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Smoothness and regularity

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Some applications of the cotangent complex to smoothness and regularity are given; in particular, the proof of a criterion for formal smoothness which was conjectured in [8] (see 2.1), and some generalisations of this criterion for the non-noetherian and noetherian cases (2.2, 2.6, 2.7). Also considered is the descent of formal smoothness.

0. All the rings considered are commutative with unity; the topologies are linear. The definitions and notations used are as in EGA, OIV, §§ 19–20, and [1]. The following facts about the cotangent complex will be needed:

Let $A \to B$ be a morphism of rings; to any $B$-module $M$ are associated the $B$-modules $H_1(A, B, M)$, $H_i(A, B, M)$, $H_i(A, B, M)$. (For the definitions, see: [8] for $i = 0, 1$; [9] for $i = 0, 1, 2$; [1] or [14] for $i \geq 0$. At least for $i = 0, 1$, the various definitions give isomorphic modules. This follows from the properties 0.1–0.4 below ([6], 3.5).) $H_i(\text{resp. } H^i)$, $i \geq 0$ is a (co)homological functor and has the properties.

0.1. $H_0(A, B, M) = \Omega^1_{B/A} \otimes_B M$ (where $\Omega^1_{B/A}$ is the module of $A$-differentials in $B$); $H^0(A, B, M) = \text{Der}_A(B, M)(= \text{the module of } A\text{-derivations of } B \text{ in } M)$; see [9], 2.3).

0.2. If $A \to B$ is surjective with kernel $b$, then $H_1(A, B, M) = b/b^2 \otimes_B M$ and $H^1(A, B, M) = \text{Hom}_B(b/b^2, M)$ ([9], 3.1.2).

0.3. If $B$ is a polynomialring over $A$, then $H_i(A, B, .) = 0 = H^i(A, B, .)$ for $i \geq 1$ ([9], 3.1.1 or [1], 16.3).

0.4. If $A \to B \to C$ are morphisms of rings and $M$ is a $C$-module, then the sequence

$$\cdots \to H_i(A, B, M) \to H_i(A, C, M) \to H_i(B, C, M) \to H_{i-1}(A, B, M) \to \cdots \to H_0(B, C, M) \to 0$$
is exact ([9], 2.3.5 or [1], 18.2), and similarly for $H^i$, with arrows reversed.

0.5. If $B$ is an $A$-algebra and $S$ a multiplicatively closed system in $B$, then the canonical morphism $H_i(A, B, M) \otimes_B S^{-1}B \cong H_i(A, S^{-1}B, S^{-1}M)$ is an isomorphism ([9], 2.3.4 or [1], 16).

0.6. If $A \to B \to C$ are morphisms of rings and $M$ a flat $C$-module, then the canonical morphism $H_i(A, B, C) \otimes_C M \cong H_i(A, B, M)$ is an isomorphism.

0.7. Let $A' \leftarrow A \rightarrow B$ be morphisms of rings, where $u$ or $v$ is flat, $B' = A' \otimes_A B$, and $M$ a $B'$-module. Then the canonical morphism $H_i(A, B, M) \cong H_i(A', B', M)$ is an isomorphism ([9], 2.3.2 or [1], 19.2).

0.8. If $A \to B$ are fields, then $H_i(A, B, \cdot) = 0 = H^i(A, B, \cdot)$ for $i \geq 2$ ([9], 3.5 or [1], 22.2) and $A \to B$ is separable iff it is formally smooth (EGA, OIV, 19.6.1 or 9, 3.5).

0.9. Let $A$ be a local, noetherian ring, $B$ its residue class field and $K$ a field which is an extension of $B$. Then the following three statements are equivalent: $A$ is regular: $H_2(A, K, K)$ is zero; $H_i(A, K, K) = 0$ for $i \geq 2$. This follows from [9], 3.2.1 or [1], 27.1 and 27.2, using 0.6 and 0.8.

The following criteria (0.10 and 1.1) are essentially the discrete and non-discrete forms of the Jacobian criterion of smoothness (EGA, OIV, 22.6.1 and 22.6.2).

0.10. A morphism of rings $A \to B$ is formally smooth in the discrete topologies if and only if $\Omega_{B/A}$ is a projective $B$-module and $H_1(A, B, B) = 0$ ([8], 9.5.7 or [9], 3.1.3).

1.1. Let $A \to B$ be a morphism of topological rings: the topology of $B$ is $c$-adic for some ideal $c$ in $B$, and that of $A$ is also $c$-adic. Let $C = B/c$. If $A \to B$ is formally smooth then $\Omega_{B/A} \otimes_B C$ is a projective $C$-module and $H_1(A, B, C) = 0$: the converse is also true if $B$ is a noetherian ring.

1.2. For $A$ and $B$ noetherian, 1.1 is proved in [2], 5.4. The proof of 1.1 in general is based on the same ideas. Let $A \to B$ be formally smooth. Then $\Omega_{B/A}$ is a formally projective $B$-module (EGA, OIV, 20.4.9), hence $\Omega_{B/A} \otimes_B C$ is a projective $C$-module (II, IX, 1.25). Let $R$ be a polynomial ring over $A$, and $R \to B$ a surjection of $A$-algebras with kernel $b$. Then the following sequence is exact (0.4 and 0.1-0.3).

$$(1.2.1) \quad 0 \to H_1(A, B, C) \to b/b^2 \otimes_C C \longrightarrow \Omega_{R/A} \otimes_B C \to \Omega_{B/A} \otimes_B C \to 0.$$ 

But $\delta_{B/R/A}$ is formally left invertible (EGA, OIV, 20.7.8 and 19.4.4), so $\delta_{B/R/A} \otimes_B C$ is injective, and $H_1(A, B, C) = 0$. 

Now, let $B$ be noetherian, $\Omega_{B/A} \otimes_B C$ be a projective $C$-module, and $H_1(A, B, C) = 0$. Let $R$ and $b$ be as above. Then ([9], 3.1.2) and (1.2.1) imply that $H_1(A, B, M) = 0$ for any $C$-module $M$, and hence for any discrete $B$-module $M$ with open annihilator. Let $A_d, B_d$ denote the rings $A, B$ with the discrete topology, and $A_t, B_t$ the rings $A, B$ with the given topologies. Then $H_1(A, B, M) = 0$ means $H_1(A_d, B_d, M) = 0$. $A_d \to B_d \to B_t$ and $M$ give the exact sequence ([2], 2 or EGA, OIV, 20.3.7)

$$0 \to H^0_1(A_d, B_t, M) \to H^0_1(A_d, B_d, M) \to H^1_1(B_d, B_t, M) \to 0$$

But $\mathfrak{u}$ is an isomorphism (EGA, OIV, 20.3.3), so $\mathfrak{v}$ also is. $B$ is noetherian, hence $H^1_1(B_d, B_t, M) = 0([2], 5.1)$: it follows that $H^1_1(A_d, B_t, M) = 0$. Then $A_d \to A_t \to B_t$ and $M$ give the exact sequence

$$H^0_1(A_d, A_t, M) \to H^1_1(A_t, B_t, M) \to H^1_1(A_d, B_t, M),$$

where the first and third terms are zero. The formal smoothness of $A_t \to B$, now follows (EGA, OIV, 19.4.4).

1.3. REMARKS. (i) I do not know if the converse part of 1.1 remains valid for $B$ a non-noetherian ring.

(ii) The criterion 1.1 can be reformulated as follows: if $B$ is noetherian, then $A \to B$ is formally smooth iff $\Omega_{B/A}$ is a formally projective $B$-module and $H_1(A, B, C) = 0$. The results of N. Radu ([11], [12], [13]) shows that the condition $H_1(A, B, C) = 0$ is superfluous in this form of 1.1 if $B$ is a laskerian local ring with the $m$-adic topology ($m$ the maximal ideal of $B$), and $A$ is a field of characteristic zero (or arbitrary characteristic if $B$ is noetherian).

2.0. It is known that, if $A \to B$ is a local morphism of local noetherian rings, formally smooth in the topologies given by the maximal ideals, then $A$ is regular if and only if $B$ is regular.

PROOF. Let $K$ be the residue class field of $B$, then the maps $A \to B \to K$ give the exact sequence (0.4):

$$H_2(A, B, K) \to H_2(A, K, K) \to H_2(B, K, K) \to H_1(A, B, K).$$

But $H_1(A, B, K) = 0(1.1)$ and $H_2(A, B, K) = 0 ([4], corollary). Now apply 0.9.

2.1. In ([8], 9.6), the following criterion for formal smoothness is stated and partially proved:

THEOREM. Let $A \to B \to C$ be local homomorphisms of local noetherian rings, $A$ and $C$ regular, $B \to C$ surjective, so that $C = B/c, c$ an ideal of $B$. Finally, let $B$ be a localisation of a finitely generated $A$-algebra. Then $B$ is...
formally smooth over $A$ if and only if the following three conditions are satisfied:

a) $B$ is regular, i.e. $c$ is a regular ideal;
b) $\Omega_{B/A} \otimes_B C$ is a projective $C$-module;c) The echaracteristic homomorphism

$$N_{C/A} \to c/c^2$$

is injective.

(The morphism from c) is $H_1(A, C, C) \to H_1(B, C, C)$ (0.2)

In [8], it was proved that the conditions are necessary. The sufficiency, however, was only proved for $A$ a field and the sufficiency in general conjectured. I gave generalisations of this criterion for the non-noetherian and noetherian case ([5]). For the noetherian case I use essentially (1.1), but this does not work in the non-noetherian case. A direct proof for 2.1 is as follows:

Let $A \to B$ be formally smooth (here the topology is arbitrary, cf. EGA, OIV, 22.6.4). Then $B$ is regular (by 2.0), $\Omega_{B/A} \otimes_B C$ is a projective $C$-module and $H_1(A, B, C) = 0$ (by 0.10). From the exact sequence (0.4),

$$H_2(B, C, C) \to H_1(A, B, C) \to H_1(A, C, C) \to H_1(B, C, C) = c/c^2,$$

it results that $f$ is injective.

Let a), b), c) be satisfied. Then $H_2(B, C, C) = 0([9], 3.2.1)$ since $c$ is generated by a $B$-regular sequence. Hence, by the above sequence and c), $H_1(A, B, C) = 0$; this and b) imply that $B$ is a formally smooth $A$-algebra for the $c$-adic topology (1.1).

We now turn to the non-noetherian case. Let $A \to A' \to B \to C$ be morphisms of rings, $u$ and $v$ surjective, $b = \text{Ker } u, c = \text{Ker } v$.

2.2. THEOREM. Suppose that $A'$ is a formally smooth $A$-algebra (in the $b$-adic topology) and that $b/b^2$ is $c$-separated (or is a $B$-module of finite type with $B$ local). Then $A \to B$ is formally smooth (in the discrete topologies) if and only if $\Omega_{B/A} \otimes_B C$ is a projective $C$-module and $H_1(A, B, C) = 0$.

PROOF. The necessity results from (0.10).

For the converse two facts are necessary.

(2.2.1) If $\Omega_{B/A} \otimes_B C$ is a projective $C$-module and $H_1(A, B, C) = 0$ then

$$\delta_{B/A \otimes B} : b/b^2 \otimes C \to \Omega_{A'/A} \otimes C$$

is left invertible. For $A \to A' \to B$ and $C$ give the exact sequence

$$0 = H_1(A, B, C) \to H_1(A', B, C) \to H_0(A', C) \to H_0(A, B, C) \to 0,$$
by 0.4 and 0.1, 0.2. Then 2.2.1 results from the projectivity of $\Omega_{B/A} \otimes_B C = H_0(A, B, C)$.

(2.2.2) Let $h : M \to N$ be a morphism of $B$-modules, $c$ an ideal of $B$, $\text{Mac}$ - separated $B$-module, and $N$ a projective $B$-module. If $h_1 = h \otimes_B B/c : M/cM \to N/cN$ is left invertible, then $h$ is also left invertible. To see this, let $g' : N/cN \to M/cM$ be a left inverse for $h_1$. Since $N$ is projective, there is a morphism $g : N \to M$, such that the composition $N \xrightarrow{g} M \to M/cM$ equals $N \to N/cN \xrightarrow{g'} M/cM$. It is obvious that $g_1 = g'$, where the subscript 1 means $\otimes_B B/c$, i.e. that $(gh)_1 = 1$. It follows that $(gh)_r = gh \otimes_B B/c^r$ is equal to 1 (see the proof of IL, XII, 2, 2, 1). Let $x \in M$; then $x - (gh)(x) \in c^r M$ for any $r \geq 1$, and so $gh = 1$.

Now let $\Omega_{B/A} \otimes_B C$ be a projective $B$-module and $H_1(A, B, C) = 0$. Then $\delta_{B/A'} \otimes_B C$ is left invertible (2.2.1); hence $\delta_{B/A'}$ is also left invertible (for $\Omega_{A'/A} \otimes_A B$ is a projective $B$-module by 0.10. Now, if $b/b^2$ is $c$-separated, apply 2.2.2; otherwise apply EGA, IV, 19.1.12). Hence $B$ is a formally smooth $A$-algebra (EGA, IV, 20.5.12).

(2.2.3) REMARKS. (i) The hypotheses of (2.2) are fulfilled if $B$ is an $A$-algebra essentially of finite presentation and $C$ any quotient ring of $B$.

(ii) Under the hypotheses of 2.2, assume $H_2(B, C, C) = 0$ (conditions for this are given in [3], 5.1, or [9], 3.2.1); then $A \to B$ is formally smooth if and only if $\Omega_{B/A} \otimes_B C$ is a projective $C$-module and $H_1(A, B, C) \to c/c^2$ is injective. (This shows that for the 'only if' part of 2.1, sufficient hypotheses on $A \to B$ are that $B$ be noetherian and an $A$-algebra essentially of finite presentation.)

2.3 COROLLARY. Let $Z \xrightarrow{i} Y \xrightarrow{h} X$ be morphisms of schemes, $i$ being an immersion and $h$ being locally of finite presentation. Then $h$ is smooth in a neighbourhood of $Z$ in $Y$ if and only if $H_0(X, Y, O_Z)$ is a flat $O_Z$-module and $H_1(X, Y, O_Z) = 0$.

PROOF. Let $z \in Z$, $y = i(z)$ and $x = h(y)$. Then

$$H_i(X, Y, O_Z)_z = H_i(O_{X,x}, O_{Y,y}, O_{Z,z}),$$

by 0.5

Since $H_0(O_{X,x}, O_{Y,y}, O_{Z,z})$ is an $O_{Z,z}$-module of finite presentation, flat means projective. Then $h_y : 0_{X,x} \to 0_{Y,y}$ is smooth for any $z \in Z$ (cf. 2.2.3 and 2.2). But this is an open property (EGA, IV, 17.5). Hence $h$ is locally smooth, i.e. it is smooth ([8] 9.5.6).

2.4. For the non-discrete topologies, 2.2 takes the following form:

PROPOSITION. Let $a$ be an ideal of $A$ s.t. $m = u(a) \supseteq c$. Suppose that $A \to A'$ is formally smooth in the $a$-adic topology and that the topology of $b/b^2$ induced by $b$ is $m$-adic. If $\Omega_{B/A} \otimes_B C$ is a projective $C$-module and
$H_1(A, B, C) = 0$, then $A \to B$ is formally smooth in the $m$-adic topology; hence $\Omega_{B/A}$ is a formally projective $B$-module.

**Proof.** From 2.2.1 it results that $\delta_{B/A'/A} \otimes_B C$ (and hence also $\delta_{B/A'/A} \otimes_B K$, where $K = B/m$) is left invertible. $\Omega_{A'/A}$ is a formally projective $A$-module in the $\alpha$-adic topology (0.10), so $\Omega_{A'/A} \otimes_A B$ is a formally projective $B$-module for the $m$-adic topology; hence $\delta_{B/A'/A}$ is formally left invertible (EGA, I, 19.1.9). Now the jacobian criterion of smoothness (EGA, I, 22, 5, 1) says that $A \to B$ is formally smooth in the $m$-adic topology.

Generalisations for the noetherian case. Let $A \to B \to C$ be morphisms of rings, $B$ and $C$ noetherian.

**2.5. Proposition.** Let $c = \text{Ker } v$, $d \supset c$ an ideal of $B$. $D' = B/d$, and $D$ a $(C/cC)$-algebra. Suppose also that the topology of $B$ is $d$-adic, that of $A$ is $(d \cap A)$-adic and that of $C$ is $dC$-adic.

i) If $d \subset R(B)(= \text{the Jacobson radical of } B)$, $A$ is regular, and $A \to B$ is formally smooth, then $B$ is regular, $\Omega_{B/A} \otimes_B D$ is a projective $D$-module and $f : H_1(A, C, D) \to H_1(B, C, D)$ is injective.

ii) Let $D' \to D$ be faithfully flat. Let $H_3(B, C, D) = 0$ (e.g. if $B \to C$ is a Koszul morphism ([9], 3.2.2); in particular, if $v$ is surjective and $c$ generated by a regular sequence ([9], 3.2.1)). If $\Omega_{B/A} \otimes_B D$ is a projective $D$-module and $f$ is injective, then $A \to B$ is formally smooth.

iii) Let $D' \to D$ be faithfully flat, $H_2(B, C, D) = 0$ and $A \to C$ formally smooth. If $d$ is maximal or $B \to C$ is formally étale, then $A \to B$ is formally smooth.

**Proof.** $A \to B \to C$ and $D$ give the exact sequence

\begin{equation}
(2.5.1) \quad H_2(B, C, D) \to H_1(A, B, D) \to H_1(A, C, D) \overset{f}{\to} H_1(B, C, D) \to \Omega_{B/A} \otimes_B D \to \Omega_{C/A} \otimes_C D \to \Omega_{C/B} \otimes_C D \to 0
\end{equation}

i) From 1.1, it results that $\Omega_{B/A} \otimes_B D'$ is a projective $D$-module and $H_1(A, B, D') = 0$. Then $\Omega_{B/A} \otimes_B D$ is a projective $D$-module and $H_1(A, B, D) = 0$. Indeed, let $R$ be a ring of polynomials over $A$, and $R \to B$ a surjection of $A$-algebras with kernel $b$. Then $A \to R \to B$ and $D'$, $D$ give the exact sequence (0.4, 0.1–0.3).

\begin{align*}
0 \to H_1(A, B, D') & \to b/b^2 \otimes D' \to \Omega_{R/A} \otimes B \to \Omega_{B/A} \otimes D' \to 0 \\
0 \to H_1(A, B, D) & \to b/b^2 \otimes D \to \Omega_{R/A} \otimes B \to \Omega_{B/A} \otimes D \to 0
\end{align*}

since $H_1(A, B, D') = 0$ and $\Omega_{B/A} \otimes_B D'$ is projective, it follows that $H_1(A, B, D) = 0$. It follows immediately that $f$ is injective (2.5.1).

Let $b \subset R(B)$. If $m$ is a maximal ideal of $B$ and $n = A \cap m$ then
$A_n \to B_m$ is formally smooth in the radicial topologies. Since $A_n$ is regular, it follows from 2.0 that $B_m$ also is.

ii) We have that $H_1(A, B, D) = H_1(A, B, D') \otimes_{D'} D$, and $\Omega_{B/A} \otimes_B D = (\Omega_{B/A} \otimes_B D') \otimes_{D'} D$. (0.6)

From the fact that $f$ is injective and $H_2(B, C, D) = 0$, it results that $H_1(A, B, D) = 0$. Hence $H_1(A, B, D') = 0$. Since $D' \to D$ is faithfully flat and $\Omega_{B/A} \otimes_B D$ is $D$-projective, $\Omega_{B/A} \otimes_B D'$ is a projective $D'$-module ([15]). Hence $A \to B$ is formally smooth (1.1).

iii) From the formal smoothness of $A \to C$, it results that $\Omega_{C/A} \otimes_C D$ is a projective $D$-module and $H_1(A, C, D) = 0$. Hence $H_1(A, B, D) = H_1(A, B, D') \otimes_{D'} D$, and $H_1(A, B, D') = 0$.

Let $\mathfrak{b}$ be maximal, i.e. $D'$ a field; then $A \to B$ is formally smooth, because of 1.1.

Let $B \to C$ be formally étale; then $H_1(B, C, D) = 0$ (1.1) and $\Omega_{C/B} = 0$ (and also $H_2(B, C, D) = 0$, if $B, C$ are local ([4])). Hence $\Omega_{C/B} \otimes_B D = 0$.

Now from 2.5.1 it results that $\Omega_{B/A} \otimes_B D = \Omega_{C/A} \otimes_C D$; hence $\Omega_{B/A} \otimes_B D'$ is a projective $D'$-module. Consequently, $A \to B$ is formally smooth (1.1).

In particular, we obtain:

2.6. Corollary. Let $A, B, C, u$ and $v$ be local and $L$ be the residue class field of $C$; the topologies are adic and given by the maximal ideals.

i) If $A$ is regular and $A \to B$ formally smooth, then $B$ is regular and $f : H_1(A, C, L) \to H_1(B, C, L)$ is injective.

ii) If $H_2(B, C, L) = 0$ (e.g. if $B \to C$ is Koszul, or if $B$ is regular and $H_2(C, L, L) = 0$ (use 0.4 and 0.9). This last occurs, for instance, if $C$ is regular, by 0.9), and if $f$ is injective, then $A \to B$ is formally smooth.

iii) If $H_2(B, C, L) = 0$ and $A \to C$ is formally smooth, then $A \to B$ is formally smooth.

2.7. Corollary. Let $A, B, C, u$ and $v$ be local, $B \to C$ surjective with kernel $\mathfrak{c}$, $\mathfrak{b} \supseteq \mathfrak{c}$ an ideal of $B$ and $D = B/\mathfrak{b}$. The topology of $B$ is $d$-adic, that of $A$ is $(\mathfrak{b} \cap A)$-adic.

i) If $A$ is regular and $A \to B$ is formally smooth, then $B$ is regular, $\Omega_{B/A} \otimes_B D$ is a projective $D$-module and $f : H_1(A, C, D) \to \mathfrak{c}/\mathfrak{c}D$ is injective.

ii) If $\mathfrak{c}$ is generated by a $B$-regular sequence (e.g. for $B$ and $C$ regular), and if $\Omega_{B/A} \otimes_B D$ is a projective $D$-module and $f$ is injective, then $A \to B$ is formally smooth.

3.0. In EGA, Olv, 19.7.1 the following smoothness criterion is given: Let $A \to B$ be a local morphism of local noetherian rings and let $k$ be the residue class field of $A$; the topologies are adic and given by the maximal ideals. Then $A \to B$ is formally smooth if and only if $A \to B$ is flat and $k \to k \otimes_A B$ is formally smooth.
The following counter-example given by N. Radu shows that $B$ must be noetherian for this criterion to be valid.

(3.0.1) Let $k$ be a perfect field and $B$ a $k$-algebra which is a non-discrete valuation ring of dimension 1; let $m$ be the maximal ideal of $B$, and $K = B/m$. Then $B \to K$ is formally étale.

Indeed, $m = m^2$; $k \to B \to K$ and $K$ give the exact sequence (0.4):

$$0 = m/m^2 \to \Omega_{B/k} \otimes_B K \xrightarrow{v_{K/B/k}} \Omega_{K/k} \to 0.$$ 

Hence $v_{K/B/k}$ is left invertible; but this means that $K$ is a formally smooth $B$-algebra with respect to $k$ (EGA, OIV, 20.5.7). On the other hand, $K$ is a formally smooth $k$-algebra (EGA, OIV, 19.6.1); hence $B \to K$ is formally smooth.

(A purely homological proof of the above criterion is given in [4].)

The following results concern the descent of formal smoothness; from (3.1) results the "only if" part of 3.0.

3.1. THEOREM. Let $A' \twoheadrightarrow A \twoheadrightarrow B$ be ring-morphisms with $B$ noetherian. Let $B' = A' \otimes_A B$. Let $B$ be local with maximal ideal $m$, and $q$ be a prime ideal of $B$ s.t. $q \cap B = m$. The topologies of $B$ and $B'_q$ are adic and given by the maximal ideals. Suppose that $u$ or $v$ is flat. Then $A' \to B_q$ is formally smooth if and only if $A \to B$ is formally smooth.

PROOF. Let $k = B/m$ and $K = B'_q/qB$. Then

$$H_1(A, B, k) \otimes_K K = H_1(A, B, K) = H_1(A', B'_q, K) = H_1(A', B'_q, K)$$

(by 0.6, 0.7 and 0.5). Now apply (1.1).

3.2. PROPOSITION. Let $A \to B$ be a morphism of topological rings, $B$ noetherian, and $a \subset A$, $b \subset B$ ideals with $aB \subset b$, such that the topology of $A$ is $a$-adic and that of $B$, $b$-adic. Then, if (1) or (2) holds, $A \to B$ is formally smooth if $A' \to B'$ is.

(1) $A' = A/a$, $B' = B/b$, $A \to B$ flat.

(2) $A'$ a faithfully flat $A$-algebra, with the $(aA)$-adic topology, and $B' = A' \otimes_A B$ (which has the $(bB)$-adic topology).

Moreover, in (2), $A \to B$ is formally étale if $A' \to B'$ is.

PROOF. Let $C = B/b$ and $C' = B'/bB'$; then $C \to C'$ is faithfully flat. Hence, $\Omega_{B/A} \otimes_B B = \Omega_{B'/A'}(0.7)$ so that

$$(\Omega_{B/A} \otimes_B C) \otimes_C C' = \Omega_{B'/A'} \otimes_{B'} C'),$$

and

$$H_1(A, B, C) \otimes_C C' = H_1(A, B, C') = H_1(A', B', C')$$

(0.6 and 0.7).
i) Let be (1). From 1.1 it results that \(\Omega_{B'/A'} \otimes B' C = \Omega_{B/A} \otimes B C\) is a projective \(C\)-module and \(H_1(A, B, C) = H_1(A', B', C) = 0\); now apply 1.1 again.

ii) Let be (2). Let \(A' \to B'\) be formally smooth; then \(\Omega_{B'/A'} \otimes B' C'\) is a projective \(C'\)-module and \(H_1(A', B', C') = 0\) (1.1). Hence, by the above equalities, \(\Omega_{B/A} \otimes \dot{B} C\) is a projective \(C\)-module ([15]) and \(H_1(A, B, C) = 0\); then \(A \to B\) is formally smooth (1.1)

Let \(A' \to B'\) be formally étale, so \(\Omega_{B'/A'} = 0\). Hence, as above, \(\Omega_{B/A} \otimes B C = 0\) and \(H_1(A, B, C) = 0\). Let \(M_n = \Omega_{B'/A'} \otimes B' / \mathfrak{b}^n B'\) and \(M_n = \Omega_{B/A} \otimes B / \mathfrak{b}^n B\); then \(M_n = M_n \otimes B / \mathfrak{b}^n B' / \mathfrak{b}^n B'\) (0.7). Then \(\hat{\Omega}_{B'/A'} = 0\) gives \(M_n = 0\), but \(B / \mathfrak{b}^n \to B' / \mathfrak{b}^n B'\) is faithfully flat, and so \(M_n = 0\). Hence \(\hat{\Omega}_{B/A} = 0\). Now use 0.10 and 1.1. (Observe that 3.1, 3.2 are formally very similar and probably both follow from a more general statement.)

3.3. REMARK. i) Let \(A' \to A \to B\) be morphisms of rings, \(B' = A' \otimes A B\) and \(A \to A'\) faithfully flat; the topologies are discrete. Then \(A' \to B'\) is formally smooth (resp. étale) iff \(A \to B\) is formally smooth (resp. étale).

Indeed \(\Omega_{B/\mathfrak{A}B} \otimes B' = \Omega_{B'/\mathfrak{A}A'}\) (0.7) and \(H_1(A, B, B) \otimes B = H_1(A', B', B')\) by 0.6 and 0.7. Now apply 0.10.

ii) Let \(X' \to X \to Y\) be morphisms of schemes, \(Y' = X' \times X Y\), and \(X' \to Y\) faithfully flat. If \(Y' \to X'\) is formally étale (resp. locally formally smooth), then \(Y \to X\) is formally étale (resp. locally formally smooth).

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