COMPOSITIO MATHEMATICA

A. BREZULEANU

Smoothness and regularity

Compositio Mathematica, tome 24, nº 1 (1972), p. 1-10
<http://www.numdam.org/item?id=CM_1972__24_1_1_0>

COMPOSITIO MATHEMATICA, Vol. 24, Fasc. 1, 1972, pag. 1-10
Wolters-Noordhoff Publishing
Printed in the Netherlands

SMOOTHNESS AND REGULARITY

by

A. Brezuleanu

5th Nordic Summerschool in Mathematics
Oslo, August 5–25, 1970

Some applications of the cotangent complex to smoothness and regularity are given; in particular, the proof of a criterion for formal smoothness which was conjectured in [8] (see 2.1), and some generalisations of this criterion for the non-noetherian and noetherian cases (2.2, 2.6, 2.7). Also considered is the descent of formal smoothness.

0. All the rings considered are commutative with unity; the topologies are linear. The definitions and notations used are as in EGA, 0_{IV}, §§ 19–20, and [1]. The following facts about the cotangent complex will be needed:

Let $A \to B$ be a morphism of rings; to any B-module M are associated the B-modules $H_i(A, B, M)$, $H^i(A, B, M)$. (For the definitions, see: [8] for $i = 0, 1$; [9] for $i = 0, 1, 2$; [1] or [14] for $i \geqq 0$. At least for $i = 0$, 1, the various definitions give isomorphic modules. This follows from the properties 0.1–0.4 below ([6], 3.5).) $\{H_i(\text{resp. } H^i), i \geqq 0\}$ is a (co) homological functor and has the properties.

0.1. $H_0(A, B, M) = \Omega_{B/A} \otimes_B M$ (where $\Omega_{B/A}$ is the module of A-differentials in B); $H^0(A, B, M) = \text{Der}_A(B, M)$(= the module of A-derivations of B in M; see [9], 2.3).

0.2. If $A \to B$ is surjective with kernel $\mathfrak{b}$, then $H_1(A, B, M) = \mathfrak{b}/\mathfrak{b}^2 \otimes_B M$ and $H^1(A, B, M) = \text{Hom}_B(b/b^2, M)$([9], 3.1.2).

0.3. If B is a polynomialring over A, then $H_i(A, B, .) = 0 = H^i(A, B, .)$ for $i \geqq 1$([9], 3.1.1 or [1], 16.3).

0.4. If $A \to B \to C$ are morphisms of rings and M is a C-module, then the sequence

$$\cdots \to H_i(A, B, M) \to H_i(A, C, M) \to H_i(B, C, M) \to H_{i-1}(A, B, M) \to \cdots \to H_0(B, C, M) \to 0$$

is exact ([9], 2.3.5 or [1], 18.2), and similarly for H^i, with arrows reversed.

0.5. If B is an A-algebra and S a multiplicatively closed system in B, then the canonical morphism $H_i(A, B, M) \otimes_B S^{-1}B \tilde{\to} H_i(A, S^{-1}B, S^{-1}M)$ is an iscmorphism ([9], 2.3.4. or [1], 16).

0.6. If $A \to B \to C$ are morphisms of rings and M a flat C-module, then the canonical morphism $H_i(A, B, C) \otimes_C M \tilde{\to} H_i(A, B, M)$ is an isomorphism.

0.7. Let $A' \overset{v}{\leftarrow} A \overset{u}{\to} B$ be morphisms of rings, where u or v is flat, $B' = A' \otimes_A B$, and M a B'-module. Then the canonical morphism $H_i(A, B, M) \tilde{\to} H_i(A', B', M)$ is an isomorphism ([9], 2.3.2. or [1], 19.2).

0.8. If $A \to B$ are fields, then $H_i(A, B, \cdot) = O = H^i(A, B, \cdot)$ for $i \geqq 2$ ([9], 3.5 or [1], 22.2) and $A \to B$ is separable iff it is formally smooth (EGA, O_{IV}, 19.6.1 or 9, 3.5).

0.9. Let A be a local, noetherian ring, B its residue class field and K a field which is an extension of B. Then the following three statements are equivalent: A is regular: $H_2(A, K, K)$ is zero; $H_i(A, K, K) = 0$ for $i \geqq 2$. This follows from [9], 3.2.1 or [1], [1], 27.1 and 27.2, using 0.6 and 0.8.

The following criteria (0.10 and 1.1) are essentially the discrete and non-discrete forms of the Jacobian criterion of smoothness (EGA, O_{IV}, 22.6.1 and 22.6.2).

0.10. A morphism of rings $A \to B$ is formally smooth in the discrete topologies if and only if $\Omega_{B/A}$ is a projective B-module and $H_1(A, B, B) = 0$ ([8], 9.5.7 or [9], 3.1.3).

1.1. Let $A \to B$ be a morphism of topological rings: the topology of B is $\mathfrak{c}$-adic for some ideal $\mathfrak{c}$ in B, and that of A is also adic. Let $C = B/\mathfrak{c}$. If $A \to B$ is formally smooth then $\Omega_{B/A} \otimes_B C$ is a projective C-module and $H_1(A, B, C) = 0$: the converse is also true if B is a noetherian ring.

1.2. For A and B noetherian, 1.1 is proved in [2], 5.4. The proof of 1.1 in general is based on the same ideas. Let $A \to B$ be formally smooth. Then $\Omega_{B/A}$ is a formally projective B-module (EGA, O_{IV}, 20.4.9), hence $\Omega_{B/A} \otimes_B C$ is a projective C-module (II, IX, 1.25). Let R be a polynomialring over A, and $R \to B$ a surjection of A-algebras with kernel $\mathfrak{b}$. Then the following sequence is exact (0.4 and 0.1–0.3).

$$(1.2.1) \quad 0 \to H_1(A, B, C) \to \mathfrak{b}/\mathfrak{b}^2 \underset{B}{\otimes} C \xrightarrow[\delta_{B/R/A} \otimes C]{} \Omega_{R/A} \otimes C \to \Omega_{B/A} \underset{B}{\otimes} C \to 0.$$

But $\delta_{B/R/A}$ is formally left invertible (EGA, O_{IV}, 20.7.8 and 19. 4.4), so $\delta_{B/R/A} \otimes_B C$ is injective, and $H_1(A, B, C) = 0$.

Now, let B be noetherian, $\Omega_{B/A} \otimes_B C$ be a projective C-module, and $H_1(A, B, C) = 0$. Let R and $\mathfrak{b}$ be as above. Then ([9], 3.1.2) and (1.2.1) imply that $H^1(A, B, M) = 0$ for any C-module M, and hence for any discrete B-module M with open annihilator. Let A_d, B_d denote the rings A, B with the discrete topology, and A_t, B_t the rings A, B with the given topologies. Then $H^1(A, B, M) = 0$ means $H^1(A_d, B_d, M) = 0$. $A_d \to B_d \to B_t$ and M give the exact sequence ([2], 2 or EGA, 0_{IV}, 20.3.7)

$$0 \to H_t^0(A_d, B_t, M) \xrightarrow{u} H_t^0(A_d, B_d, M) \to H_t^1(B_d, B_t, M) \xrightarrow{v} H_t^1(A_d, B_t, M) \to H^1(A_d, B_d, M) = 0$$

But u is an isomorphism (EGA, 0_{IV}, 20.3.3), so v also is. B is noetherian, hence $H_t^1(B_d, B_t, M) = 0$([2], 5.1): it follows that $H_t^1(A_d, B_t, M) = 0$. Then $A_d \to A_t \to B_t$ and M give the exact sequence

$$H_t^0(A_d, A_t, M) \to H_t^1(A_t, B_t, M) \to H_t^1(A_d, B_t, M),$$

where the first and third terms are zero. The formal smoothness of $A_t \to B_t$ now follows (EGA, 0_{IV}, 19.4.4).

1.3. REMARKS. (i) I do not know if the converse part of 1.1 remains valid for B a non-noetherian ring.

(ii) The criterion 1.1 can be reformulated as follows: it B is noetherian, then $A \to B$ is formally smooth iff $\Omega_{B/A}$ is a formally projective B-module and $H_1(A, B, C) = 0$. The results of N. Radu ([11], [12], [13]) shows that the condition $H_1(A, B, C) = 0$ is superfluous in this form of 1.1 if B is a laskerian local ring with the $\mathfrak{m}$-adic topology ($\mathfrak{m}$ the maximal ideal of B), and A is a field of characteristic zero (or arbitrary characteristic if B is noetherian).

2.0. It is known that, if $A \to B$ is a local morphism of local noetherian rings, formally smooth in the topologies given by the maximal ideals, *then A is regular if and only if B is regular.*

PROOF. Let K be the residue class field of B, then the maps $A \to B \to K$ give the exact sequence (0.4):

$$H_2(A, B, K) \to H_2(A, K, K) \to H_2(B, K, K) \to H_1(A, B, K).$$

But $H_1(A, B, K) = 0$(1.1) and $H_2(A, B, K) = 0$ ([4], corollary). Now apply 0.9.

2.1. In ([8], 9.6), the following criterion for formal smoothness is stated and partially proved:

THEOREM. *Let $A \to B \to C$ be local homomorphisms of local noetherian rings, A and C regular, $B \to C$ surjective, so that $C = B/\mathfrak{c}$, $\mathfrak{c}$ an ideal of B. Finally, let B be a localisation of a finitely generated A-algebra. Then B is* *formally smooth over A if and only if the following three conditions are satisfied*:

a) *B is regular, i.e.* $\mathfrak{c}$ *is a regular ideal*;

b) $\Omega_{B/A} \otimes_B C$ *is a projective C-module*;

c) *The echaracteristic homomorphism*

$$N_{C/A} \to \mathfrak{c}/\mathfrak{c}^2$$

is injective.

(The morphism from *c*) is $H_1(A, C, C) \to H_1(B, C, C)$ (0.2)

In [8], it was proved that the conditions are necessary. The sufficiency, however, was only proved for A a field and the sufficiency in general conjectured. I gave generalisations of this criterion for the non-noetherian and noetherian case ([5]). For the noetherian case *I* use essentially (1.1), but this does not work in the non-noetherian case. A direct proof for 2.1 is as follows:

Let $A \to B$ be formally smooth (here the topology is arbitrary, cf. EGA, 0_{IV}, 22.6.4). Then B is regular (by 2.0), $\Omega_{B/A} \otimes_B C$ is a projective C-module and $H_1(A, B, C) = 0$ (by 0.10). From the exact sequence (0.4),

$$H_2(B, C, C) \to H_1(A, B, C) \to H_1(A, C, C) \xrightarrow{f} H_1(B, C, C) = \mathfrak{c}/\mathfrak{c}^2,$$

it results that f is injective.

Let a), b), c) be satisfied. Then $H_2(B, C, C) = 0$([9], 3.2.1) since $\mathfrak{c}$ is generated by a B-regular sequence. Hence, by the above sequence and c), $H_1(A, B, C) = 0$; this and b) imply that B is a formally smooth A-algebra for the $\mathfrak{c}$-adic topology (1.1).

We now turn to the non-noetherian case. Let $A \to A' \xrightarrow{u} B \xrightarrow{v} C$ be morphisms of rings, u and v surjective, $\mathfrak{b} = \operatorname{Ker} u$, $\mathfrak{c} = \operatorname{Ker} v$.

2.2. Theorem. *Suppose that A' is a formally smooth A-algebra* (in the $\mathfrak{b}$-adic topology) *and that* $\mathfrak{b}/\mathfrak{b}^2$ *is* $\mathfrak{c}$-*separated* (*or is a B-module of finite type with B local*). *Then* $A \to B$ *is formally smooth* (in the discrete topologies) *if and only if* $\Omega_{B/A} \otimes_B C$ *is a projective C-module and* $H_1(A, B, C) = 0$.

Proof. The necessity results from (0.10).

For the converse two facts are necessary.

(2.2.1) *If* $\Omega_{B/A} \otimes_B C$ *is a projective C-module and* $H_1(A, B, C) = 0$ *then*

$$\delta_{B/A/A} \underset{B}{\otimes} C : b/b^2 \underset{B}{\otimes} C \to \Omega_{A'/A} \underset{A'}{\otimes} C$$

is left invertible. For $A \to A' \to B$ and C give the exact sequence

$$0 = H_1(A, B, C) \to H_1(A', B, C) \xrightarrow[\delta_{B/A'/A} \otimes_B C]{} H_0(A, A', C) \to$$
$$\to H_0(A, B, C) \to 0,$$

by 0.4 and 0.1, 0.2. Then 2.2.1 results from the projectivity of $\Omega_{B/A} \otimes_B C = H_0(A, B, C)$.

(2.2.2) *Let $h : M \to N$ be a morphism of B-modules, $\mathfrak{c}$ an ideal of B, M a $\mathfrak{c}$ - separated B-module, and N a projective B-module. If $h_1 = h \otimes_B B/\mathfrak{c} : M/\mathfrak{c}M \to N/\mathfrak{c}N$ is left invertible, then h is also left invertible.* To see this, let $g' : N/\mathfrak{c}N \to M/\mathfrak{c}M$ be a left inverse for h_1, Since N is projective, there is a morphism $g : N \to M$, such that the composition $N \xrightarrow{g} M \to M/\mathfrak{c}M$ equals $N \to N/\mathfrak{c}N \xrightarrow{g'} M/\mathfrak{c}M$. It is obvious that $g_1 = g'$, where the subscript 1 means $\otimes_B B/\mathfrak{c}$, i.e. that $(gh)_1 = 1$. It follows that $(gh)_r = gh \otimes_B B/\mathfrak{c}^r$ is equal to 1 (see the proof of IL, XII, 2, 2, 1). Let $x \in M$; then $x-(gh)(x) \in \mathfrak{c}^r M$ for any $r \geqq 1$, and so $gh = 1$.

Now let $\Omega_{B/A} \otimes_B C$ be a projective B-module and $H_1(A, B, C) = 0$. Then $\delta_{B/A'/A} \otimes_B C$ is left invertible (2.2.1); hence $\delta_{B/A'/A}$ is also left invertible (for $\Omega_{A'/A} \otimes_{A'} B$ is a projective B-module by 0.10. Now, if $\mathfrak{b}/\mathfrak{b}^2$ is $\mathfrak{c}$-separated, apply 2.2.2; otherwise apply EGA, 0_{IV}, 19.1.12). Hence B is a formally smooth A-algebra (EGA, 0_{IV}, 20.5.12).

2.2.3. REMARKS. (i) The hypotheses of (2.2) are fulfilled if B is an A-algebra essentially of finite presentation and C any quotient ring of B.

(ii) Under the hypotheses of 2.2, assume $H_2(B, C, C) = 0$ (conditions for this are given in [3], 5.1, or [9], 3.2.1); then $A \to B$ is formally smooth if and only if $\Omega_{B/A} \otimes_B C$ is a projective C-module and $H_1(A, B, C) \to \mathfrak{c}/\mathfrak{c}^2$ is injective. (This shows that for the 'only if' part of 2.1, sufficient hypotheses on $A \to B$ are that B be noetherian and an A-algebra essentially of finite presentation.)

2.3 COROLLARY. *Let $Z \xrightarrow{i} Y \xrightarrow{h} X$ be morphisms of schemes, i being an immersion and h being locally of finite presentation. Then h is smooth in a neighbourhood of Z in Y if and only if $H_0(X, Y, O_Z)$ is a flat O_Z-Module and $H_1(X, Y, O_Z) = 0$.*

PROOF. Let $z \in Z$, $y = i(z)$ and $x = h(y)$. Then

$$H_i(X, Y, O_Z)_z = H_i(O_{X,x}, O_{Y,y}, O_{Z,z}), \text{ by } 0.5$$

Since $H_0(O_{X,x}, O_{Y,y}, O_{Z,z})$ is an $O_{Z,z}$-module of finite presentation, flat means projective. Then $h_y : O_{X,x} \to O_{Y,y}$ is smooth for any $z \in Z$ (cf. 2.2.3 and 2.2). But this is an open property (EGA, IV, 17.5). Hence h is locally smooth, i.e. it is smooth ([8] 9.5.6).

2.4. For the non-dicrete topologies, 2.2 takes the following form:

PROPOSITION. *Let $\mathfrak{a}$ be an ideal of A* s.t. $\mathfrak{m} = u(\mathfrak{a}) \supset \mathfrak{c}$. *Suppose that $A \to A'$ is formally smooth in the $\mathfrak{a}$-adic topology and that the topology of $\mathfrak{b}/\mathfrak{b}^2$ induced by $\mathfrak{b}$ is $\mathfrak{m}$-adic. If $\Omega_{B/A} \otimes_B C$ is a projective C-module and*

$H_1(A, B, C) = 0$, then $A \to B$ is formally smooth in the $\mathfrak{m}$-adic topology; hence $\Omega_{B/A}$ is a formally projective B-module.

PROOF. From 2.2.1 it results that $\delta_{B/A'/A} \otimes_B C$ (and hence also $\delta_{B/A'/A} \otimes_B K$, where $K = B/\mathfrak{m}$) is left invertible. $\Omega_{A'/A}$ is a formally projective A-module in the $\mathfrak{a}$-adic topology (0.10), so $\Omega_{A'/A} \otimes_A B$ is a formally projective B-module for the $\mathfrak{m}$-adic topology; hence $\delta_{B/A'/A}$ is formally left invertible (EGA, 0_{IV}, 19.1.9). Now the jacobian criterion of smoothness (EGA, 0_{IV}, 22, 5, 1) says that $A \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.

Generalisations for the noetherian case. Let $A \xrightarrow{u} B \xrightarrow{v} C$ be morphisms of rings, B and C noetherian.

2.5. PROPOSITION. *Let $\mathfrak{c} = \operatorname{Ker} v$, $\mathfrak{d} \supset \mathfrak{c}$ an ideal of B. $D' = B/\mathfrak{d}$, and D a $(C/\mathfrak{d}C)$-algebra. Suppose also that the topology of B is $\mathfrak{d}$-adic, that of A is $(\mathfrak{d} \cap A)$-adic and that of C is $\mathfrak{d}C$-adic.*

i) *If $\mathfrak{d} \subset R(B)$(=* the Jacobson radical of *B), A is regular, and $A \to B$ is formally smooth, then B is regular, $\Omega_{B/A} \otimes_B D$ is a projective D-module and $f: H_1(A, C, D) \to H_1(B, C, D)$ is injective.*

ii) *Let $D' \to D$ be faithfully flat. Let $H_2(B, C, D) = 0$* (e.g. if $B \to C$ is a Koszul morphism ([9], 3.2.2); in particular, if v is surjective and $\mathfrak{c}$ generated by a regular sequence ([9], 3.2.1)). *If $\Omega_{B/A} \otimes_B D$ is a projective D-module and f is injective, then $A \to B$ is formally smooth.*

iii) *Let $D' \to D$ be faithfully flat, $H_2(B, C, D) = 0$ and $A \to C$ formally smooth. If $\mathfrak{d}$ is maximal or $B \to C$ is formally étale, then $A \to B$ is formally smooth.*

PROOF. $A \to B \to C$ and D give the exact sequence

$$(2.5.1) \qquad H_2(B, C, D) \to H_1(A, B, D) \to H_1(A, C, D) \xrightarrow{f} H_1(B, C, D) \to \\ \to \Omega_{B/A} \otimes_B D \to \Omega_{C/A} \otimes_C D \to \Omega_{C/B} \otimes_C D \to 0$$

i) From 1.1, it results that $\Omega_{B/A} \otimes_B D'$ is a projective D-module and $H_1(A, B, D') = 0$. Then $\Omega_{B/A} \otimes_B D$ is a projective D-module and $H_1(A, B, D) = 0$. Indeed, let R be a ring of polynomials over A, and $R \to B$ a surjection of A-algebras with kernel $\mathfrak{b}$. Then $A \to R \to B$ and D', D give the exact sequence (0.4, 0.1–0.3).

$$0 \to H_1(A, B, D') \to b/b^2 \otimes_B D' \to \Omega_{R/A} \otimes_R D' \to \Omega_{B/A} \otimes_B D' \to 0$$

$$0 \to H_1(A, B, D) \to b/b^2 \otimes_B D \to \Omega_{R/A} \otimes_R D \to \Omega_{B/A} \otimes_B D \to 0$$

since $H_1(A, B, D') = 0$ and $\Omega_{B/A} \otimes_B D'$ is projective, it follows that $H_1(A, B, D) = 0$. It follows immediately that f is injective (2.5.1).

Let $\mathfrak{d} \subset R(B)$. If $\mathfrak{m}$ is a maximal ideal of B and $\mathfrak{n} = A \cap \mathfrak{m}$ then

$A_{\mathfrak{n}} \to B_{\mathfrak{m}}$ is formally smooth in the radicial topologies. Since $A_{\mathfrak{n}}$ is regular, it follows from 2.0 that $B_{\mathfrak{m}}$ also is.

ii) We have that $H_1(A, B, D) = H_1(A, B, D') \otimes_{D'} D$, and $\Omega_{B/A} \otimes_B D = (\Omega_{B/A} \otimes_B D') \otimes_{D'} D$. (0.6)

From the fact that f is injective and $H_2(B, C, D) = 0$, it results that $H_1(A, B, D) = 0$. Hence $H_1(A, B, D') = 0$. Since $D' \to D$ is faithfully flat and $\Omega_{B/A} \otimes_B D$ is D-projective, $\Omega_{B/A} \otimes_B D'$ is a projective D'-module ([15]). Hence $A \to B$ is formally smooth (1.1).

iii) From the formal smoothness of $A \to C$, it results that $\Omega_{C/A} \otimes_C D$ is a projective D-module and $H_1(A, C, D) = 0$(1.1). Since $H_2(B, C, D) = 0$, we find from 2.5.1 that $H_1(A, B, D) = 0$. But $H_1(A, B, D) = H_1(A, B, D') \otimes_{D'} D$, hence $H_1(A, B, D') = 0$.

Let $\mathfrak{d}$ be maximal, i.e. D' a field; then $A \to B$ is formally smooth, because of 1.1.

Let $B \to C$ be formally étale; then $H_1(B, C, D) = 0$(1.1) and $\hat{\Omega}_{C/B} = 0$ (and also $H_2(B, C, D) = 0$, if B, C are local ([4])). Hence $\Omega_{C/B} \otimes_B D = 0$. Now from 2.5.1 it results that $\Omega_{B/A} \otimes_B D = \Omega_{C/A} \otimes_C D$; hence $\Omega_{B/A} \otimes_B D'$ is a projective D'-module. Consequently, $A \to B$ is formally smooth (1.1). In particular, we obtain:

2.6. Corollary. *Let A, B, C, u and v be local and L be the residue class field of C; the topologies are adic and given by the maximal ideals.*

i) *If A is regular and $A \to B$ formally smooth, then B is regular and $f: H_1(A, C, L) \to H_1(B, C, L)$ is injective.*

ii) *If $H_2(B, C, L) = 0$* (e.g. if $B \to C$ is Koszul, or if B is regular and $H_3(C, L, L) = 0$ (use 0.4 and 0.9). This last occurs, for instance, if C is regular, by 0.9), *and if f is injective, then $A \to B$ is formally smooth.*

iii) *If $H_2(B, C, L) = 0$ and $A \to C$ is formally smooth, then $A \to B$ is formally smooth.*

2.7. Corollary. *Let A, B, C, u and v be local, $B \to C$ surjective with kernel $\mathfrak{c}$, $\mathfrak{d} \supset \mathfrak{c}$ an ideal of B and $D = B/\mathfrak{d}$. The topology of B is $\mathfrak{d}$-adic, that of A is $(\mathfrak{d} \cap A)$-adic.*

i) *If A is regular and $A \to B$ is formally smooth, then B is regular, $\Omega_{B/A} \otimes_B D$ is a projective D-module and f: $H_1(A, C, D) \to \mathfrak{c}/\mathfrak{d}\mathfrak{c}$ is injective.*

ii) *If $\mathfrak{c}$ is generated by a B-regular sequence* (e.g. for B and C regular), *and if $\Omega_{B/A} \otimes_B D$ is a projective D-module and f is injective, then $A \to B$ is formally smooth.*

3.0. In EGA, 0_{IV}, 19.7.1 the following smoothness criterion is given: Let $A \to B$ be a local morphism of local noetherian rings and let k be the residue class field of A; the topologies are adic and given by the maximal ideals. Then $A \to B$ is formally smooth if and only if $A \to B$ is flat and $k \to k \otimes_A B$ is formally smooth.

The following counter-example given by *N.* Radu shows that B must be noetherian for this criterion to be valid.

(3.0.1) *Let k be a perfect field and B a k-algebra which is a non-discrete valuation ring of dimension* 1; *let $\mathfrak{m}$ be the maximal ideal of B, and $K = B/\mathfrak{m}$. Then $B \to K$ is formally étale.*

Indeed, $\mathfrak{m} = \mathfrak{m}^2$; $k \to B \to K$ and K give the exact sequence (0.4):

$$0 = \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \underset{B}{\otimes} K \xrightarrow{v_{K/B/k}} \Omega_{K/k} \to 0.$$

Hence $v_{K/B/k}$ is left invertible; but this means that K is a formally smooth B-algebra with respect to k(EGA, O_{IV}, 20.5.7). On the other hand, K is a formally smooth k-algebra (EGA, O_{IV}, 19.6.1); hence $B \to K$ is formally smooth.

(A purely homological proof of the above criterion is given in [4].)

The following results concern the descent of formal smoothness; from (3.1) results the 'only if' part of 3.0.

3.1. THEOREM. *Let $A' \overset{v}{\leftarrow} A \overset{u}{\rightarrow} B$ be ring-morphisms with B noetherian. Let $B' = A' \otimes_A B$. Let B be local with maximal ideal $\mathfrak{m}$, and $\mathfrak{q}$ be a prime ideal of B'* s.t. $\mathfrak{q} \cap B = \mathfrak{m}$. *The topologies of B and $B'_{\mathfrak{q}}$ are adic and given by the maximal ideals. Suppose that u or v is flat. Then $A' \to B_{\mathfrak{q}}$ is formally smooth if and only if $A \to B$ is formally smooth.*

PROOF. Let $k = B/\mathfrak{m}$ and $K = B'_{\mathfrak{q}}/\mathfrak{q}B$. Then

$$H_1(A, B, k) \underset{k}{\otimes} K = H_1(A, B, K) = H_1(A', B'_{\mathfrak{q}}, K) = H_1(A', B'_{\mathfrak{q}}, K)$$

(by 0.6, 0.7 and 0.5). Now apply (1.1).

3.2. PROPOSITION. *Let $A \to B$ be a morphism of topological rings, B noetherian, and $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$ ideals with $\mathfrak{a} B \subset \mathfrak{b}$, such that the topology of A is $\mathfrak{a}$-adic and that of B, $\mathfrak{b}$-adic. Then, if* (1) *or* (2) *holds, $A \to B$ is formally smooth if $A' \to B'$ is.*

(1) *$A' = A/\mathfrak{a}$, $B' = B/\mathfrak{b}$, $A \to B$ flat.*

(2) *A' a faithfully flat A-algebra, with the $(\mathfrak{a}A)$ – adic topology, and* $B' = A' \otimes_A B$ (which has the $(\mathfrak{b}B)$ – adic topology).

Moreover, in (2), *$A \to B$ is formally étale if $A' \to B'$ is.*

PROOF. Let $C = B/\mathfrak{b}$ and $C' = B'/\mathfrak{b}B'$; then $C \to C'$ is faithfully flat. Hence, $\Omega_{B/A} \otimes_B B = \Omega_{B'/A'}$(0.7) so that

$$(\Omega_{B/A} \underset{B}{\otimes} C) \underset{C}{\otimes} C' = \Omega_{B'/A'} \underset{B'}{\otimes} C'),$$

and

$$H_1(A, B, C) \underset{C}{\otimes} C' = H_1(A, B, C') = H_1(A', B', C')$$

(0.6 and 0.7).

i) Let be (1). From 1.1 it results that $\Omega_{B'/A'} \otimes_{B'} C (= \Omega_{B/A} \otimes_B C)$ is a projective C-module and $H_1(A, B, C) = H_1(A', B', C) = 0$; now apply 1.1 again.

ii) Let be (2). Let $A' \to B'$ be formally smooth; then $\Omega_{B'/A'} \otimes_{B'} C'$ is a projective C'-module and $H_1(A', B', C') = 0$ (1.1). Hence, by the above equalities, $\Omega_{B/A} \otimes_B C$ is a projective C-module ([15]) and $H_1(A, B, C) = 0$; then $A \to B$ is formally smooth (1.1)

Let $A' \to B'$ be formally étale, so $\hat{\Omega}_{B'/A'} = 0$ $H_1(A', B', C') = 0$(0.10 and 1.1). Hence, as above, $\Omega_{B/A} \otimes_B C = 0$ and $H_1(A, B, C) = 0$. Let $M'_n = \Omega_{B'/A'} \otimes_{B'} B'/\mathfrak{b}^n B'$ and $M_n = \Omega_{B/A} \otimes_B B/\mathfrak{b}^n$; then $M'_n = M_n \otimes_{B/\mathfrak{b}^n} B'/\mathfrak{b}^n B'$ (0.7. Then $\hat{\Omega}_{B'/A'} = 0$ gives $M'_n = 0$, but $B/\mathfrak{b}^n \to B'/\mathfrak{b}^n B'$ is faithfully flat, and so $M_n = 0$. Hence $\hat{\Omega}_{B/A} = 0$. Now use 0.10 and 1.1. (Observe that 3.1, 3.2 are formally very similar and probably both follow from a more general statement.)

3.3. Remark. i) *Let $A' \leftarrow A \to B$ be morphisms of rings, $B' = A' \otimes_A B$ and $A \to A'$ faithfully flat; the topologies are discrete. Then $A' \to B'$ is formally smooth* (resp. *étale*) *iff $A \to B$ is formally smooth* (resp. *étale*).

Indeed $\Omega_{B/A} \otimes_B B' = \Omega_{B'/A'}$ (0.7) and $H_1(A, B, B) \otimes_B B = H_1(A', B, B') = H_1(A', B', B')$, by 0.6 and 0.7. Now apply 0.10.

ii) *Let $X' \to X \leftarrow Y$ be morphisms of schemes, $Y' = X' \times_X Y$, and $X' \to Y$ faithfully flat. If $Y' \to X'$ is formally étale* (resp. *locally formally smooth*), *then $Y \to X$ is formally étale* (resp. *locally formally smooth*).

BIBLIOGRAPHY

M. André
[1] Méthodes simpliciales en Algèbre Homologique et Algèbre Commutative, Lecture Notes in Math., No. 32 (1967).

M. André
[2] Groupes des Cohomologies pour les Algèbres Topologiques, Comm. Math. Helv., 43 (1968), p. 235–255

M. André
[3] On the vanishing of the second homology group of a commutative algebra, Lecture Notes in Math., (1968).

M. André
[4] Démonstration homologique d'un théorème sur les algèbres lisses, Battelle Institut, 1969.

A. Brezuleanu
[5] Sur un critère de lissité formelle, I, II, C.R. Acad. Sc. Paris, t. 269, 270.

A. Brezuleanu
[6] Thèse, Bucharest, 1970.

A. Grothendieck et Dieudonné
EGA Eléments de Géométrie Algébrique, Publ. Math. No. 20 et 32.

A. Grothendieck
[8] Catégories cofibrées additive et complexe cotangent relatif, Lecture Notes in Math., No. 79 (1968).

S. Lichtenbaum and M. Schlessinger
[9] The cotangent complex of a morphism, Trans. Amer. Math. Soc. 1967, pp 41–70.

N. Radu
IL Inele locale, vol. I, II, Editura Academici R.S. Romania, 1968, 1970.

N. Radu
[11] Une caractérisation des algèbres noethériennes régulières sur un corps de caractéristiques zéro, C.R. Acad. Sc. Paris, t. 270, p. 851–853.

N. Radu
[12] Un critère différentiel de lissité formelle pour une algèbre locale noethérienne sur un corps, C.R. Acad. Sc. Paris, t. 270.

N. Radu
[13] Sur la décomposition primaire des idéaux différentiels (to appear).

D. G. Quillen
[14] mimeographed notes.

Gruson et Raynaud
[15] Descente de la projectivité (to appear).

(Oblatum 5–X–1970)

A. Brezuleanu,
Mathematisches Institut,
Universität Bonn,
Beringstrasse 1,
53–BONN,
Germany.

This paper also appears in 'Algebraic Geometry, Oslo, 1970', Proceedings of the 5th Nordic Summer-School in Mathematics, Wolters-Noordhoff Publishing, 1972.