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UPON THE GENERALISED INVERSE OF A FORMAL POWER SERIES WITH VECTOR VALUED COEFFICIENTS

by

P. Wynn

In a recent paper [1] the author has derived a general result concerning the existence and uniqueness of the inverse of a formal power series whose coefficients are elements of an associative ring. Presented in terms of square matrices, this result is as follows: let $p\{T_\nu|z\} = \sum_{\nu=0}^{\infty} T_\nu z^\nu$ be a formal power series whose coefficients $\{T_\nu\}$ are $n \times n$ matrices $(1 \leqq n < \infty)$, with T_0 non-singular, and whose variable z is scalar; then there exists a similar formal power series $p\{\hat{T}_\nu|z\}$ which is uniquely determined by the formal equation

$$p\{T_\nu|z\}p\{\hat{T}_\nu|z\} = I \tag{1}$$

where I is the $n \times n$ unit matrix; furthermore the series $p\{\hat{T}_\nu|z\}$ also satisfies the formal equation

$$p\{\hat{T}_\nu|z\}p\{T_\nu|z\} = I \tag{2}$$

and is uniquely determined by it. The coefficients $\{\hat{T}_\nu\}$ are determined by equating coefficients of similar powers of z in relationship (1); one has

$$\sum_{\nu=0}^{r} T_\nu \hat{T}_{r-\nu} = \begin{matrix} I \\ 0 \end{matrix} \qquad \begin{matrix} (r = 0) \\ (r = 1, 2, \cdots) \end{matrix} \tag{3}$$

so that

$$\hat{T}_0 = T_0^{-1}$$

$$\hat{T}_r = -\hat{T}_0 \sum_{\nu=0}^{r-1} T_\nu \hat{T}_{r-\nu}. \qquad (r = 1, 2, \cdots)$$

Similarly, from relationship (2)

$$\hat{T}_r = -\{\sum_{\nu=0}^{r-1} \hat{T}_{r-\nu} T_\nu\}\hat{T}_0. \qquad (r = 1, 2, \cdots)$$

It is not assumed that either of the power series $p\{T_\nu|z\}$ or $p\{\hat{T}_\nu|z\}$ should converge, or converge asymptotically, or should exhibit any other such property; relationships (3) simply mean that given a sufficient number of coefficients $\{T_\nu\}$, an arbitrarily large number of coefficients $\{\hat{T}_\nu\}$ can be derived.

As is well known ([2]–[5]), every square or rectangular matrix A of finite dimension has a generalised inverse A^+ uniquely determined by the four equations

$$(4)\quad AA^+A = A,\quad A^+AA^+ = A^+,\quad (AA^+)^* = AA^+,\quad (A^+A)^* = A^+A,$$

where the asterisk denotes the complex conjugate transpose. The suggestion naturally prompts itself that the generalised inverse $p\{\hat{T}_\nu|z\}$ of a formal power series $p\{T_\nu|z\}$ with rectangular matrix coefficients (in particular, with square matrix coefficients and T_0 not restricted to being non-singular) might also uniquely be determined by the use of four equations of the form (4) in which $p\{T_\nu|z\}$ and $p\{\hat{T}_\nu|z\}$ replace A and A^+ respectively, and the equations are to be understood as formal equations among formal power series, i.e. that the four relationships

$$(5)\qquad p\{T_\nu|z\}p\{\hat{T}_\nu|z\}p\{T_\nu|z\} = p\{T_\nu|z\}$$

$$(6)\qquad p\{\hat{T}_\nu|z\}p\{T_\nu|z\}p\{\hat{T}_\nu|z\} = p\{\hat{T}_\nu|z\}$$

$$(7)\qquad p\{T_\nu|z\}p\{\hat{T}_\nu|z\} = p\{G_\nu|z\},\quad G_\nu^* = G_\nu \qquad (\nu = 0, 1, \cdots)$$

$$(8)\qquad p\{\hat{T}_\nu|z\}p\{T_\nu|z\} = p\{H_\nu|z\},\quad H_\nu^* = H_\nu \qquad (\nu = 0, 1, \cdots)$$

uniquely determine the series $p\{\hat{T}_\nu|z\}$.

We shall first show that this is not, in general, so. The equations to determine $\hat{T}_0$ are (4) with A, A^+ replaced by $T_0, \hat{T}_0$ respectively, and hence $\hat{T}_0 = T_0^+$. Equating coefficients of z throughout relationship (5), we find that

$$(9)\qquad T_0\hat{T}_0T_1 + T_0\hat{T}_1T_0 + T_1\hat{T}_0T_0 = T_1$$

i.e.

$$(10)\qquad T_0\hat{T}_1T_0 = T_1 - T_0\hat{T}_0T_1 - T_1\hat{T}_0T_0.$$

The general matrix equation

$$(11)\qquad AXB = C$$

has a solution if and only if $AA^+CB^+B = C$. Hence, from equation (10), $\hat{T}_1$ can be constructed if and only if

$$(12)\qquad (I - T_0\hat{T}_0)T_1(\hat{T}_0T_0 - I) = 0.$$

One can easily devise examples of pairs of matrices T_0 and T_1 such that condition (12) does not hold, and hence the general process upon which the determination of the series $p\{\hat{T}_\nu|z\}$ is tentatively to be based has already broken down.

It is, however, clear that in two special cases the difficulty described in the preceding paragraph does not arise.

The first is that in which T_0 and T_1 are square matrices of finite dimension, T_0 being non-singular. We then have $\hat{T}_0 = T_0^{-1}$, and condition (12) is automatically fulfilled. In this case, however, the four conditions (5)–(8) serve only to determine the inverse power series $p\{\hat{T}_\nu|z\}$ uniquely determined by either of the formulae (1) and (2). It is easily verified that the sets of relationships among the coefficients $\{T_\nu\}$ and $\{\hat{T}_\nu\}$ resulting from formulae (5) and (6) reduce in the case under consideration to those deriving from (1) and (2). Furthermore, using equation (1), we find that in the notation of condition (7), $G_0 = I$, $G_\nu = 0$ $(\nu = 1, 2, \cdots)$. Hence, condition (7) is satisfied as is also, for similar reasons, condition (8).

The second case in which conditions (5)–(8) serve uniquely to determine an inverse series $p\{\hat{T}_\nu|z\}$ is that in which the coefficients of the series $p\{T_\nu|z\}$ are either row vectors or column vectors of finite dimension, with $T_0 \neq 0$; and it is the purpose of this paper to show that this is so. With regard to the difficulty associated with condition (12) we remark that, as is easily verified, if T_0 is a non-zero row vector $\hat{T}_0 = T_0^+ = T_0^*(T_0 T_0^*)^{-1}$, so that in this case $T_0 \hat{T}_0 = 1$; if T_0 is a non-zero column vector $\hat{T}_0 = T_0^+ = (T_0^* T_0)^{-1} T_0^*$, so that we now have $\hat{T}_0 T_0 = 1$. In both of these cases, therefore, condition (12) is satisfied.

We first consider in extenso the case in which the coefficients of the series $p\{T_\nu|z\}$ are row vectors:

THEOREM 1. *Let the coefficients of the formal series $p\{T_\nu|z\}$ be row vectors of finite dimension with complex elements, with $T_0 \neq 0$, and let z be a complex scalar; then the formal power series $p\{\hat{T}_\nu|z\}$ is uniquely determined by conditions* (5)–(8), *and its coefficients $\{\hat{T}_\nu\}$ are column vectors of the same dimension as the $\{T_\nu\}$ and may be constructed by means of the recursion*

$$(13) \qquad \hat{T}_0 = T_0^*(T_0 T_0^*)^{-1}$$

$$\left.\begin{aligned}
&(14) \qquad X_r = \sum_{\nu=1}^{r} T_\nu \hat{T}_{r-\nu} \\
&(15) \qquad Y_r = \{T_r - \sum_{\nu=1}^{r-1} (X_\nu + T_0 Y_\nu) T_{r-\nu}\}^* (T_0 T_0^*)^{-1} \\
&(16) \qquad \hat{T}_r = -\hat{T}_0 X_r + (I - \hat{T}_0 T_0) Y_r .
\end{aligned}\right\} \qquad (r = 1, 2, \cdots)$$

PROOF. We have already shown that conditions (5)–(8) lead to the formula $\hat{T}_0 = T_0^+$, so that formula (13) is correct.

It has also been shown that condition (5) leads to equation (10) for $\hat{T}_1$, and that this equation is soluble. If the general matrix equation (11) is soluble, its solution is

$$X = A^{+}CB^{+} + Y - A^{+}AYBB^{+}, \tag{17}$$

where Y is an arbitrary matrix whose dimensions are such as to make formula (17) meaningful. Hence, in the case under consideration in which $T_0^{+} = \hat{T}_0$ and $T_0\hat{T}_0 = 1$, $\hat{T}_1$ has the form

$$\hat{T}_1 = -\hat{T}_0 X_1 + (I - \hat{T}_0 T_0)Y' \tag{18}$$

where

$$X_1 = T_1\hat{T}_0, \tag{19}$$

I is the unit matrix having the same dimension as $\hat{T}_0 T_0$, and Y' is a column vector of the same dimension as $\hat{T}_0$ and is yet to be determined. The equation analogous to (9) determined from condition (6) is

$$\hat{T}_0 T_0 \hat{T}_1 + \hat{T}_0 T_1 \hat{T}_0 + \hat{T}_1 T_0 \hat{T}_0 = \hat{T}_1$$

which, since $T_0\hat{T}_0 = 1$, reduces to

$$\hat{T}_0\{T_0\hat{T}_1 + T_1\hat{T}_0\} = 0. \tag{20}$$

Using formulae (18) and (19), we have

$$T_0\hat{T}_1 + T_1\hat{T}_0 = -T_0\hat{T}_0 T_1\hat{T}_0 + T_0(I - \hat{T}_0 T_0)Y' + T_1\hat{T}_0$$

so that

$$T_0\hat{T}_1 + T_1\hat{T}_0 = 0 \tag{21}$$

whatever value Y' turns out to have, and condition (20) is satisfied. It also follows from formula (21) that, in the notation of condition (7), G_1 is the zero 1×1 matrix, and hence $G_1^* = G_1$.

We must now discuss the use of condition (8) in the final determination of $\hat{T}_1$: it is required that the square matrix $\hat{T}_0 T_1 + \hat{T}_1 T_0$ be $*$-symmetric, i.e. that

$$\hat{T}_0 T_1 - \hat{T}_0 X_1 T_0 + Y'T_0 - \hat{T}_0 T_0 Y'T_0$$

be $*$-symmetric. The most general form of the column vector $Y' - \hat{T}_0 T_0 Y'$ satisfying this single requirement is

$$\begin{aligned}(I - \hat{T}_0 T_0)Y' &= (T_1^* - T_0^* X_1^*)(T_0 T_0^*)^{-1} + T_0^*(T_0 T_0^*)^{-1}\alpha \\ &= T_1^*(T_0 T_0^*)^{-1} - \hat{T}_0 X_1^* + \hat{T}_0\alpha,\end{aligned} \tag{22}$$

where α is an undetermined finite scalar. However, by premultiplying equation (22) throughout by T_0, we derive

$$T_0 T_1^*(T_0 T_0^*)^{-1} - X_1^* + \alpha = 0.$$

Hence

$$(I - \hat{T}_0 T_0)Y' = (I - \hat{T}_0 T_0)T_1^*(T_0 T_0^*)^{-1}.$$

This equation may be solved for Y', and its general solution is

$$Y' = T_1^*(T_0 T_0^*)^{-1} + \hat{T}_0 T_0 Y'',$$

where Y'' is an arbitrary vector having the same dimension as Y'. From formula (18) we now have

$$\begin{aligned}\hat{T}_1 &= -\hat{T}_0 X_1 + (I - \hat{T}_0 T_0) T_1^*(T_0 T_0^*)^{-1} + (I - \hat{T}_0 T_0)\hat{T}_0 T_0 Y'' \\ &= -\hat{T}_0 X_1 + (I - \hat{T}_0 T_0) Y_1\end{aligned}$$

where Y_1 is obtained by setting $r = 1$ in formula (15). In conclusion, we have shown that the coefficient $\hat{T}_r$ is uniquely determined by the recursion of formulae (13)–(16) when $r = 1$. We also wish to remark that, since

$$X_1 + T_0 Y_1 = T_1 \hat{T}_0 + \hat{T}_0^* T_1^*,$$

the 1×1 matrix $X_1 + T_0 Y_1$ is $*$-symmetric.

We now assume that formulae (14)–(16) are valid when r is replaced by $1, 2, \cdots, r-1$ and, furthermore, that

$$\sum_{\nu=0}^{r'} T_\nu \hat{T}_{r'-\nu} = 0 \qquad (r' = 1, 2, \cdots, r-1) \tag{23}$$

and, in the notation of formulae (14) and (15), that

$$(X_\nu + T_0 Y_\nu)^* = (X_\nu + T_0 Y_\nu). \quad (\nu = 1, 2, \cdots, r-1) \tag{24}$$

Equating coefficients of z^r in relationship (5), we have

$$\sum_{\nu=0}^{r} T_\nu \sum_{\nu'=0}^{r-\nu} \hat{T}_{\nu'} T_{r-\nu-\nu'} = T_r$$

or, using formulae (23),

$$T_0 \hat{T}_0 T_r + \{\sum_{\nu=0}^{r} T_\nu \hat{T}_{r-\nu}\} T_0 = T_r$$

i.e.

$$T_0 \hat{T}_r T_0 = -X_r T_0 \tag{25}$$

where X_r is given by formulae (14). This equation, regarded as an equation in the unknown column vector $\hat{T}_r$, is soluble if

$$-T_0 \hat{T}_0 X_r T_0 \hat{T}_0 T_0 = -X_r T_0,$$

a relationship which is clearly satisfied; the general solution of equation (25) is then given by the formula

$$\hat{T}_r = -\hat{T}_0 X_r + (I - \hat{T}_0 T_0) Y' \tag{26}$$

where Y' is an arbitrary vector of the same dimension as $\hat{T}_0$. Equating coefficients of z^r in relationship (6), it is required that

$$\sum_{\nu=0}^{r} \hat{T}_\nu \sum_{\nu'=0}^{r-\nu} T_{\nu'} \hat{T}_{r-\nu-\nu'} = \hat{T}_r$$

or, again using formulae (23), that

$$\hat{T}_0 \sum_{\nu=0}^{r} T_\nu \hat{T}_{r-\nu} + \hat{T}_r T_0 \hat{T}_0 = \hat{T}_r \tag{27}$$

i.e. that

$$\hat{T}_0 X_r - \hat{T}_0 T_0 \hat{T}_0 X_r + \hat{T}_0 T_0 (I - \hat{T}_0 T_0) Y' = 0$$

a relationship that is clearly satisfied, independent of Y'. We note that by premultiplying equation (27) throughout by T_0, we immediately derive the formula

$$\sum_{\nu=0}^{r} T_\nu \hat{T}_{r-\nu} = 0.$$

Thus, in the notation of condition (7), $G_r = 0$ and hence $G_r^* = G_r$.

Turning to the last of the conditions which $\hat{T}_r$ must satisfy, it is required that the square matrix $\sum_{\nu=0}^{r} \hat{T}_\nu T_{r-\nu}$ be $*$-symmetric; this matrix is, from formula (26) and those of formulae (16) that have been assumed true, equal to

$$\hat{T}_0 T_r + \sum_{\nu=1}^{r-1} \{-\hat{T}_0 X_\nu + (I - \hat{T}_0 T_0) Y_\nu\} T_{r-\nu} - \hat{T}_0 (X_r + T_0 Y') T_0 + Y' T_0. \tag{28}$$

By use of those of formulae (15) that have been assumed true, we find that

$$\sum_{\nu=1}^{r-1} Y_\nu T_{r-\nu} = \{\sum_{\nu=1}^{r-1} T_\nu^* T_{r-\nu} - \sum_{\nu=1}^{r-2} \sum_{\nu'=1}^{r-\nu-1} T_{\nu'}^* (X_\nu + T_0 Y_\nu)^* T_{r-\nu-\nu'}\} (T_0 T_0^*)^{-1}. \tag{29}$$

It is clear from formulae (24) that the matrix on the right hand side of equation (29) is $*$-symmetric. Hence the condition that the matrix (28) should be $*$-symmetric reduces to the equivalent condition that the matrix

$$T_0^* Y_r^* - \hat{T}_0 X_r T_0 + \{(I - \hat{T}_0 T_0) Y'\} T_0$$

should be $*$-symmetric, where Y_r is given by formula (15). The most general form of the matrix $(I - \hat{T}_0 T_0) Y'$ which satisfies this condition in isolation is

$$(I - \hat{T}_0 T_0) Y' = Y_r - \hat{T}_0 X_r^* + \hat{T}_0 \alpha \tag{30}$$

where α is an undetermined finite scalar. However, by premultiplying equation (30) throughout by T_0, we see that we must also have

$$T_0 Y_r - X_r^* + \alpha = 0,$$

so that

$$(I - \hat{T}_0 T_0) Y' = (I - \hat{T}_0 T_0) Y_r.$$

This equation may be solved for the vector Y', and its general solution is

$$Y' = Y_r + \hat{T}_0 T_0 Y''$$

where Y'' is an arbitrary column vector of the same dimension as Y'. From formula (26) we now derive

$$\begin{aligned}\hat{T}_r &= -\hat{T}_0 X_r + (I - \hat{T}_0 T_0) Y_r + (I - \hat{T}_0 T_0) \hat{T}_0 T_0 Y'' \\ &= -\hat{T}_0 X_r + (I - \hat{T}_0 T_0) Y_r.\end{aligned}$$

Hence we have shown that formula (16) holds as it stands.

It remains to show that the number $X_r + T_0 Y_r$ is real, i.e. that the 1×1 matrix $X_r + T_0 Y_r$ is $*$-symmetric. The matrix $\sum_{v=0}^{r} T_v \hat{T}_{r-v}$ is, from condition (8), $*$-symmetric. In this sum we replace the vectors $\{\hat{T}_v\}$ by their equivalent expressions given by formulae (16) and, furthermore, the vector Y_r occurring in the term $-\hat{T}_0 T_0 Y_r$ of formula (16) by its equivalent expression given by formula (15); in this way, we know that the matrix

$$\begin{aligned}&\hat{T}_0 T_r + \sum_{v=1}^{r-1} \{-\hat{T}_0 X_v + (I - \hat{T}_0 T_0) Y_v\} T_{r-v} - \hat{T}_0 (X_r + T_0 Y_r) T_0 \\ &\quad + \{T_r - \sum_{v=1}^{r-1} (X_v + T_0 Y_v) T_{r-v}\}^* (T_0 T_0^*)^{-1} T_0\end{aligned}$$

is $*$-symmetric. The matrices $\hat{T}_0 T_r + T_r (T_0 T_0^*)^{-1} T_0$ and

$$\sum_{v=1}^{r-1} [\hat{T}_0 \{X_v + T_0 Y_v\} T_{r-v} + T_{r-v}^* \{X_v^* + Y_v^* T_0^*\} (T_0 T_0^*)^{-1} T_0]$$

are, by inspection, $*$-symmetric; furthermore the matrix $\sum_{v=1}^{r-1} Y_v T_{r-v}$ has, with the aid of equation (29), already been proved to be $*$-symmetric. Thus we are left with the knowledge that the matrix $T_0^* (X_r + T_0 Y_r) T_0$ is $*$-symmetric. Since $T_0 \neq 0$, the 1×1 matrix $X_r + T_0 Y_r$ is $*$-symmetric.

The result of the theorem now follows by induction.

In the case in which the coefficients $\{T_v\}$ are column vectors, we have

THEOREM 2. *Let the coefficients of the formal series $p\{T_v|z\}$ be column vectors of finite dimension with complex elements, with $T_0 \neq 0$, and let z be a complex scalar; then the formal power series $p\{\hat{T}_v|z\}$ is uniquely determined by conditions* (5)–(8), *and its coefficients $\{\hat{T}_v\}$ are row vectors of the same dimension as the $\{T_v\}$ and may be constructed by means of the recursion*

$$\left.\begin{aligned}\hat{T}_0 &= (T_0^* T_0)^{-1} T_0^* \\ X_r &= \sum_{\nu=1}^{r} \hat{T}_{r-\nu} T_\nu \\ Y_r &= (T_0^* T_0)^{-1}\{T_r - \sum_{\nu=1}^{r-1} T_{r-\nu}(X_\nu + Y_\nu T_0)\}^* \\ \hat{T}_r &= -X_r \hat{T}_0 + Y_r(I - T_0 \hat{T}_0).\end{aligned}\right\} \qquad (r = 1, 2, \cdots)$$

Proof. The proof of this theorem is analogous to that of Theorem 1; the details are therefore omitted. We remark only that in this case we have

$$\sum_{\nu=0}^{r} \hat{T}_\nu T_{r-\nu} = 0 \qquad (r = 1, 2, \cdots)$$

and

$$(X_r + Y_r T_0)^* = X_r + Y_r T_0. \qquad (r = 1, 2, \cdots)$$

It follows from the symmetry of conditions (5)–(8) that the inverse of the inverse of a formal power series with vector valued coefficients is the original series.

We conclude by remarking that in many applications of formal power series one is concerned with series of the form $p\{\mu; T_\nu|z\} = \sum_{\nu=\mu}^{\infty} T_\nu z^\nu$, and that the inverse of such a series has the form $p\{-\mu; \hat{T}_\nu|z\}$. For the sake of brevity we have dealt consistently with the case in which $\mu = 0$. However, our theory can immediately be extended to the more general case merely by writing

$$p\{\mu; T_\nu|z\} = z^\mu p\{0; T_{\nu+\mu}|z\}$$

and

$$p\{-\mu; \hat{T}_\nu|z\} = z^{-\mu} p\{0; \hat{T}_{\nu-\mu}|z\}.$$

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