mu y\, \tau(2^{t+1} * \bar{\alpha}(y)) > 0$. From (51) and (53):

$$(54)\quad y_t = \varphi(t).$$

By Lemma 1(b) and (vii): $(x)A(\alpha, x)$. So by (4) $(x)(U(\mu y T_n^m(f, \bar{\alpha}(y), \bar{\Psi}(y), x)) = 0)$. Then easily from (10), etc.: (55) $(\overline{Ey})$ Ref$(\Psi, \bar{\alpha}(y))$. By (22) with (53) and (55): (56) $(Ey)_{y \leq y_t} T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)$. Then (57) $\beta(t)+1 = \tau(2^{t+1} * \bar{\alpha}(y_t))$ [(51), (54)] $= 1 + (U(\mu y_{y \leq y_t} T_n^{m+1}(e, \bar{\varepsilon}(y), \bar{\alpha}(y), \bar{\Psi}(y), t)))_0$ [(22) with (53), (55)] $= 1 + (\varphi(\varepsilon, \alpha, \Psi, t))_0$ [(3), (56)].

Now from (v) and (vii) with (2): $(\varphi[\varepsilon, \alpha, \Psi])_1$ realizes-Ψ, α, $(\varphi[\varepsilon, \alpha, \Psi])_0$ $\beta \in C^i$ & $B(\beta)$. Then using (57), $(\varphi[\varepsilon, \alpha, \Psi])_{1,1}$ realizes-Ψ, α, β $B(\beta)$.

So, to realize (*) we use $\chi[\varepsilon, \Psi] = \Lambda\alpha\Lambda\pi\Lambda\beta\Lambda\varphi\Lambda\zeta\Lambda\eta(\varphi[\varepsilon, \alpha, \Psi])_{1,1}$.

IV. In conclusion, to realize-Ψ C we use $\Lambda\varepsilon\langle\tau, \langle\varepsilon_h[\tau], \chi[\varepsilon, \Psi]\rangle\rangle$.

5. Normal forms in I^+

Preparatory to the normal form result in Theorem 2, we must establish a number of lemmas.

LEMMA 6(a). *Let $A(\alpha, \Psi)$ be a formula of I containing no quantifiers except bounded ones and containing as free variables exactly α, Ψ. There is a prime formula $S(z, \Psi)$ containing as free variables exactly z, Ψ such that*

$$\vdash_{\mathbf{B}} \forall\alpha A(\alpha, \Psi) \sim \forall z S(z, \psi)$$

PROOF. By *FIM* *D, *E, etc. we can find a prime formula $P(\alpha, \Psi)$ with $A(\alpha, \Psi) \sim P(\alpha, \Psi)$. By Lemma 6(b) of [3] there is a prime formula S' such that $P(\alpha, \Psi) \sim \forall x S'(\overline{\langle\alpha, \Psi\rangle}(x)) \sim \forall x S'(\prod_{i<x} p_i \exp(1+\langle\alpha, \Psi\rangle(i))) \sim \forall x S'(\prod_{i<x} p_i \exp(1+(2^{\alpha(i)} * \langle\Psi\rangle(i)))$. Let $S(z, \Psi)$ be $S'(\prod_{i<(z)_0} P_i \exp(1+(2^{(z)_{i+1} \dot{-} 1} * \langle\Psi\rangle(i)))$. We shall deduce $\forall\alpha\forall x S'(\prod_{i<x} p_i \exp(1+(2^{\alpha(i)} * \langle\Psi\rangle(i)))) \sim \forall z S(z, \Psi)$, abbreviated (i) $\sim$ (ii). Assuming (i) and putting $\alpha = \lambda i(z)_{i+1} \dot{-} 1$ and $x = (z)_0$ we deduce $S(z, \Psi)$, whence (ii). Assuming (ii) and, for reductio ad absurdum, $\neg S'(\prod_{i<x} p_i \exp(1+(2^{\alpha(i)} * \langle\Psi\rangle(i))))$ we can let $z = 2^x * \bar{\alpha}(x)$, whence $\neg S'(\prod_{i<(z)_0} p_i \exp(1+(2^{(z)_{i+1} \dot{-} 1} * \langle\Psi\rangle(i))))$, contradicting (ii); so by *158, etc., (i).

LEMMA 6(b). *Let $A(\alpha, \Psi)$ be as in part* (a). *Then for every C-index h*:

$$\vdash^+ \forall\alpha^h A(\alpha, \Psi) \sim \forall\alpha A(\alpha, \Psi).$$

PROOF. For $\supset$, obtain P and S' as in the proof of (a), so that (i) $\forall\alpha, \Psi[A(\alpha, \Psi) \sim \forall x S'(\overline{\langle\alpha, \Psi\rangle}(x))]$. Assume (ii) $\forall\alpha^h A(\alpha, \Psi)$ and (iii) $\exists\alpha\neg A(\alpha, \Psi)$, whence assume $\neg A(\alpha, \Psi)$. By (i), $\neg\forall x S'(\overline{\langle\alpha, \Psi\rangle}(x))$ whence by M, $\exists x\neg S'(\overline{\langle\alpha, \Psi\rangle}(x))$. Assume (iv) $\neg S'(\overline{\langle\alpha, \Psi\rangle}(x))$. Via Lemma 5, assume (v) $\beta \in C^h$ & $\bar{\beta}(x) = \bar{\alpha}(x)$. Then by (iv): $\neg S'(\overline{\langle\beta, \Psi\rangle}(x))$. So $\neg\forall x S'(\overline{\langle\beta, \Psi\rangle}(x))$, whence from (i): (vi) $\neg A(\beta, \Psi)$. But (v) and (vi) contradict (ii), so rejecting (iii): $\neg\exists\alpha\neg A(\alpha, \Psi)$, whence $\forall\alpha A(\alpha, \Psi)$.

LEMMA 7. *Suppose $A(x)$, C are formulas containing no quantifiers except bounded ones, C not containing x. Then*

$$\vdash^+ (\forall x A(x) \supset C) \supset \exists x(A(x) \supset C).$$

PROOF. $\subset$: Use *98a.

$\supset$: Assume $\forall x A(x) \supset C$. Then $\neg C \supset \neg\forall x A(x)$.

By M, (i) $\neg C \supset \exists x\neg A(x)$. CASE 1: C. Then $A(x) \supset C$. whence $\exists x(A(x) \supset C)$. CASE 2: $\neg C$. Then by (i), $\neg A(x)$. Thence $A(x) \supset C$. So $\exists x(A(x) \supset C)$.

LEMMA 8.

$$\vdash^{+}\forall\alpha^{h}\exists xB(\alpha, x) \supset \exists\tau^{(0:h)}\forall\alpha^{h}\forall y[(\tau(2^{*}\bar{\alpha}(y)) > 0 \\ \& \forall z_{z<y}\tau(2^{*}\bar{\alpha}(z)) = 0) \supset B(\alpha, \tau(2^{*}\bar{\alpha}(y)) \dot{-} 1)].$$

PROOF. Assume $\forall\alpha^{h}\exists xB(\alpha, x)$. Thence easily $\forall\alpha^{h}\exists\beta B(\alpha, \beta(0))$. Applying C, assume

$$\tau \in C^{(0:h)} \;\&\; \forall\alpha^{h}\forall\beta\forall\varphi[\forall t(\beta(t)+1 = \tau(2^{t+1} * \bar{\alpha}(\varphi(t))) \\ \& \forall z_{z<\varphi(t)}\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) = 0) \supset B(\alpha, \beta(0))].$$

Assume $\alpha \in C^{h}$. From $\tau \in C^{(0:h)}$: $\forall t\exists y\,\tau(2^{t+1} * \bar{\alpha}(y)) > 0$. Then by *2.2 assume $\forall t\,\tau(2^{t+1} * \bar{\alpha}(\psi(t))) > 0$. Introduce φ and $\beta : \forall t\varphi(t) = \mu y_{y \leqq \psi(t)}$ $\tau(2^{t+1} * \bar{\alpha}(y)) > 0$, $\forall t\beta(t) = \tau(2^{t+1} * \bar{\alpha}(\varphi(t))) \dot{-} 1$. Thence $\forall t(\beta(t)+1 = \tau(2^{t+1} * \bar{\alpha}(\varphi(t)))$ & $\forall z_{z<\varphi(t)}\tau(2^{t+1} * \bar{\alpha}(\varphi(t))) = 0)$. So $B(\alpha, \beta(0))$. Also $\beta(0) = \tau(2 * \bar{\alpha}(\varphi(0))) \dot{-} 1$, and $\varphi(0) = \mu y_{y \leqq \psi(0)}\tau(2 * \bar{\alpha}(y)) > 0$, where $\tau(2 * \bar{\alpha}(\psi(0))) > 0$. So, assuming $\tau(2 * \bar{\alpha}(y)) > 0$ & $\forall z_{z<y}\tau(2 * \bar{\alpha}(z)) = 0$, we deduce $y = \varphi(0)$. So $\beta(0) = \tau(2 * \bar{\alpha}(y))) \dot{-} 1$.

LEMMA 9. *Let $E(\tau, \alpha, \delta, \varphi, t)$ and $F(\delta, y)$ be formulas of I^{+} containing no quantifiers except bounded ones. Then for any C-indices h and j there is a C-index k and there is a prime formula $S(\tau, x)$ such that*

$$\vdash^{+}\exists\tau^{h}\forall\alpha^{j}\forall\delta\forall\varphi[\forall tE(\tau, \alpha, \delta, \varphi, t) \supset \forall yF(\delta, y)] \sim \exists\tau^{k}\forall xS(\tau, x).$$

PROOF.

$\vdash^{+}\exists\tau^{h}\forall\alpha^{j}\forall\delta\forall\varphi[\forall tE(\tau, \alpha, \delta, \varphi, t) \supset \forall yF(\delta, y)] \sim$
$\exists\tau^{h}\forall\alpha^{j}\forall\delta\forall\varphi\forall y[\forall tE(\tau, \alpha, \delta, \varphi, t) \supset F(\delta, y)]$ [* 95]
$\sim \exists\tau^{h}\forall\alpha^{j}\forall\delta\forall\varphi\forall y\exists t\,[E(\tau, \alpha, \delta, \varphi, t) \supset F(\delta, y)]$ [Lemma 7] $\sim$
$\exists\tau^{h}\forall\alpha^{(j,0,0,0)}\exists tH(\tau, \alpha, t)$ [where $H(\tau, \alpha, t)$ abbreviates
$E(\tau, (\alpha)_{0,0,0}, (\alpha)_{0,0,1}, (\alpha)_{0,1}, t) \supset F((\alpha)_{0,0,1}, (\alpha)_{1}(0))]$
$\sim \exists\tau^{h}\exists\eta^{(0:(j,0,0,0))}\forall\alpha^{(j,0,0,0)}\forall yJ(\tau, \eta, \alpha, y)$
[Lemma 8, with $J(\tau, \eta, \alpha, y)$ abbreviating
$(\eta(2 * \bar{\alpha}(y)) > 0 \;\&\; \forall z_{z<y}\eta(2 * \bar{\alpha}(z)) = 0) \supset H(\tau, \alpha, \eta(2 * \bar{\alpha}(y)) \dot{-} 1)]$
$\sim \exists\tau^{(h,(0:(j,0,0,0)))}\forall\alpha^{(j,0,0,0,0)}J((\tau)_{0}, (\tau)_{1}, (\alpha)_{1}, (\alpha)_{0}(0))$
$\sim \exists\tau^{(h,(0:(j,0,0,0)))}\forall\alpha J((\tau)_{0}, (\tau)_{1}, (\alpha)_{1}, (\alpha)_{0}(0))$
[Lemma 6(b), observing that only bounded quantifiers appear in H and finally in J] $\sim \exists\tau^{k}\forall xS(\tau, x)$ [Lemma 6(a), and letting $k = (h, (0 : (j, 0, 0, 0)))$]. By Lemma 6(a), S is prime.

THEOREM 2. *For every formula A of I^+ there is a C-index h and there is a prime formula A' such that (for α, x variables not free in A):*

$$\vdash^+ A \sim \exists\alpha^h \forall x A'.$$

PROOF. By induction on the number of logical symbols in A.

BASIS. A is prime. Let h be 0 and A' be A.

IND. STEP. We have cases A is $B \& C$, $B \vee C$, etc. By ind. hyp. $B \sim \exists\alpha^j \forall x B'$ and $C \sim \exists\gamma^i \forall x C'$ for prime formulas B', C'.

CASE 1. A is $B \& C$. Then $A \sim B \& C \sim \exists\alpha^j \forall x B'(\alpha, x) \& \exists\gamma^i \forall x C'(\gamma,x)$ [ind. hyp.] $\sim \exists\alpha^j \exists\gamma^i \forall x(B'(\alpha, x) \& C'(\gamma, x))$ [*31, *87] $\sim \exists\alpha^{(i,j)} \forall x(B'((\alpha)_1, x) \& C'((\alpha)_0, x))$. So let $h = (i,j)$ and let $A'(\alpha, x)$ be a prime formula such that $A'(\alpha, x) \sim B'((\alpha)_1, x) \& C'((\alpha)_0, x)$ [*FIM*. *14.1, *D, *E].

CASE 2. A is $B \vee C$. Then $A \sim B \vee C \sim \exists\alpha^j \forall x B'(\alpha, x) \vee \exists\gamma^i \forall x C'(\gamma, x) \sim \exists\alpha\ [(\alpha \in C^j \& \forall x B'(\alpha, x)) \vee \exists\gamma^i \forall x C'(\gamma, x)]$ [*90] $\sim \exists\alpha^j\ [\forall x B'(\alpha, x) \vee \exists\gamma^i \forall x C'(\gamma, x)]$ [$\supset$: by cases prior to $\exists$-elim., using Lemma 5 in second case] $\sim \exists\alpha^j \exists\gamma^i\ [\forall x B'(\alpha, x) \vee \forall x C'(\gamma, x)]$ [similarly] $\sim \exists\alpha^j \exists\gamma^i \exists z[(z = 0 \supset \forall x B'(\alpha, x)) \& (z \neq 0 \supset \forall x C'(\gamma, x))]$ [$\subset$: by cases $(z = 0, z \neq 0)$] $\sim \exists\alpha^j \exists\gamma^i \exists z \forall x[(z = 0 \supset B'(\alpha, x)) \& (z \neq 0 \supset C'(\gamma, x))]$ [*95, *87] $\sim \exists\alpha^{(i,j,0)} \forall x\ [((\alpha)_1(0) = 0 \supset B'((\alpha)_{0,1}, x)) \& ((\alpha)_1(0) \neq 0 \supset C'((\alpha)_{0,0}, x))]$. Let $h = (i,j,0)$ and again find $A'(\alpha, x)$ by *FIM* *14.1, *D, *E, etc.

CASE 3. A is $B \supset C$. Then $A \sim B \supset C \sim \exists\alpha^j \forall x B'(\alpha, x) \supset \exists\gamma^i \forall x C'(\gamma, x) \sim \forall\alpha^j [\forall x B'(\alpha, x) \supset \exists\gamma^i \forall x C'(\gamma, x)]$ [*95, *5] $\sim \exists\tau^h \forall\alpha^j \forall\delta \forall\varphi [\forall t G(\tau, \alpha, \delta, \varphi(t), t) \supset (\forall x B'(\alpha, x) \supset \forall y C'(\delta, y))]$ [axiom C, with $h = (i:j)$] $\sim \exists\tau^h \forall\alpha^j \forall\delta \forall\varphi [\forall t(G(\tau, \alpha, \delta, \varphi(t), t) \& B'(\alpha, t)) \supset \forall y C'(\delta, y)]$ [*4, *5, *87, *95] $\sim \exists\tau^k \forall x S(\tau, x)$ for prime S [Lemma 9].

CASE 4. A is $\neg B$. Then $A \sim B \supset 1 = 0$. Use Case 3.

CASE 5. A is $\exists\beta B(\beta)$. Then $A \sim \exists\beta B(\beta) \sim \exists\beta \exists\alpha^j \forall x B'(\beta, \alpha, x) \sim \exists\alpha^{(0,j)} \forall x B'((\alpha)_0, (\alpha)_1, x)$.

CASE 6. A is $\exists x B(x)$. Then $A \sim \exists x B(x) \sim \exists\beta B(\beta(0))$. Use Case 5.

CASE 7. A is $\forall\beta B(\beta)$. Then $A \sim \forall\beta \exists\alpha^j \forall x B'(\beta, \alpha, x) \sim \exists\tau^{(j:0)} \forall\beta \forall\delta \forall\varphi [\forall t G(\tau, \beta, \delta, \varphi(t), t)) \supset \forall x B'(\beta, \delta, x)]$ [axiom C] $\sim \exists\tau^k \forall x S(\tau, x)$ for prime S [Lemma 9, with $j = 0$].

CASE 8: A is $\forall x B(x)$. Then $A \sim \forall x B(x) \sim \forall\alpha B(\alpha(0))$. Use Case 7.

REFERENCES

ERRETT BISHOP
[1] Mathematics as a numerical language, Intuitionism and Proof Theory, edited by John Myhill, A. Kino and R. E. Vesley, North-Holland, Amsterdam, (1970), 53–71.

KURT GÖDEL
[2] Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes, Dialectica 12 (1958), 280–287.

S. C. KLEENE
[3] Classical extensions of intuitionistic mathematics, Proceedings of the 1964 International Congress for Logic, Methodology and Philosophy of Science, edited by Y. Bar-Hillel, North-Holland, Amsterdam, (1965), 31–44.

S. C. KLEENE
[4] Countable functionals, Constructivity in Mathematics, edited by A. Heyting, North-Holland, Amsterdam, (1959), 81–100.

S. C. KLEENE
[5] Introduction to Metamathematics, Van Nostrand, New York, 1952.

S. C. KLEENE AND R. E. VESLEY
[6] The Foundations of Intuitionistic Mathematics, North-Holland, Amsterdam, 1965.

G. KREISEL
[7] Interpretation of analysis by means of constructive functionals of finite types, Constructivity in Mathematics, edited by A. Heyting, North-Holland, Amsterdam (1959), 101–128.

G. KREISEL
[8] On weak completeness of intuitionistic predicate logic, Journal of Symbolic Logic, vol. 27 (1962), 139–158.

JOHN MYHILL
[9] Lecture notes for a seminar in logic, S.U.N.Y. at Buffalo, Spring 1969, and University of Michigan, Fall 1969.

M. YASUGI
[10] Intuitionistic analysis and Gödel's interpretation, Journal of the Mathematical Society of Japan, vol. 15 (1963), 101–112.

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